# On a robust a posteriori error estimator for a diffusion-reaction problem

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# Outline

- Singularly perturbed problem: $-\Delta u + \kappa^2 u = f$
- Standard equilibrated residuals
  - upper bound with no constant
  - elementwise local
  - non-robust if $\kappa \to \infty$
  - not computable
- Improvements
  - robust fluxes
  - computable
- Conclusions

# The problem

- Classical formulation: $\kappa = \text{const.} > 0$

$$
\begin{aligned}
-\Delta u + \kappa^2 u = f \quad & \text{in } \Omega \subset \mathbb{R}^2 \\
u = 0 \quad & \text{on } \partial\Omega
\end{aligned}
$$

- Weak formulation:
  $V = H_0^1(\Omega), \quad B(u,v) = \int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x + \int_\Omega \kappa^2 uv \,\mathrm{d}x$

$$
u \in V : \quad B(u,v) = \int_\Omega fv \,\mathrm{d}x \quad \forall v \in V
$$

- Linear triangular FEM:
  $V_h = \{v_h \in V : v_h|_K \in P^1(K), K \in \mathcal{T}_h\}$

$$
u_h \in V_h : \quad B(u_h, v_h) = \int_\Omega f v_h \,\mathrm{d}x \quad \forall v_h \in V_h
$$

# Notation

- Goal $|||e||| \leq \eta, \qquad e = u - u_h$
- Residual equation

$$e \in V : \quad B(e, v) = \int_\Omega f v \, \mathrm{d}x - B(u_h, v) \quad \forall v \in V$$

- $B_K(u, v) = \int_K \nabla u \cdot \nabla v \, \mathrm{d}x + \int_K \kappa^2 u v \, \mathrm{d}x$
- $|||v|||^2 = B(v, v)$
  $|||v|||_K^2 = B_K(v, v)$
- $H_E^1(K) = \{v \in H^1(K) : v = 0 \text{ on } \partial K \cap \partial\Omega\}$

# Equilibrated residuals

- $g_K|_\gamma \in P^1(\gamma)$, $\gamma \subset \partial K$, $K \in \mathcal{T}_h$, $\quad g_K \approx \dfrac{\partial u|_K}{\partial n_K}$ on $\partial K$
- $g_K|_\gamma + g_{K^*}|_\gamma = 0$ for $\gamma = \partial K \cap \partial K^*$
- $\varepsilon_K \in H^1_E(K)$ :

$$B_K(\varepsilon_K, v) = \int_K f v \,\mathrm{d}x - B_K(u_h, v) + \int_{\partial K} g_K v \,\mathrm{d}s \quad \forall v \in H^1_E(K)$$

- $e = u - u_h$

$$B(e,v) = \sum_{K \in \mathcal{T}_h} \left( \int_K f v \,\mathrm{d}x - B_K(u_h, v) + \int_{\partial K} g_K v \,\mathrm{d}s \right)$$

$$= \sum_{K \in \mathcal{T}_h} B_K(\varepsilon_K, v) \le \left( \sum_{K \in \mathcal{T}_h} |||\varepsilon_K|||_K^2 \right)^{\frac{1}{2}} |||v|||$$

- $|||e|||^2 \le \sum_{K \in \mathcal{T}_h} |||\varepsilon_K|||_K^2$

# Non-robustness

$$|||e|||^2 \leq \sum_{K \in \mathcal{T}_h} |||\varepsilon_K|||_K^2$$

Example (M.A. 1999)

$$-u'' + \kappa^2 u = \cos \pi x \quad \text{in } (-1/2, 1/2) \qquad u(x) = \frac{\cos \pi x}{\pi^2 + \kappa^2}$$

$$u(\pm 1/2) = 0$$

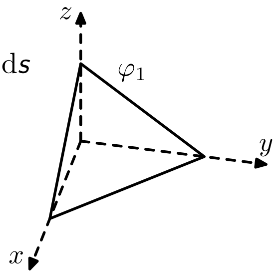
| $\kappa$ | $I_{\text{eff}}(g_K^{\text{equilib}})$ | $I_{\text{eff}}(g_K^{\text{robust}})$ |
|---|---|---|
| 1 | 1.00 | 1.00 |
| $10^1$ | 1.00 | 1.00 |
| $10^2$ | 1.00 | 1.00 |
| $10^3$ | 1.03 | 1.00 |
| $10^4$ | 2.73 | 1.00 |
| $10^5$ | 8.08 | 1.00 |
| $10^6$ | 25.37 | 1.00 |

