1 Introduction

In [?], Laczkovich proved

Fact 1.1. *Given a Polish space and an infinite sequence $\vec{B}$ of its Borel subsets, then*

1. *either there is an infinite set $a \subset \omega$ such that $\limsup_a \vec{B}$ is countable,*
2. *or there is an inifnite set $a \subset \omega$ such that $\liminf_a \vec{B}$ is uncountable.*

Later, Komjáth improved this to include sequences of analytic sets. Following further work of ???, I define

Definition 1.2. Let I be a σ-ideal on a Polish space X, the ideal has the *Laczkovich-Komjáth*, or LK property, if for every infinite sequence $\vec{B}$ of analytic sets, either there is an infinite set $a \subset \omega$ with $\limsup_a \vec{B} \in I$ or there is an infinite set $a \subset \omega$ such that $\liminf_a \vec{B} \notin I$.

2 Negative results

The first concern: is the LK property for sequences of analytic sets truly stronger than the formulation with just sequences of Borel sets? It turns out that the answer is negative for a large and well-researched class of σ-ideals:

Definition 2.1. A σ-ideal I on a Polish space X is $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$ if for every analytic set $A \subset 2^\omega \times X$ the set $\{y \in 2^\omega : A_y \in I\}$ is coanalytic.

For example, the ideals of countable, meager or Lebesgue null sets are $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$ by classical results of ??. [?] gives many more examples and relates this property to forcing properties of the quotient P_I.

Proposition 2.2. *Suppose that the ideal I is $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$. If the LK property fails for a sequence of analytic sets, then it fails for a sequence of Borel sets.*

Proof. Note that the property $\phi(\vec{B}) = \forall a \subset \omega \ \liminf_a B_n \in I$ is a $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$ property for countable sequences of sets. By the first reflection theorem [?], whenever $\vec{B}$ is a sequence of analytic sets with $\phi(\vec{B})$, then there is a sequence $\vec{C}$ of their Borel supersets with $\phi(\vec{C})$. Clearly, if $\vec{B}$ witnessed the failure of LK-property, so does the sequence $\vec{C}$. □

The question of further reduction of Borel rank of the offending sequence of Borel sets remains open. In all specific cases discussed in this paper, these sets can be chosen to be closed.

Proposition 2.3. *Suppose that I is a σ-ideal on a Polish space X such that the quotient forcing P_I is proper. If I has the LK property, then P_I does not add an independent real.*

Proof. Suppose that $B \in P_I$ is a condition and $\dot{y} \in 2^\omega$ is a P_I-name. Use the properness assumeption to strengthen B if necessary to find a Borel function $f : B \to 2^\omega$ such that $B \Vdash \dot{y} = \dot{f}(\dot{x}_{gen})$. Consider the sets $B_n = \{x \in B : f(x)(n) = 0\}$ for $n \in \omega$. By the LK property, there are two cases. Either there is an infinite set $a \subset \omega$ such that $C = \limsup_{n\in a} B_n \notin I$; in this case $C \Vdash$ for all but finitely many numbers $n \in a$, $\dot{y}(n) = 0$. Or, there is an infinite set $a \subset \omega$ such that $C = B \setminus \limsup_{n\in a} B_n \notin I$; in this case $C \Vdash$ for all but finitely many numbers $n \in a$, $\dot{y}(n) = 1$. In either case, the sequence $\dot{y}$ is not independent. □

This proposition shows that ideals such as the meager sets or the null sets do not have the LK property. One forcing adding no independent reals is the Sacks forcing associated with the σ-ideal of countable sets. The LK property of this ideal is exactly the contents of the results of Laczkovich and Komjáth. Another forcing adding no independent reals is the Miller forcing, associated with the σ-ideal generated by the compact subsets of the Baire space ω^ω. There, the LK property fails since the forcing adds an unbounded real:

Proposition 2.4. *(LC) Suppose that I is a universally Baire σ-ideal on a Polish space X such that the quotient forcing P_I is proper and preserves Baire category. If I has the LK property, then P_I does not add an unbounded real.*

I do not know if the long list of assumtpions is necessary, in particular if the preservation of Baire category can be removed from the list. The large cardinals and definability assumptions are used to secure determinacy in a certain infinite game; in such cases as the Miller forcing, the necessary winning strategy can be easily constructed manually.

