The problem of adaptivity for hp-FEM

Tomáš Vejchodský (vejchod@math.cas.cz)

Mathematical Institute, Academy of Sciences
Žitná 25, 115 67 Prague 1
Czech Republic

Outline

- *h*-FEM
- *p*-FEM
- *hp*-FEM
- *hp* adaptivity

Model problem

Classical formulation: $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

Model problem

Classical formulation: $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

$$-\int_\Omega \Delta u v \,\mathrm{d}x = \int_\Omega f v \,\mathrm{d}x$$

Model problem

Classical formulation: $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

$$-\int_\Omega \Delta u v \,\mathrm{d}x = \int_\Omega f v \,\mathrm{d}x$$

$$-\int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x = \int_\Omega f v \,\mathrm{d}x$$

Model problem

Classical formulation: $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

Weak formulation: $u \in H_0^1(\Omega)$

$$\underbrace{\int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x}_{a(u,v)} = \underbrace{\int_\Omega f v \,\mathrm{d}x}_{F(v)} \quad \forall v \in H_0^1(\Omega)$$

Model problem

Classical formulation: $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

Weak formulation: $u \in H_0^1(\Omega)$

$$\underbrace{\int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x}_{a(u,v)} = \underbrace{\int_\Omega f v \,\mathrm{d}x}_{F(v)} \quad \forall v \in H_0^1(\Omega)$$

Galerkin formulation: $u_{hp} \in V_{hp} \subset H_0^1(\Omega)$

$$a(u_{hp}, v_{hp}) = F(v_{hp}) \quad \forall v_{hp} \in V_{hp}$$

Galerkin method

Galerkin formulation: $u_{hp} \in V_{hp} \subset H_0^1(\Omega)$

$$a(u_{hp}, v_{hp}) = F(v_{hp}) \quad \forall v_{hp} \in V_{hp}$$

$\{\varphi_1, \varphi_2, \ldots, \varphi_N\}$ – basis of V_{hp}

Galerkin method

Galerkin formulation: $u_{hp} \in V_{hp} \subset H_0^1(\Omega)$

$$a(u_{hp}, v_{hp}) = F(v_{hp}) \quad \forall v_{hp} \in V_{hp}$$

$\{\varphi_1, \varphi_2, \ldots, \varphi_N\}$ – basis of V_{hp}

$$u_{hp} = \sum_{i=1}^{N} c_i \varphi_i$$

Galerkin method

Galerkin formulation: $u_{hp} \in V_{hp} \subset H_0^1(\Omega)$

$$a(u_{hp}, v_{hp}) = F(v_{hp}) \quad \forall v_{hp} \in V_{hp}$$

$\{\varphi_1, \varphi_2, \dots, \varphi_N\}$ – basis of V_{hp}

$$\begin{aligned} u_{hp} &= \sum_{i=1}^{N} c_i \varphi_i \\ a\Big(\sum_{i=1}^{N} c_i \varphi_i, \varphi_j\Big) &= F(\varphi_j) \quad j = 1, 2, \dots, N \end{aligned}$$

Galerkin method

Galerkin formulation: $u_{hp} \in V_{hp} \subset H_0^1(\Omega)$

$$a(u_{hp}, v_{hp}) = F(v_{hp}) \quad \forall v_{hp} \in V_{hp}$$

$\{\varphi_1, \varphi_2, \ldots, \varphi_N\}$ – basis of V_{hp}

$$\begin{aligned}
u_{hp} &= \sum_{i=1}^{N} c_i \varphi_i \\
a\Big(\sum_{i=1}^{N} c_i \varphi_i, \varphi_j\Big) &= F(\varphi_j) \quad j = 1, 2, \ldots, N \\
\sum_{i=1}^{N} c_i \underbrace{a(\varphi_i, \varphi_j)}_{A_{ji}} &= \underbrace{F(\varphi_j)}_{F_j} \quad j = 1, 2, \ldots, N
\end{aligned}$$

Galerkin method

Galerkin formulation: $u_{hp} \in V_{hp} \subset H_0^1(\Omega)$

$$a(u_{hp}, v_{hp}) = F(v_{hp}) \quad \forall v_{hp} \in V_{hp}$$

$\{\varphi_1, \varphi_2, \dots, \varphi_N\}$ – basis of V_{hp}

$$\begin{aligned}
u_{hp} &= \sum_{i=1}^{N} c_i \varphi_i \\
a\Big(\sum_{i=1}^{N} c_i \varphi_i, \varphi_j\Big) &= F(\varphi_j) \quad j = 1, 2, \dots, N \\
\sum_{i=1}^{N} c_i \underbrace{a(\varphi_i, \varphi_j)}_{A_{ji}} &= \underbrace{F(\varphi_j)}_{F_j} \quad j = 1, 2, \dots, N \\
Ac &= F
\end{aligned}$$

Finite Element Method

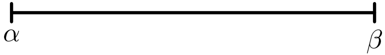
The lowest order approximation

$$V_{hp} = \{v_{hp} \in H_0^1(\Omega) : v_{hp}|_{K_i} \in P^1(K_i), i = 1, 2, \ldots, M\}$$

Finite Element Method

The lowest order approximation

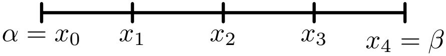
$$V_{hp} = \{v_{hp} \in H_0^1(\Omega) : v_{hp}|_{K_i} \in P^1(K_i), i = 1, 2, \ldots, M\}$$



Finite Element Method

The lowest order approximation

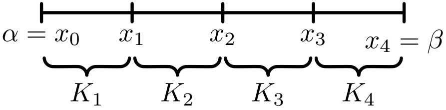
$$V_{hp} = \{v_{hp} \in H_0^1(\Omega) : v_{hp}|_{K_i} \in P^1(K_i), i = 1, 2, \ldots, M\}$$



Finite Element Method

The lowest order approximation

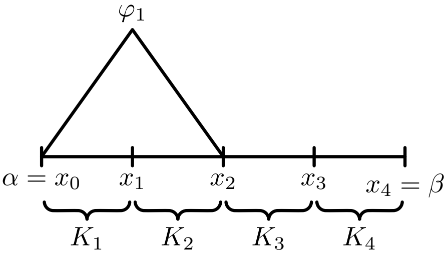
$$V_{hp} = \{v_{hp} \in H_0^1(\Omega) : v_{hp}|_{K_i} \in P^1(K_i), i = 1, 2, \ldots, M\}$$



Finite Element Method

The lowest order approximation

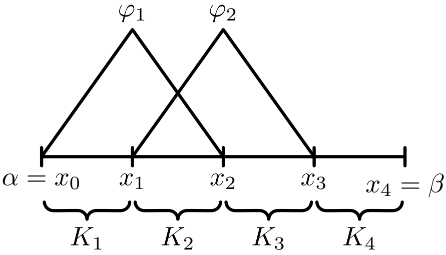
$$V_{hp} = \{v_{hp} \in H_0^1(\Omega) : v_{hp}|_{K_i} \in P^1(K_i), i = 1, 2, \ldots, M\}$$



Finite Element Method

The lowest order approximation

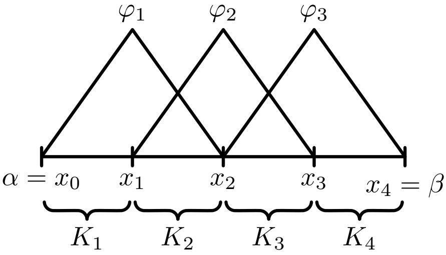
$$V_{hp} = \{ v_{hp} \in H_0^1(\Omega) : v_{hp}|_{K_i} \in P^1(K_i), i = 1, 2, \ldots, M \}$$



Finite Element Method

The lowest order approximation

$$V_{hp} = \{v_{hp} \in H_0^1(\Omega) : v_{hp}|_{K_i} \in P^1(K_i), i = 1, 2, \ldots, M\}$$