# Construction of fluxes

The first order equilibration condition:

$$0 = \int_K f\varphi_i \,\mathrm{d}x - B_K(u_h, \varphi_i) + \int_{\partial K} g_K^{\text{equilib}} \varphi_i \,\mathrm{d}s$$

$\varphi_i$ ... basis of $P^1(K) \cap H^1_E(K)$.

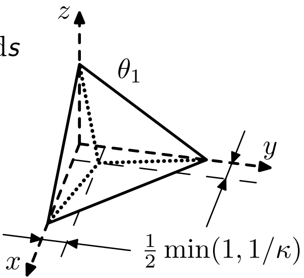



Robust fluxes:

$$0 \approx \int_K f\theta_i \,\mathrm{d}x - B_K(u_h, \theta_i) + \int_{\partial K} g_K^{\text{robust}} \theta_i \,\mathrm{d}s$$

$\theta_i \approx \mathcal{E}\varphi_i$ ... approximate minimum energy extension of $\varphi_i$

$\mathcal{E}v \in H^1(K)$:
$\mathcal{E}v = v$ on $\partial K$
$B_K(\mathcal{E}v, w) = 0 \quad \forall w \in H^1_0(K)$



# Estimation of local errors

$$\varepsilon_K \in H^1_E(K), \quad v \in H^1_E(K),$$

$$B_K(\varepsilon_K, v) = \int_K fv \,\mathrm{d}x - B_K(u_h, v) + \int_{\partial K} g_K v \,\mathrm{d}s$$

# Estimation of local errors

$$\varepsilon_K \in H^1_E(K), \quad v \in H^1_E(K), \quad \boldsymbol{\sigma}_K \in \mathbf{H}(\mathrm{div}, K),$$

$$B_K(\varepsilon_K, v) = \int_K f v \,\mathrm{d}x - B_K(u_h, v) + \int_{\partial K} g_K v \,\mathrm{d}s + \underbrace{\int_K \mathrm{div}\,\boldsymbol{\sigma}_K v \,\mathrm{d}x + \int_K \boldsymbol{\sigma}_K \cdot \nabla v \,\mathrm{d}x - \int_{\partial K} \boldsymbol{\sigma}_K \cdot n_K v \,\mathrm{d}s}_{=0}$$

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$$\begin{aligned} B_K(\varepsilon_K, v) = &\int_K f v \,\mathrm{d}x - \int_K \nabla u_h \cdot \nabla v \,\mathrm{d}x - \int_K \kappa^2 u_h v \,\mathrm{d}x \\ &+ \int_{\partial K} g_K v \,\mathrm{d}s \\ &+ \int_K \mathrm{div}\, \boldsymbol{\sigma}_K v \,\mathrm{d}x + \int_K \boldsymbol{\sigma}_K \cdot \nabla v \,\mathrm{d}x - \int_{\partial K} \boldsymbol{\sigma}_K \cdot n_K v \,\mathrm{d}s \end{aligned}$$

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$$r = f + \Delta u_h - \kappa^2 u_h$$

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$$r = f + \Delta u_h - \kappa^2 u_h$$

$$\boldsymbol{\sigma}_K \cdot n_K = g_K - \frac{\partial u_h}{\partial n_K} \quad \text{on } \partial K$$

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$$\varepsilon_K \in H^1_E(K), \quad v \in H^1_E(K), \quad \boldsymbol{\sigma}_K \in \mathbf{H}(\mathrm{div}, K),$$

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$$+ \int_K \boldsymbol{\sigma}_K \cdot \nabla v \, \mathrm{d}x$$