Proof. The assumptions imply that Player I has a winning strategy σ in the following two player infinite game: in it, Player I and II alternate to produce Borel sets $C_n \in P_I$ and $D_n \in P_I$ respectively with the demand that $D_n \subset C_n$. Player II wins if the *result* of the play, the set $\limsup_n D_n$, does not belong to I.

Now suppose that $\dot{y}$ is a P_I-name for an unbounded real; I must produce a failure of the LK property. By induction on $n \in \omega$ build finite sets T_n of partial finite plays of the game according to the strategy σ in which Player II makes the last move and

- $T_0 = \{0\}$;
- the last move of every play in T_{n+1} decides $\dot{y} \restriction n$;
- every play in $\bigcup_{m\in n} T_m$ has a one move extension in T_n.

Let $B_n \subset X$ be the union of the last moves of Player II in all plays in the set T_n, for every number $n \in \omega$. Obviously, these are Borel sets. If $a \subset \omega$ is an infinite set, then the first item shows that the set $\liminf_{n\in a} B_n$, if I-positive, would force a ground model bound on the function $\dot{y}$, which is impossible. And the second item shows that there is an infinite play τ according to the strategy σ

such that $\tau \restriction i \in T_{n_i}$ for every number $i \in \omega$, where n_i is the i-th element of the set a. Since the result of the play τ must be I-positive, so must $\limsup_{n \in a} B_n$. Thus the sequence $B_n : n \in \omega$ witnesses the failure of the LK property.

□

In search for bounding proper partial orders that do not add independent reals one immediately encounters iterations and products of Sacks forcing. It turns out that σ-ideals associated with such posets also never have the LK property:

Proposition 2.5. *(LC) Suppose that I is a universally Baire σ-ideal on a Polish space X such that the quotient forcing P_I is proper and preserves Baire category. If I has the LK property, then P_I adds exactly one generic real degree.*

The list of assumptions again seems to contain some unnecessary items.

Proof. Suppose that the assumptions hold and the σ-ideal I has the LK property. Suppose that $B \in P_I$ is a condition and $\dot{y} \in 2^\omega$ is a P_I-name. I must show that either $\dot{x}_{gen} \in V[\dot{y}]$ or $\dot{y} \in V$. Passing to a stronger condition if necessary I may find a Borel function $f : B \to 2^\omega$ such that $B \Vdash \dot{y} = \dot{f}(\dot{x}_{gen})$. I must show that either there is a Borel I-positive set on which the function f is constant or a Borel I-positive set on which the function f is one-to-one.

Suppose that the former alternative fails. The first step in the proof is the extraction of a perfect set of conditions such that their f-images are pairwise disjoint. To do that, look at the ideal J on 2^ω of all Borel sets C such that $B \Vdash \dot{y} \notin C$. If the ideal J were c.c.c. then P_J is a definable c.c.c. forcing and therefore adds an independent real by ??? Moreover, $\dot{y}$ is a P_I-name for P_J-generic. Thus P_I adds an independent real, contradicting the LK property of I by Proposition ??. Now, the forcing P_I is bounding by the previous proposition and the LK property. Follow the proof of ??[**?**] to find a perfect collection of compact I-positive sets $C_b : b \in [\omega]^{\aleph_0}$ whose f-images are pairwise disjoint.

For the remander of the proof assume $X = [0, 1]$. Fix a function $g \in [0, 1]^\omega$ such that for every point $x \in [0, 1]$ there are infinitely many $n \in \omega$ such that $|g(n) - x| < 1/n$. Consider the sets $B_n \subset X$ defined by $x \in B_n$ if there is a (unique) infinite set $b \subset \omega$ such that $x \in C_b$, and $|g(i) - x| < 1/i$ where $i \in \omega$ is such that n is between the i-th and $i+1$-th element of the set b. Whenever $a \subset \omega$ is infinite, the set $\limsup_{n \in a} B_n$ is I-positive, since it contains the set C_a. The LK property then implies the existence of an infinite set $a \subset \omega$ such that $C = \liminf_{n \in a} B_n \notin I$. It is not difficult to see that for every set $b \subset \omega$, the set C can pick at most one point of the set C_b, namely the limit of the numbers $g(i)$ where i varies over all those numbers for which there is a number $n \in a$ between i-th and $i+1$-st element of b. Since the f-images of the sets C_b are pairwise disjoint, $f \restriction C$ is one-to-one, completing the proof.