Finite Element Method

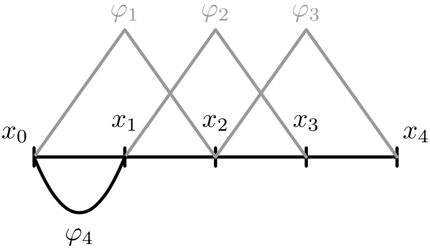
The second order approximation

$$V_{hp} = \{v_{hp} \in H_0^1(\Omega) : v_{hp}|_{K_i} \in P^2(K_i), i = 1, 2, \ldots, M\}$$

Finite Element Method

The second order approximation

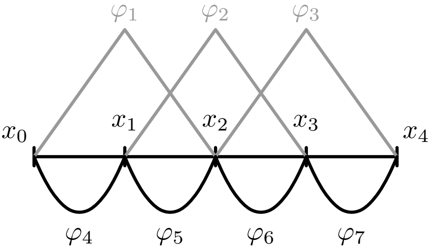
$$V_{hp} = \{v_{hp} \in H_0^1(\Omega) : v_{hp}|_{K_i} \in P^2(K_i), i = 1, 2, \dots, M\}$$



Finite Element Method

The second order approximation

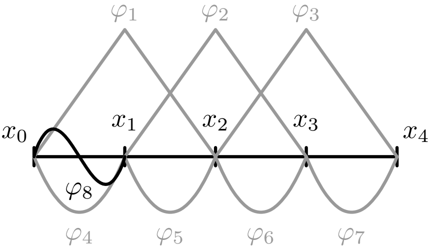
$$V_{hp} = \{v_{hp} \in H_0^1(\Omega) : v_{hp}|_{K_i} \in P^2(K_i), i = 1, 2, \ldots, M\}$$



Finite Element Method

The third order approximation

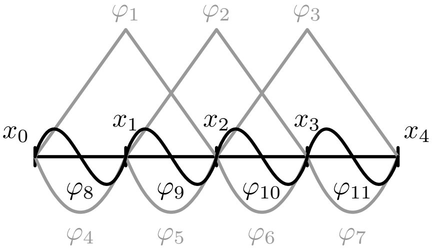
$$V_{hp} = \{v_{hp} \in H_0^1(\Omega) : v_{hp}|_{K_i} \in P^3(K_i), i = 1, 2, \ldots, M\}$$



Finite Element Method

The third order approximation

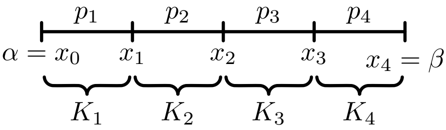
$$V_{hp} = \{ v_{hp} \in H_0^1(\Omega) : v_{hp}|_{K_i} \in P^3(K_i), i = 1, 2, \ldots, M \}$$



Finite Element Method

The variable order approximation (*hp*-FEM)

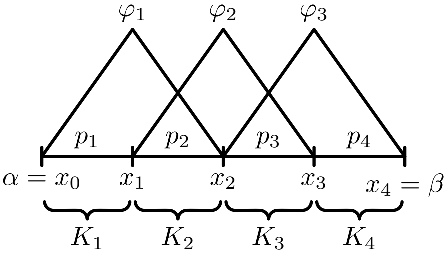
$$V_{hp} = \{v_{hp} \in H_0^1(\Omega) : v_{hp}|_{K_i} \in P^{p_i}(K_i), i = 1, 2, \ldots, M\}$$



Finite Element Method

The variable order approximation (*hp*-FEM)

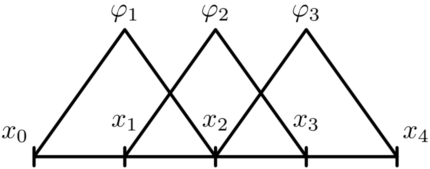
$$V_{hp} = \{v_{hp} \in H_0^1(\Omega) : v_{hp}|_{K_i} \in P^{p_i}(K_i), i = 1, 2, \dots, M\}$$



Finite Element Method

The variable order approximation (*hp*-FEM)

$$V_{hp} = \{v_{hp} \in H_0^1(\Omega) : v_{hp}|_{K_i} \in P^{p_i}(K_i), i = 1, 2, \ldots, M\}$$

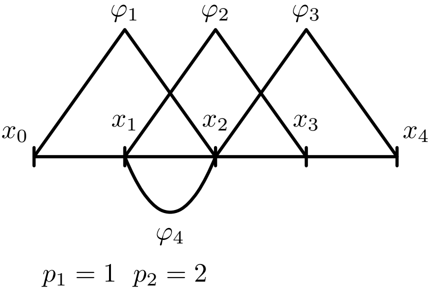


$p_1 = 1$

Finite Element Method

The variable order approximation (*hp*-FEM)

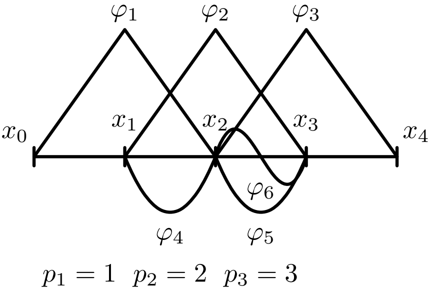
$$V_{hp} = \{v_{hp} \in H_0^1(\Omega) : v_{hp}|_{K_i} \in P^{p_i}(K_i), i = 1, 2, \ldots, M\}$$



Finite Element Method

The variable order approximation (*hp*-FEM)

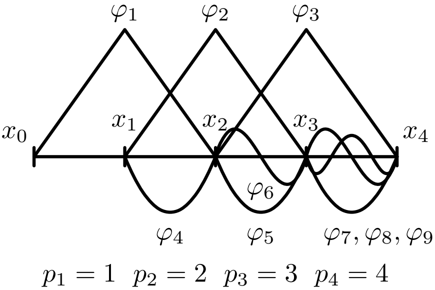
$$V_{hp} = \{v_{hp} \in H_0^1(\Omega) : v_{hp}|_{K_i} \in P^{p_i}(K_i), i = 1, 2, \ldots, M\}$$



Finite Element Method

The variable order approximation (*hp*-FEM)

$$V_{hp} = \{v_{hp} \in H_0^1(\Omega) : v_{hp}|_{K_i} \in P^{p_i}(K_i), i = 1, 2, \dots, M\}$$



h-FEM

$$\|u - u_{hp}\| \leq C(u)h$$

$$h = \max \operatorname{diam}(K_i)$$
$$N = N_{\mathrm{DOF}} = \dim V_{hp}$$

Step 1

h-FEM

$$\|u - u_{hp}\| \leq C(u)h$$

$$h = \max \operatorname{diam}(K_i)$$
$$N = N_{\mathrm{DOF}} = \dim V_{hp}$$

Step 2

h-FEM

$$\|u - u_{hp}\| \leq C(u)h$$

$$h = \max \operatorname{diam}(K_i)$$
$$N = N_{\mathrm{DOF}} = \dim V_{hp}$$

Step 3

h-FEM

$$\|u - u_{hp}\| \le C(u)h$$

$$h = \max \operatorname{diam}(K_i)$$
$$N = N_{\mathrm{DOF}} = \dim V_{hp}$$

Step 4

h-FEM

$$\|u - u_{hp}\| \le C(u)h \qquad h = \max \operatorname{diam}(K_i)$$
$$N = N_{\mathrm{DOF}} = \dim V_{hp}$$