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$$\frac{1}{2} \|\kappa v\|^2_{0,K} + \frac{1}{2} \|\nabla v\|^2_{0,K} = \frac{1}{2} |||v|||^2_K$$

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$$|||e|||^2 \le \sum_{K \in \mathcal{T}_h} |||\varepsilon_K|||^2_K \le \sum_{K \in \mathcal{T}_h} \left( \left\| \frac{1}{\kappa}(r + \mathrm{div}\, \boldsymbol{\sigma}_K) \right\|^2_{0,K} + \|\boldsymbol{\sigma}_K\|^2_{0,K} \right)$$

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# Estimation of local errors

$$\varepsilon_K \in H^1_E(K), \quad v \in H^1_E(K), \quad \boldsymbol{\sigma}_K \in \mathbf{H}(\mathrm{div}, K),$$

$$|||\varepsilon_K|||^2_K \le \left\| \frac{1}{\kappa}(r + \mathrm{div}\, \boldsymbol{\sigma}_K) \right\|^2_{0,K} + \|\boldsymbol{\sigma}_K\|^2_{0,K} \equiv \eta^2_K(\boldsymbol{\sigma}_K)$$

$$|||e|||^2 \le \sum_{K \in \mathcal{T}_h} |||\varepsilon_K|||^2_K \le \sum_{K \in \mathcal{T}_h} \left( \left\| \frac{1}{\kappa}(r + \mathrm{div}\, \boldsymbol{\sigma}_K) \right\|^2_{0,K} + \|\boldsymbol{\sigma}_K\|^2_{0,K} \right) \equiv \sum_{K \in \mathcal{T}_h} \eta^2_K(\boldsymbol{\sigma}_K)$$

$$r = f + \Delta u_h - \kappa^2 u_h$$

$$\boldsymbol{\sigma}_K \cdot n_K = g_K - \frac{\partial u_h}{\partial n_K} \quad \text{on } \partial K$$

# Solution of local problems

Minimize $\eta_K^2(\boldsymbol{\sigma}_K)$ over $\mathbf{W}^p(K) \subset \mathbf{H}(\mathrm{div}, K)$

$$\mathbf{W}^p(K) = \left\{\boldsymbol{\sigma} \in [P^p(K)]^2 : \boldsymbol{\sigma} \cdot n_K = g_K - \partial u_h / \partial n_K\right\}$$
$$\mathbf{W}_0^p(K) = \left\{\boldsymbol{\sigma} \in [P^p(K)]^2 : \boldsymbol{\sigma} \cdot n_K = 0\right\}$$

$\boldsymbol{\sigma}_K = \boldsymbol{\sigma}_0 + \boldsymbol{\sigma}_1,$

$\boldsymbol{\sigma}_0 \in \mathbf{W}_0^p(K)$

$\boldsymbol{\sigma}_1 \in [P^1(K)]^2 \cap \mathbf{W}^p(K)$ uniquely given

Find $\boldsymbol{\sigma}_0 \in \mathbf{W}_0^p(K)$ :

$$\int_K \mathrm{div}\,\boldsymbol{\sigma}_0 \,\mathrm{div}\,\mathbf{w}\,\mathrm{d}x + \int_K \kappa^2 \boldsymbol{\sigma}_0 \mathbf{w}\,\mathrm{d}x$$
$$= -\int_K r\,\mathrm{div}\,\mathbf{w}\,\mathrm{d}x - \int_K \mathrm{div}\,\boldsymbol{\sigma}_1 \,\mathrm{div}\,\mathbf{w}\,\mathrm{d}x - \int_K \kappa^2 \boldsymbol{\sigma}_1 \mathbf{w}\,\mathrm{d}x$$
$$\forall \mathbf{w} \in \mathbf{W}_0^p(K)$$