□

The above propositions do not cover all reasons for which the LK property may fail; see the following example obtained directly from the definitions:

Example 2.6. Let $B_n : n \in \omega$ be an independent collection of clopen subsets of the Cantor space 2^ω. Consider the σ-ideal generated by all sets $C \subset 2^\omega$ such that there is $m \in \omega$ such that for every $k \in \omega$ there are m many sets in the collection, indexed by numbers greater than k, whose union covers the set C. This construction fits into the Hausdorff submeasure scheme of [**?**], and so the quotient forcing P_I is proper, bounding, adds no independent reals, and adds one generic real degree. In addition, the σ-ideal is generated by closed sets and therefore the quotient P_I also preserves Baire category.

It is quite obvious that the sequence $B_n : n \in \omega$ witnesses the failure of the LK property. Let $a \subset \omega$ be infinite and consider the sets $\limsup_a B_n$, and $\liminf_a B_n$. The latter belongs to the ideal I by the definitions. I must prove that the former is I-positive. Suppose that $C = \bigcup_m C_m$ is a set in the ideal I, written as a countable union of sets such that for every $k \in \omega$ there is a set $b_{m,k} \subset \omega \setminus k$ of size m such that $C_m \subset \bigcup_{n \in b_{m,k}} B_n$. By induction on $i \in \omega$ choose numbers $n_i \in a$ such that $b_{i,n_i+1} \subset n_{i+1}$, and find a point $x \in \bigcap_i B_{n_i} \setminus \bigcup_{n \in b_{i,n_{i+1}}, i \in \omega} B_n$. Then $x \in \limsup_a B_n \setminus C$ and the set $\limsup_a B_n$ is I-positive as desired.

Another negative example of a quite different flavor:

Example 2.7. Let E_0 be the equivalence on 2^ω defined by xE_0y iff $x\Delta y$ is finite, and let I be the σ-ideal generated by those Borel sets that meet every E_0 equivalence class in at most one point. Then I does not have the LK property.

In order to prove this, for a set $B \subset 2^\omega$ and a finite binary sequence $t \in 2^{<\omega}$ let $B \circ t$ be the set of all sequences obtained from elements of B by rewriting the appropriate initial segment with t. Construct closed sets $B_n : n \in \omega$ so that

- $\forall t \in 2^{<\omega}, t \neq 0\ \forall^\infty n \in \omega\ B_n \cap B_n \circ t = 0$;
- $\forall b \subset \omega$ finite $\forall f : b \to 2^{<\omega} \forall^\infty n \in \omega$ if $\bigcap_{m \in b} B_m \circ f(m) \notin I$ then $\bigcap_{m \in b} B_m \circ f(m) \cap B \notin I$.

The construction is easy. Enumerate $2^{<\omega}$ as $t_m : m \in \omega$.

Note that in the end, whenever $a \subset \omega$ is infinite, $\liminf_a \vec{B} \in I$, since it is a Borel set meeting each E_0 class in at most one point by the first item. It is also true that $\limsup_a \vec{B} \notin I$, but this must be proved more carefully.

Question 2.8. Suppose $n < m$ are natural numbers. Does the ideal of sets of σ-finite n-dimensional Hausdorff measure on $\mathbb{R}^m$ have the LK property?

3 Positive results

There are two classes of ideals for which I can confirm the LK property, both studied in [**?**].

Definition 3.1. A σ-ideal I on a Polish space X is generated by a σ-compact collection of compact sets if there are compact sets $K_n : n \in \omega$ in the hyperspace $K(X)$ such that the elements of $\bigcup_n K_n$ σ-generated the ideal I.

It turns out that in this situation One can find a single compact set $K \subset K(X)$ whose elements generate the σ-ideal I. A typical example of ideals in this class is the ideal of countable sets; a more sophisticated example is the ideal of sets of σ-finite packing measure mass in a compact metric space. The quotient forcings P_I arising from the ideals in this class have been studied in [**?**].