Step 4 

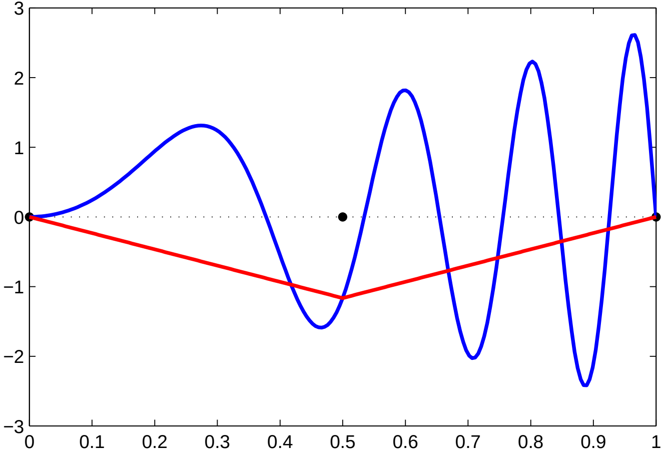
Example: $u(x) = \sin(7\pi x^2)\exp(x) \qquad x \in (0,1)$

h-FEM

$$\|u - u_{hp}\| \leq C(u)h$$

$$h = \max \operatorname{diam}(K_i)$$
$$N = N_{\mathrm{DOF}} = \dim V_{hp}$$

Step 1 

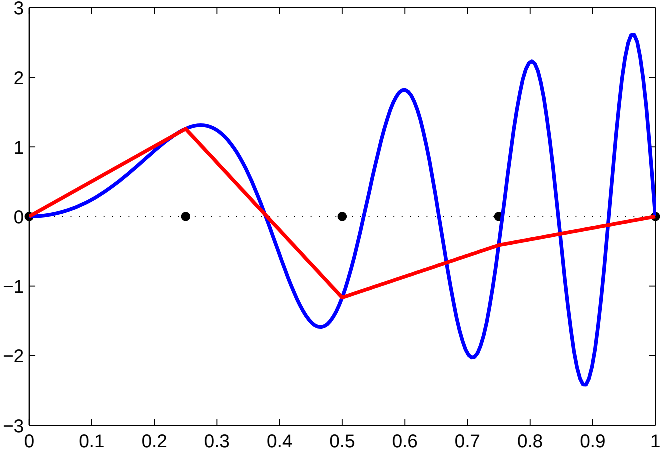


h-FEM

$$\|u - u_{hp}\| \leq C(u)h$$

$$h = \max \operatorname{diam}(K_i)$$
$$N = N_{\mathrm{DOF}} = \dim V_{hp}$$

Step 2

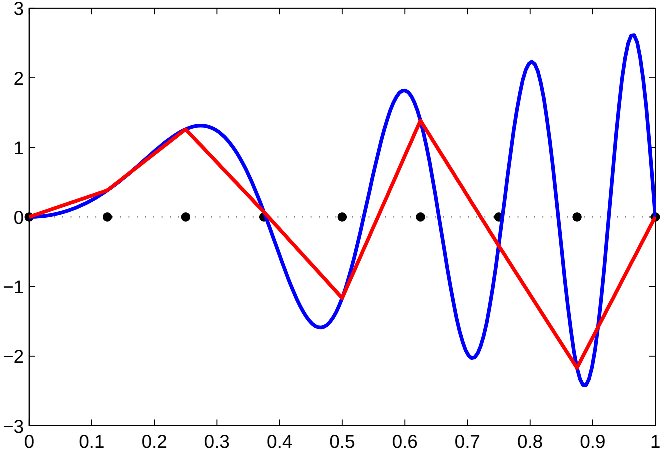


h-FEM

$$\|u - u_{hp}\| \leq C(u)h$$

$$h = \max \operatorname{diam}(K_i)$$
$$N = N_{\mathrm{DOF}} = \dim V_{hp}$$

Step 3 

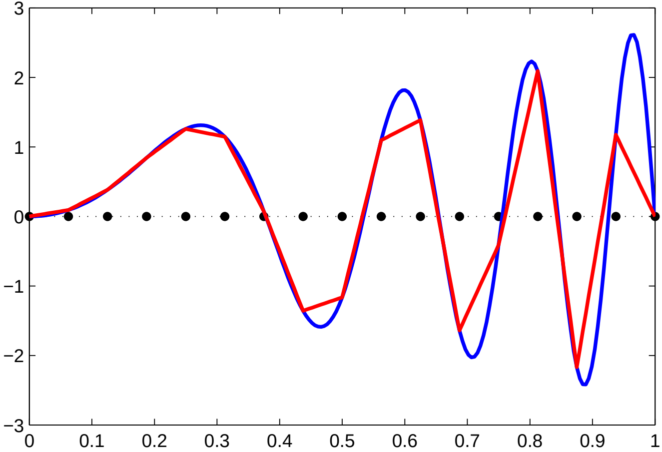


h-FEM

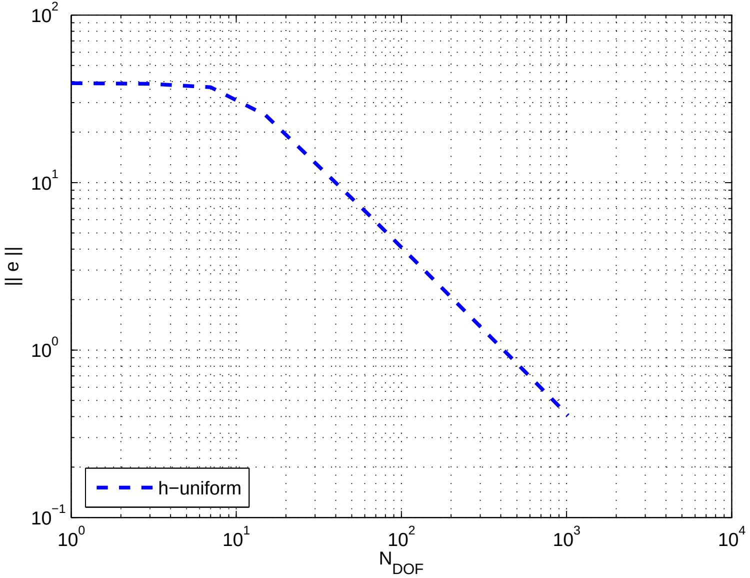
$$\|u - u_{hp}\| \leq C(u)h$$

$$h = \max \operatorname{diam}(K_i)$$
$$N = N_{\mathrm{DOF}} = \dim V_{hp}$$

Step 4 



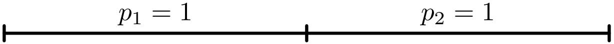
h-FEM convergence history



p-FEM

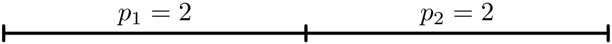
$$\|u - u_{hp}\| \le C(u)h^p$$

Step 1



p-FEM

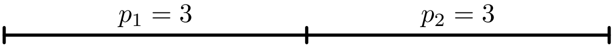
$$\|u - u_{hp}\| \leq C(u)h^p$$

Step 2 

p-FEM

$$\|u - u_{hp}\| \leq C(u)h^p$$

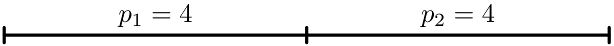
Step 3



p-FEM

$$\|u - u_{hp}\| \leq C(u)h^p$$

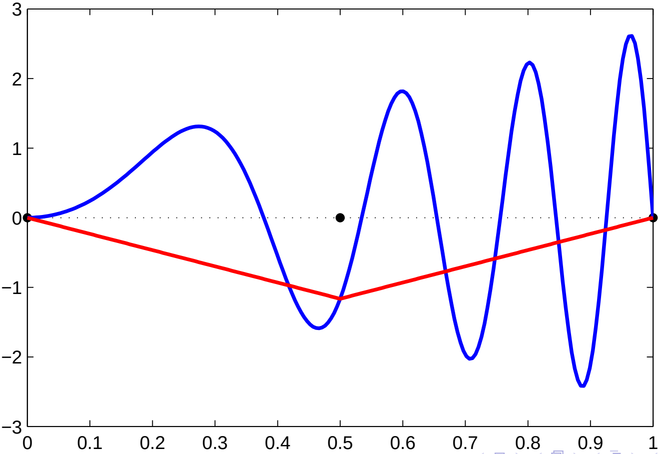
Step 4



p-FEM

$$\|u - u_{hp}\| \leq C(u)h^p$$

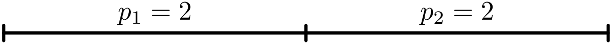
Step 1 $\quad p_1 = 1 \quad p_2 = 1$

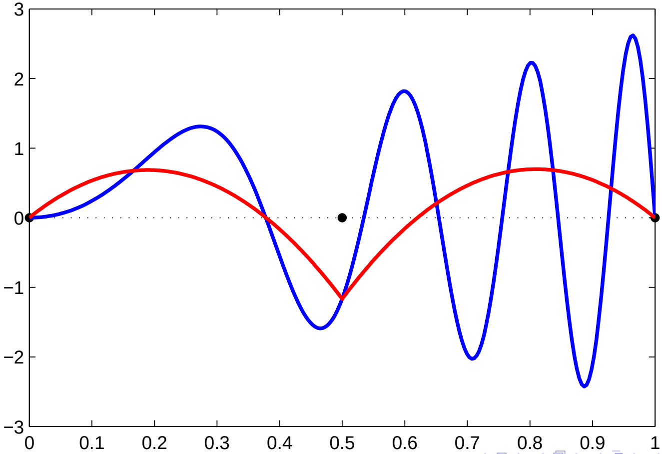


p-FEM

$$\|u - u_{hp}\| \leq C(u)h^p$$

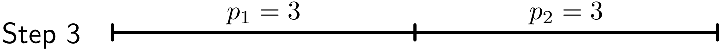
Step 2

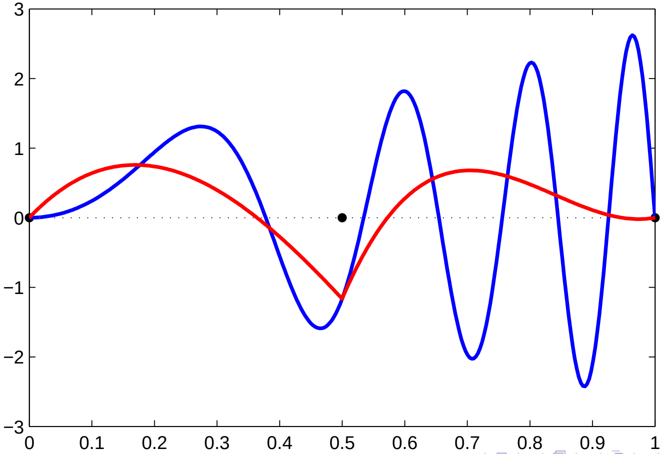




p-FEM

$$\|u - u_{hp}\| \leq C(u)h^p$$

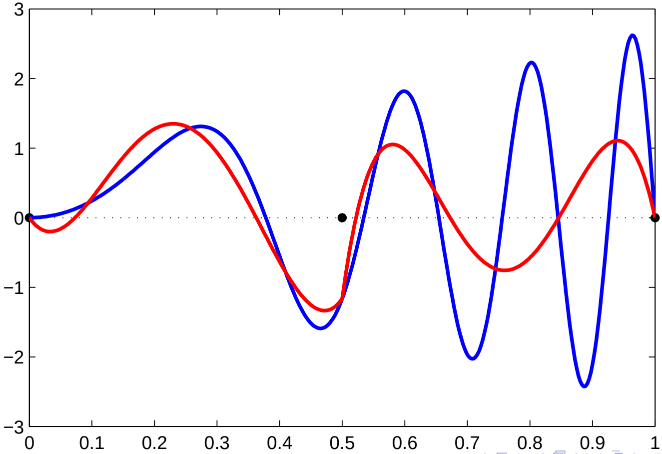




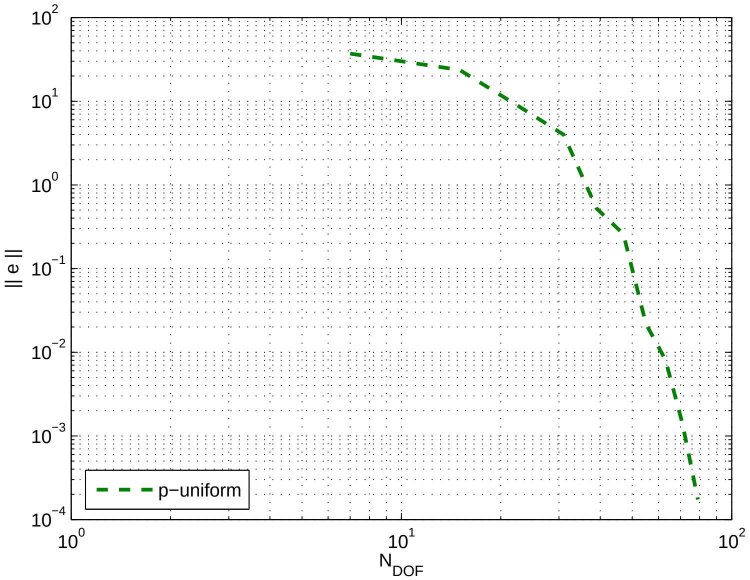
p-FEM

$$\|u - u_{hp}\| \leq C(u)h^p$$

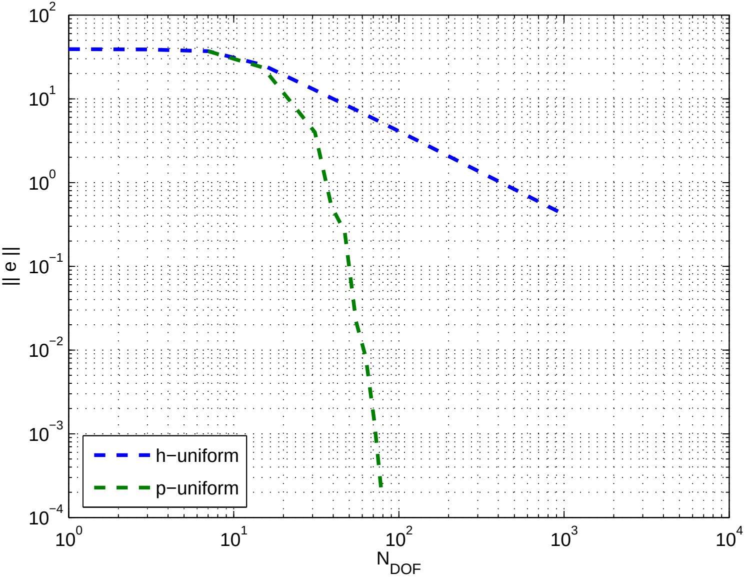
Step 4 $p_1 = 4$ $p_2 = 4$



p-FEM convergence history



h-FEM and *p*-FEM convergence comparison



Adaptivity

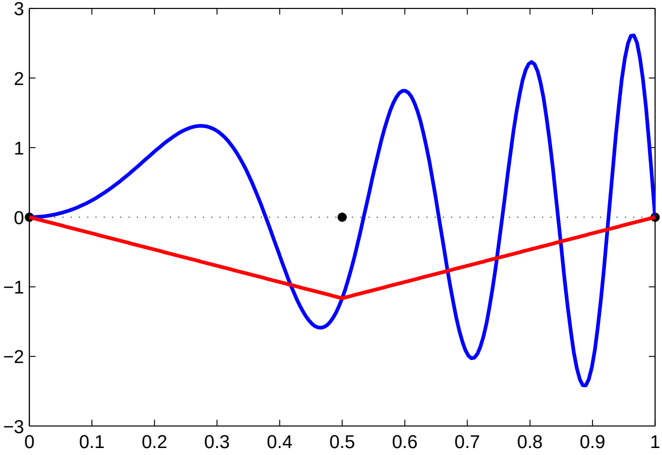
1. Solve the problem on the given mesh.
2. Estimate error on each element.
3. If error tolerance achieved then stop.
4. Mark elements with big error.
5. Adapt marked elements.
6. Go to 1.

h-adaptivity

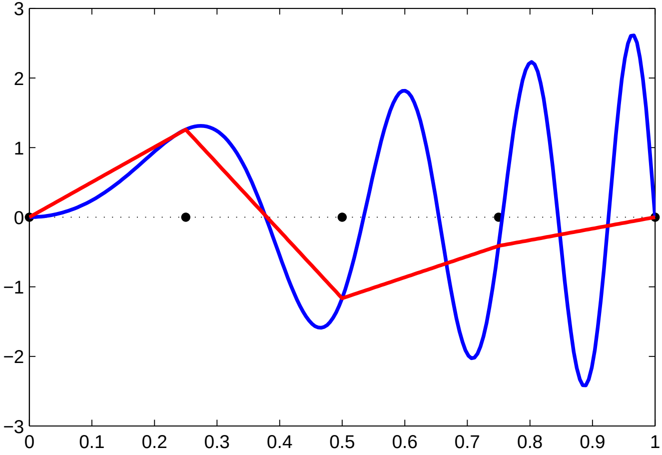
1. Solve the problem on the given mesh.
2. Estimate error on each element. A number on each element.
3. If error tolerance achieved then stop.
4. Mark elements with big error. If $\|e_k\| \geq \mathcal{E}^{\mathrm{avg}}$.
5. Adapt marked elements.
   Split the element into two!!!