Proposition 3.2. *Every σ-ideal generated by a σ-compact collection of compact subsets of a Polish space X has the LK property.*

Proof. The proof uses Mathias forcing ??? consisting of pairs $p = \langle c_p, a_p \rangle$ such that $c, a \subset \omega$ are a finite and infinite set respectively, and $q \leq p$ if $c_p \subset c_q$, $a_q \subset a_p$, and $c_q \setminus c_p \subset a_p$. This forcing is proper and adds a generic infinite set $\dot{a}_{gen} \subset \omega$ which is the union of the first coordinates in the generic filter.

Before we embark on the proof, I will review several properties of Mathias forcing. First of all, given a formula ϕ of the forcing language and a condition p, ϕ can be decided by a *direct extension* of p, that is, a condition $q \leq p$ with $c_p = c_q$. This has the following consequence. Whenever $K \subset K(X)$ is a compact set in the hyperspace closed under subsets, and $\dot{C}$ a name for an element of K, then I can pass to a direct extension $q \leq p$ which *almost decides $\dot{C}$ in the direction of D* in the sense that for every basic open set $O \subset X$ there is a tail $a_O \subset a_q$ such that $\langle c_q, a_O \rangle$ decides the statement $\dot{C} \cap O = 0$, and $D = X \setminus \bigcup\{O \subset X : O$ is basic open and the decision was negative$\}$. A compactness argument shows that necessarily $D \in K$.

Another observation: whenever $\vec{B}$ is a sequence of analytic subsets of the space X, the sets $\liminf_{\dot{a}_{gen}} \vec{B}$ and $\limsup_{\dot{a}_{gen}} \vec{B}$ do not depend on finite changes of the set $\dot{a}_{gen}$, and therefore if a condition forces a statement about these two sets and perhaps some ground model parameters, then so do all finite variations of this condition. This means that if ϕ is a formula of the forcing language using these two sets and perhaps some ground model parameters and $p \in P$ is a condition forcing $\exists n\ \phi(n)$, then there is a direct extension of the condition p deciding the number n: first, find an arbitrary extension $q \leq p$ deciding the value of n and then replace c_q with c_p.

Suppose that $\vec{B}$ is a sequence of analytic sets. I will show that either there is a condition $p \in P$ forcing $\limsup_{\dot{a}_{gen}} \vec{B} \in I$, or there is a condition $p \in P$ forcing $\liminf_{\dot{a}_{gen}} \vec{B} \notin I$. The proposition will then follow from a standard absoluteness argument. Let M be a countable elementary submodel of a large structure, and let $a \subset \omega$ be an M-generic set for the Mathias forcing compatible with the condition p. By the forcing theorem, $M \models \limsup_a \vec{B} \in I$ or $\liminf_a \vec{B} \notin I$. The ideal I is $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$, and therefore this statement carries over to V by analytic absoluteness.

Suppose then for contradiction that the empty condition forces $\liminf_{\dot{a}_{gen}} \vec{B} \notin I$ and $\liminf_{dotagen} \vec{B} \in I$. Let $K_i : i \in \omega$ be the compact subsets of the hyperspace $K(X)$ whose elements generate the σ-ideal I; without loss of generality, the sets K_i are closed under subsets, and increase with respect to inclusion. I can find names $\dot{C}_i : i \in \omega$ such that $P \Vdash \forall i\ \dot{C}_i \in \dot{K}_i$ and $\liminf_{\dot{a}_{gen}} \vec{B} \subset \bigcup_i \dot{C}_i$. Fix also continuous functions $f_n : \omega^\omega \to X$ such that $B_n = \text{rng}(f_n)$. Now, by induction on $i \in \omega$ build

- numbers n_i and infinite sets $a_i \subset \omega$ such that $n_0 < n_1 < \dots$, $a_0 \supset a_1 \supset \dots$ and $n_{i+1} \in a_i$;
- finite sequences $t_i^j : j \leq i$ of natural numbers such that for fixed j, the sequences t_i^j increase with respect to inclusion;
- basic open sets O_i;

so that the condition $\langle 0, \{n_j : j \in i\} \cup a_i \rangle \in I$ forces $O_i \cap \dot{C}_i = 0$ and $\limsup_{\dot{a}_{gen}} \vec{B} \cap \bigcap_{j \in i} O_j \cap \bigcap_{j \in i} f''_{n_j} O_{t_i^j} \notin I$. If this can be done, in the end there will be a unique point $x \in \bigcap_{i,j} f''_{n_j} O_{t_i^j}$, and the condition $\langle 0, \{n_i : i \in \omega\} \rangle$ will force $\check{x} \in \liminf_{\dot{a}_{gen}} \vec{B} \setminus \bigcup_i \dot{C}_i$, contradicting the choice of the names $\dot{C}_i$.

The induction process is easy. Start with setting $a_0 = \omega$. Suppose that the numbers $n_j : j \in i$, sequences $t_i^j : j \in i$, the open sets $O_j : j \in i$, and the set a_i have been constructed. Find an infinite set $b \subset a_i$ such that for every set $c \subset \{n_j : j \in i\}$, the condition $\langle c, b \rangle$ almost decides the set $\dot{C}_i$ in the direction of a set $D_c \in K_i$. Thinning out the set b if necessary, I can find a basic open set O_i disjoint from all the sets D_c and sequences $t_{i+1}^j : j \in i$ properly extending $t_i^j : j \in i$ such that the condition $\langle 0, b \rangle$ forces $\limsup_{\dot{a}_{gen}} \vec{B} \cap \bigcap_{j \in i+1} O_j \cap \bigcap_{j \in i} f''_{n_j} O_{t_{i+1}^j} \notin I$. Passing to a tail of b, I can make sure that for every set $c \subset \{n_j : j \in i\}$, $\langle c, b \rangle \Vdash \dot{O}_i \cap \dot{C}_i = 0$. Finally, thinning out b to some further infinite set a_{i+1}, I can find a number $n_i \in b$ such that $\langle c, a_{i+1} \rangle \Vdash \limsup_{\dot{a}_{gen}} \vec{B} \cap \bigcap_{j \in i+1} O_j \cap \bigcap_{j \in i} f''_{n_j} O_{t_{i+1}^j} \cap \vec{B}(n_i) \notin I$. Let $t_{i+1}^i = 0$ and proceed with the induction process.

□

Definition 3.3. A capacity ϕ on a compact metric space X is *Ramsey* if for every $\varepsilon > 0$ and $\delta > 0$ and every sequence $B_n : n \in \omega$ of Borel subsets of X of ϕ-mass $< \varepsilon$, there are distinct numbers $n \neq m$ such that $\phi(B_n \cup B_m) < \varepsilon + \delta$.

Examples of Ramsey capacities are not so easy to come by. Clearly, the outer Lebesgue measure is not Ramsey, as any stochastically independent sequence of sets of measure $1/2$ shows. [**?**] constructs a number of Ramsey capacities. It turns out that the Hausdorff content in the Davies-Rogers example of Hausdorff measure with only zero and infinite values is a Ramsey capacity.

Proposition 3.4. *If ϕ is a Ramsey capacity on a compact metric space, then the ideal of sets of ϕ-mass zero has the LK property.*

Proof. This is in fact a direct consequence of ???[**?**]: in the Mathias forcing extension, every set can be covered by a ground model open set of arbitrarily close ϕ-mass.

Suppose for contradiction that $\vec{B}$ is a sequence of analytic sets violating the LK property. In particular, the Mathias poset P forces $\phi(\liminf_{\dot{a}_{gen}} \vec{B}) = 0$ and $\phi(\limsup_{\dot{a}_{gen}} \vec{B}) > 0$. Passing to a stronger condition $p \in P$, I may find a real number $\varepsilon > 0$ and a ground model open set O of mass $< \varepsilon$ such that $p \Vdash \liminf_{\dot{a}_{gen}} \vec{B} \subset O \wedge \phi(\limsup_{\dot{a}_{gen}} \vec{B}) > \varepsilon$. Let $a \subset \omega$ be a Mathias generic set consistent with the condition p, and work in $V[a]$. Let $x \in \limsup_a \vec{B} \setminus O$ be a point, and let $b \subset a$ be an infinite set consistent with the condition p such that $x \in \liminf_b \vec{B}$. Now $V[a] \models \liminf_b \vec{B} \setminus O \neq 0$, and by analytic absoluteness, $V[b] \models \liminf_b \vec{B} \setminus O \neq 0$. However, the set b is also Mathias generic by the geometric criterion, and so the latter statement contradicts what was forced by the condition p!

□

References