   
6. Go to 1.

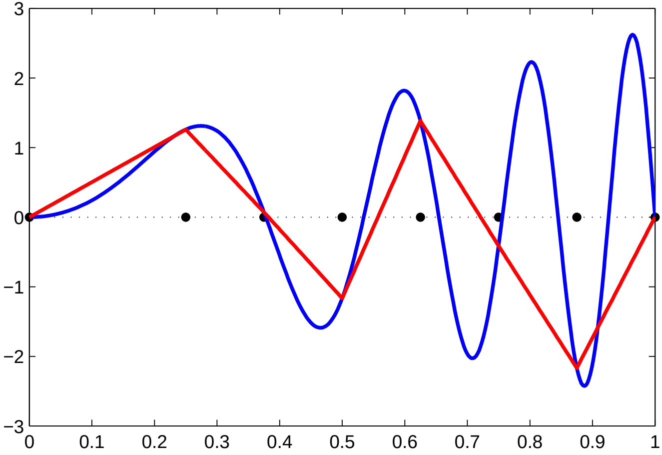
h-adaptivity



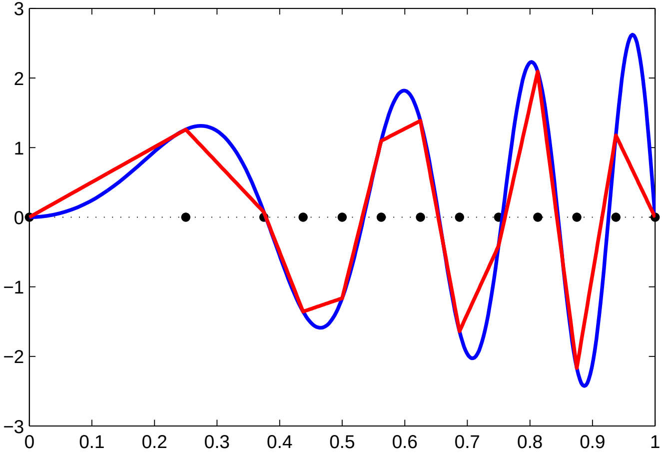
h-adaptivity



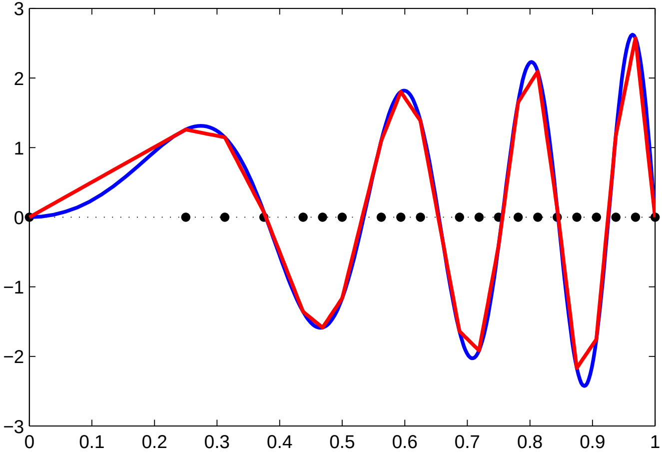
h-adaptivity



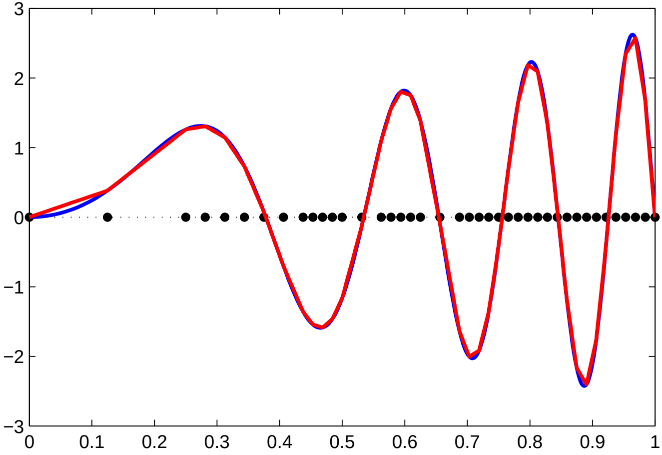
h-adaptivity



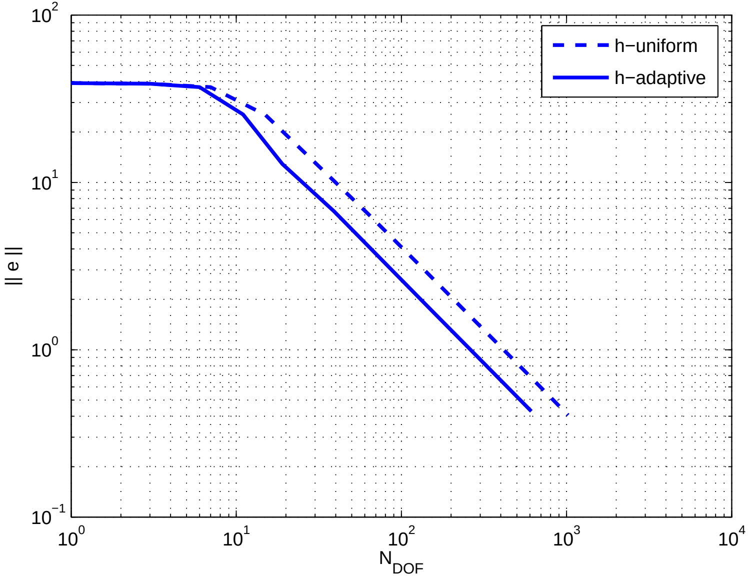
h-adaptivity



h-adaptivity



h-adaptive FEM convergence history

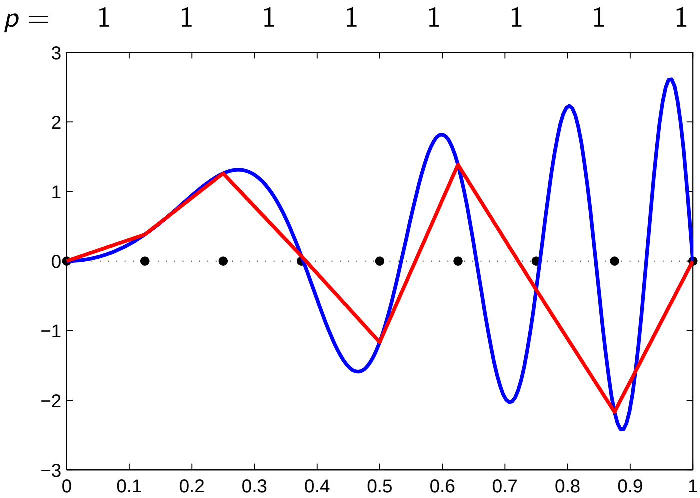


p-adaptivity

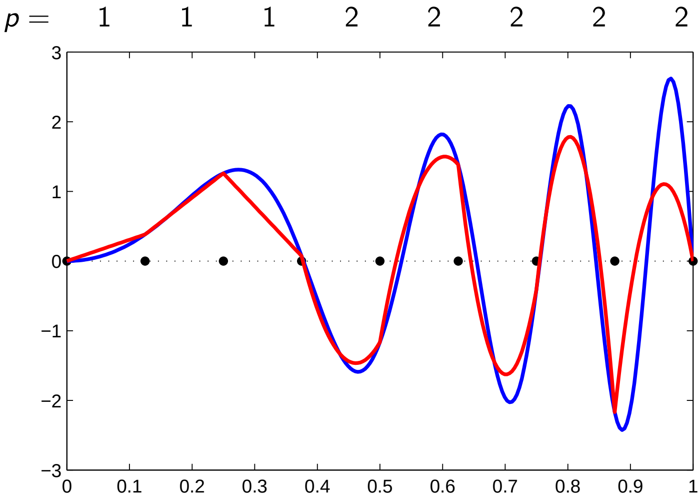
1. Solve the problem on the given mesh.
2. Estimate error on each element. A number on each element.
3. If error tolerance achieved then stop.
4. Mark elements with big error. If $\|e_k\| \geq \mathcal{E}^{\mathrm{avg}}$.
5. Adapt marked elements.
   Increase polynomial degree!!!
   

6. Go to 1.

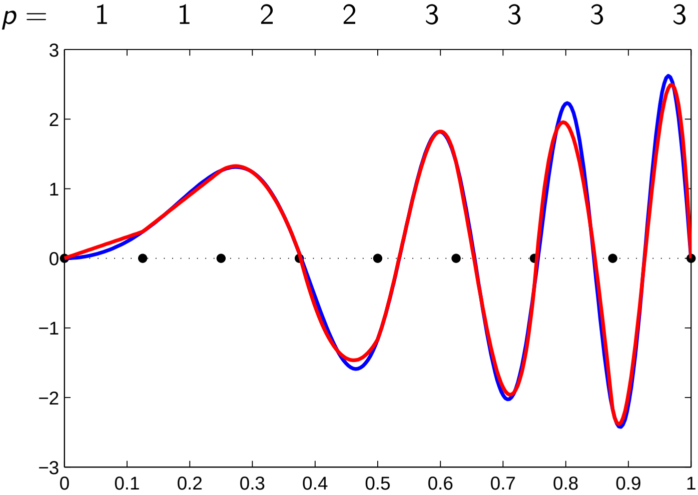
p-adaptivity



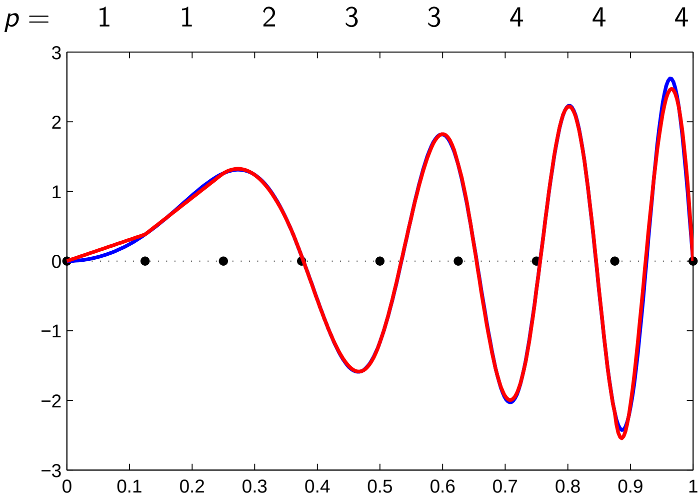
p-adaptivity



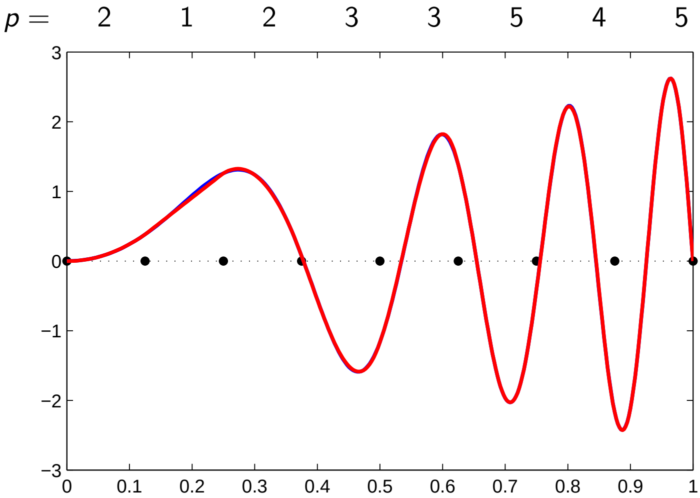
p-adaptivity



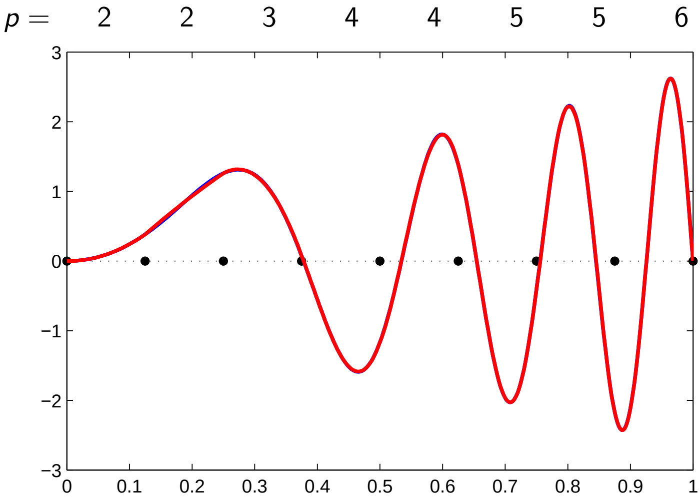
p-adaptivity



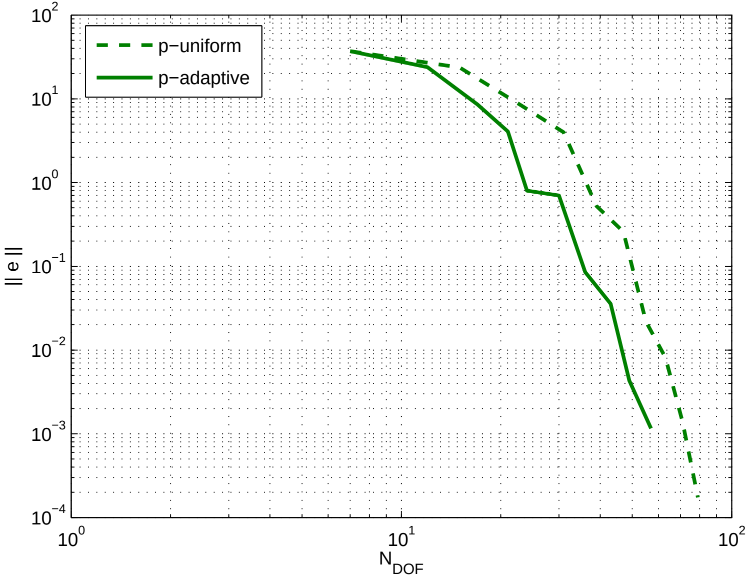
p-adaptivity



p-adaptivity

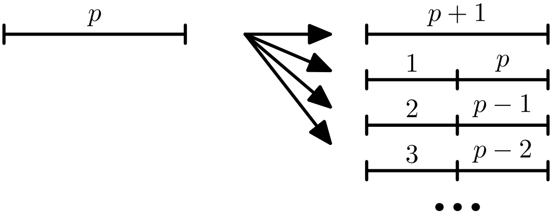


p-adaptive FEM convergence history



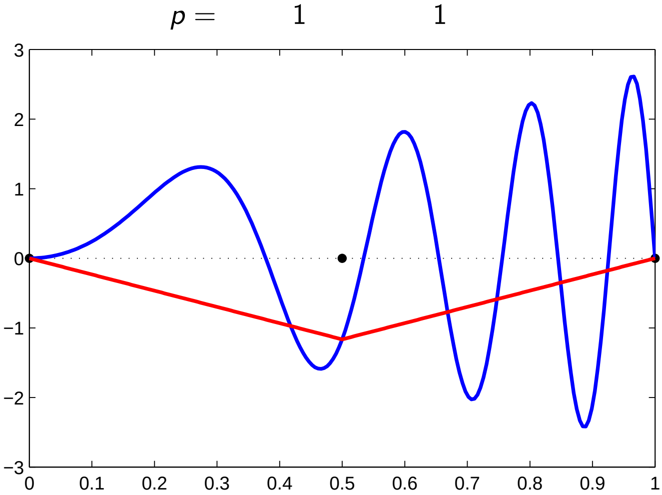
hp-adaptivity

1. Solve the problem on the given mesh.
2. Estimate error on each element. A function on each element!!!
3. If error tolerance achieved then stop.
4. Mark elements with big error. If $\|e_k\| \geq \mathcal{E}^{\text{avg}}$.
5. Adapt marked elements.
   Split the element and modify polynomial degrees!!!

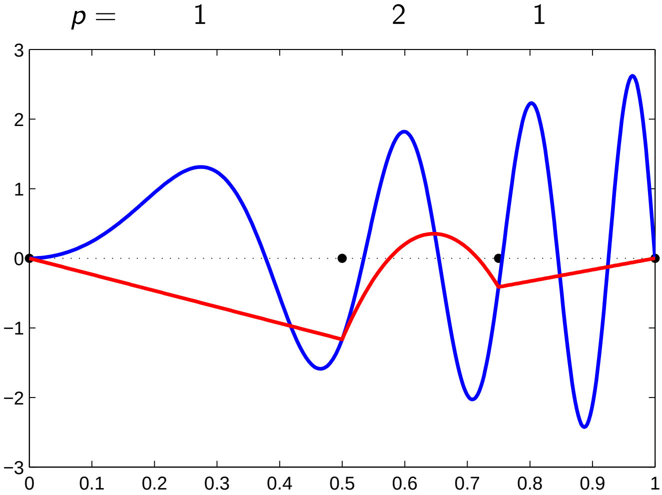


6. Go to 1.

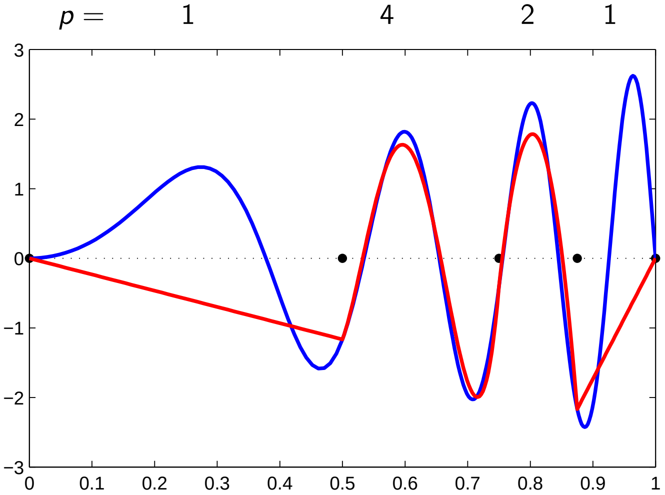
hp-adaptivity



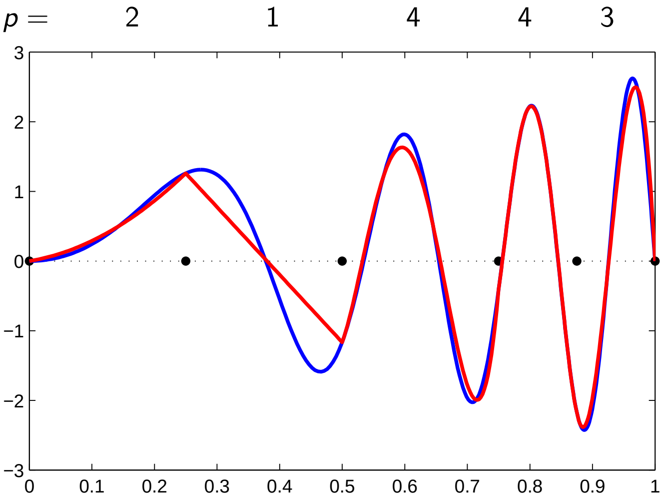
hp-adaptivity



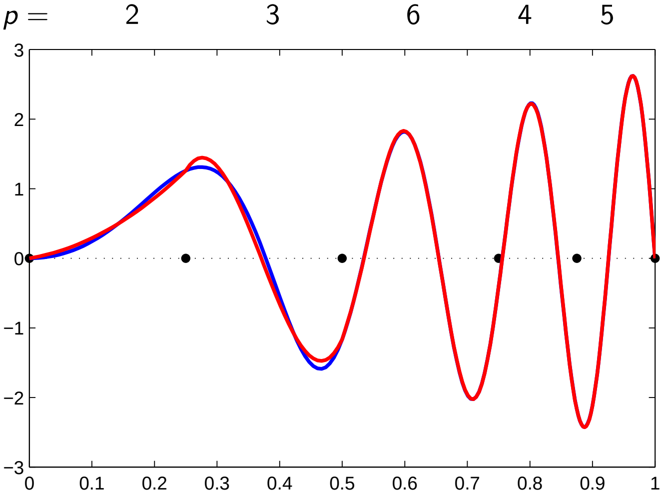
hp-adaptivity



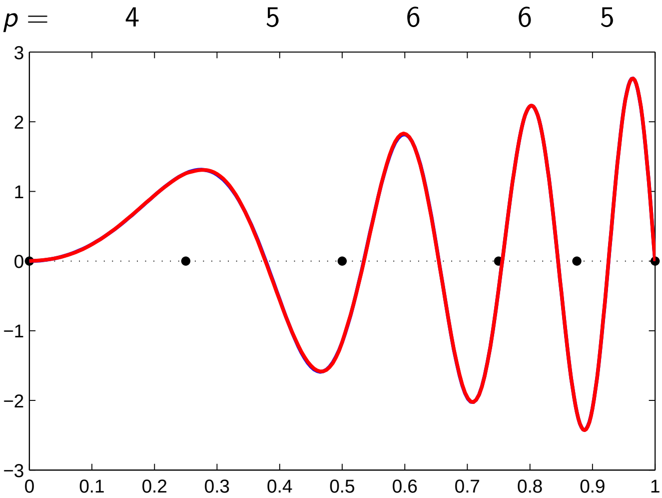
hp-adaptivity



hp-adaptivity

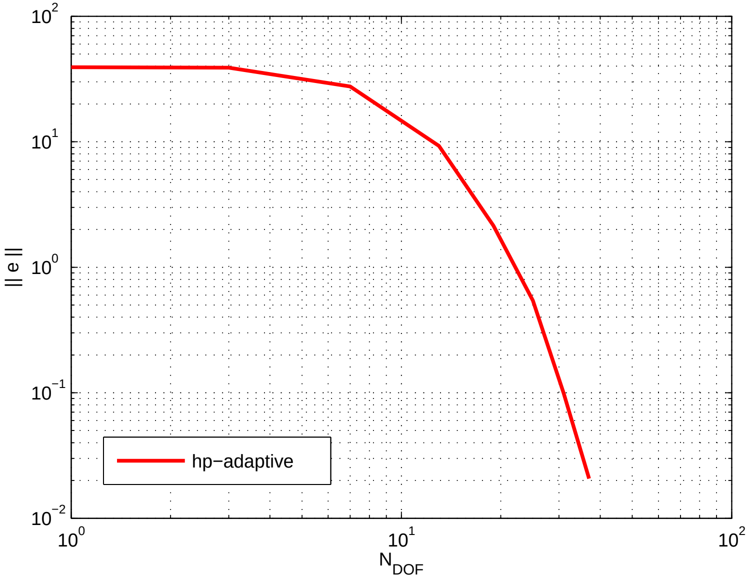


hp-adaptivity

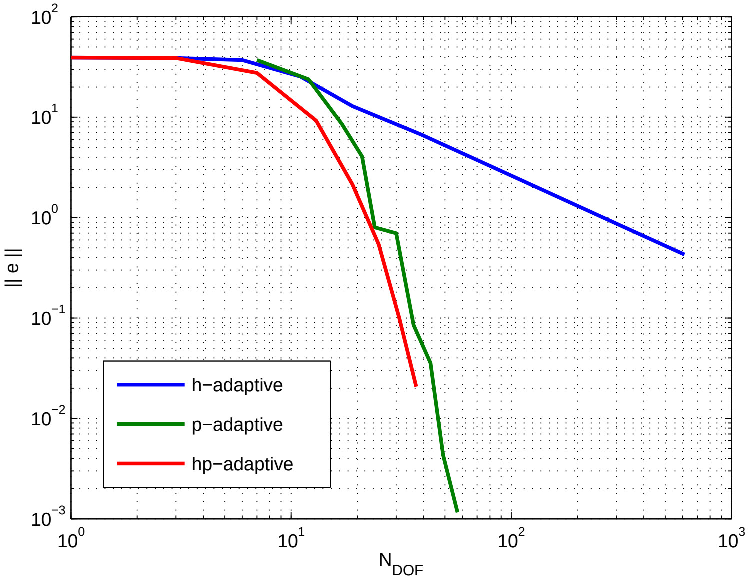


hp-adaptive FEM convergence history

$$\|u - u_{hp}\| \le C_1 \exp(-C_2\sqrt{N_{\mathrm{DOF}}})$$

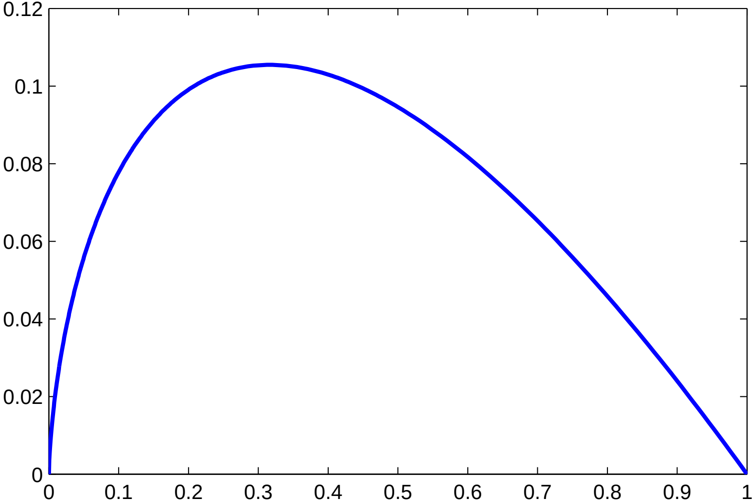


Adaptive FEM convergence comparison

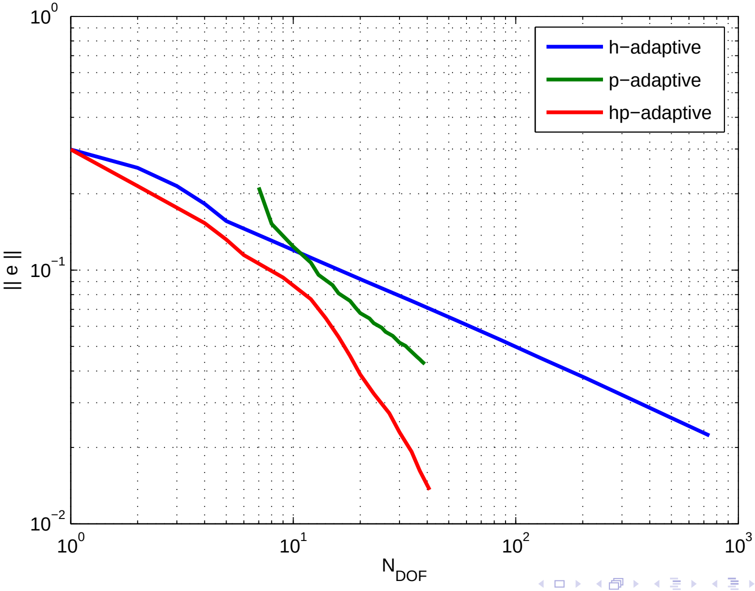


Problem with singularity

$$u(x) = x^{3/4} - x$$



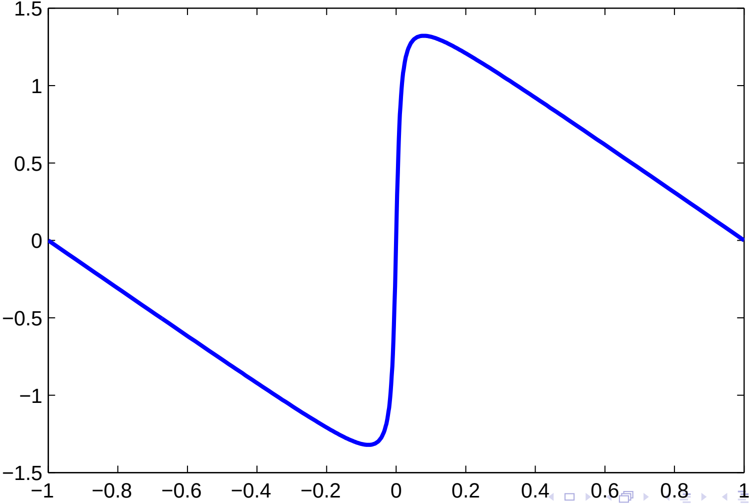
Problem with singularity



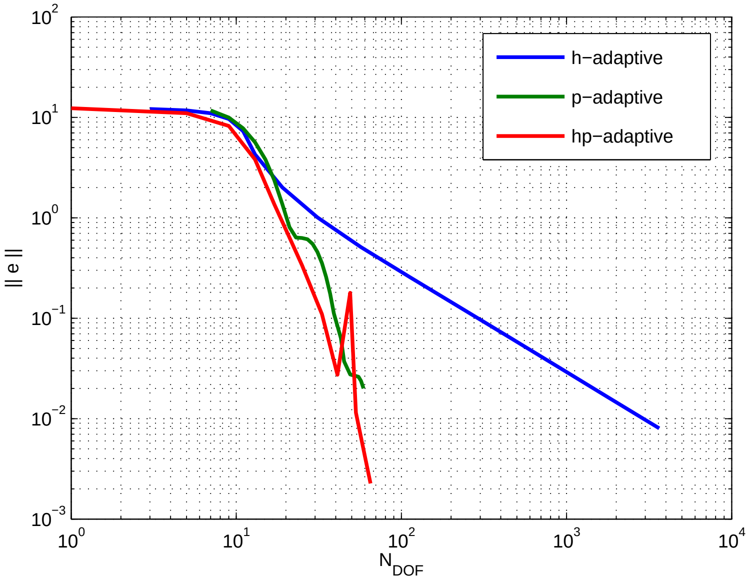
Internal layer

$$u(x) = \arctan(100x) - \ell(x),$$

where $\ell(x)$ is linear such that $u(-1) = u(1) = 0$.



Internal layer



Conclusions

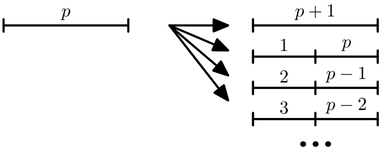
- *h*-FEM
  - Algebraic convergence.

Conclusions

- *h*-FEM
  - Algebraic convergence.
- *p*-FEM
  - Exponential convergence if u is smooth.
  - Algebraic convergence if u has singularities.

Conclusions

- *h*-FEM
  - Algebraic convergence.
- *p*-FEM
  - Exponential convergence if u is smooth.
  - Algebraic convergence if u has singularities.
- *hp*-FEM
  - Exponential convergence even if u has singularities.
  - Technically much more involved then *h*- or *p*-FEM.
  - Open problem: optimal choice of the best adaptive possibility



Thank you for your attention

Tomáš Vejchodský (vejchod@math.cas.cz)

Mathematical Institute, Academy of Sciences
Žitná 25, 115 67 Prague 1
Czech Republic