# Data oscillations

$$\overline{f}_K \in P^p(K): \quad \int_K (f - \overline{f}_K) w \,\mathrm{d}x = 0 \quad \forall w \in P^p(K)$$

$$B_K(\varepsilon_K, v) = \underbrace{\int_K \overline{f}_K v \,\mathrm{d}x - B_K(u_h, v) + \int_{\partial K} g_K v \,\mathrm{d}s}_{T2} + \underbrace{\int_K (f - \overline{f}_K) v \,\mathrm{d}x}_{T1}$$

$$T1 \leq \underbrace{\min(h_K/\pi, 1/\kappa) \left\| f - \overline{f}_K \right\|_{0,K}}_{\mathrm{osc}_K(f)} \| v \|_K \qquad v \in H^1_E(K)$$

$$T2 \leq \left\| \frac{1}{\kappa}(\overline{r} + \operatorname{div} \boldsymbol{\sigma}_K) \right\|_{0,K} \| \kappa v \|_{0,K} + \| \boldsymbol{\sigma}_K \|_{0,K} \| \nabla v \|_{0,K}$$

$$\| \varepsilon_K \|_K^2 \leq \left( \left\| \frac{1}{\kappa}(\overline{r} + \operatorname{div} \boldsymbol{\sigma}_K) \right\|_{0,K} + \mathrm{osc}_K(f) \right)^2 + \left( \| \boldsymbol{\sigma}_K \|_{0,K} + \mathrm{osc}_K(f) \right)^2$$

$$\equiv \overline{\eta}_K^2(\boldsymbol{\sigma}_K)$$

# Numerical examples

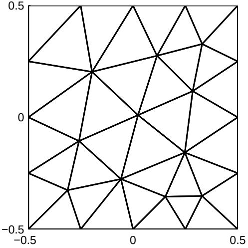
$$-\Delta u + \kappa^2 u = f \quad \text{in } \Omega$$
$$u = 0 \quad \text{on } \partial\Omega$$

## Example (A)

$$\Omega = (-1/2, 1/2)^2$$

$$f = \cos(\pi x)\cos(\pi y)$$

$$u = \frac{\cos(\pi x)\cos(\pi y)}{\pi^2 + \kappa^2}$$

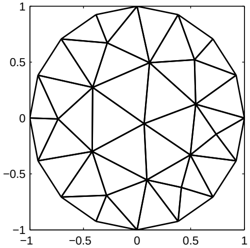



## Example (B)

$$\Omega = \{(x, y) : r < 1\}$$

$$f = 1 \qquad r = \sqrt{x^2 + y^2}$$

$$u = \frac{1}{\kappa^2}\left(1 - \frac{I_0(\kappa r)}{I_0(\kappa)}\right)$$



# Results

$\overline{f}_K \in P^1(K)$

minimization of $\overline{\eta}_K^2(\boldsymbol{\sigma}_K)$ over $\mathbf{W}^2(K)$

Example (A)

| $\kappa$ | $I_{\text{eff}}$ |
|---|---|
| $10^{-3}$ | 1.40890 |
| $10^{-2}$ | 1.40890 |
| $10^{-1}$ | 1.40903 |
| 1 | 1.41545 |
| 10 | 1.57538 |
| $10^2$ | 1.58316 |
| $10^3$ | 1.51070 |
| $10^4$ | 1.50251 |
| $10^5$ | 1.50185 |
| $10^6$ | 1.50178 |

Example (B)

| $\kappa$ | $I_{\text{eff}}$ |
|---|---|
| $10^{-3}$ | 1.04518 |
| $10^{-2}$ | 1.04518 |
| $10^{-1}$ | 1.04521 |
| 1 | 1.05175 |
| 10 | 1.40464 |
| $10^2$ | 1.02094 |
| $10^3$ | 1.00588 |
| $10^4$ | 1.00690 |
| $10^5$ | 1.00701 |
| $10^6$ | 1.00703 |

# Conclusions

$$-\Delta u + \kappa^2 u = f \quad \text{in } \Omega \subset \mathbb{R}^2$$
$$u = 0 \quad \text{on } \partial\Omega$$

$$|\!|\!| u - u_h |\!|\!|^2 \leq \sum_{K \in \mathcal{T}_h} \left( \left\| \frac{1}{\kappa}(r + \operatorname{div} \boldsymbol{\sigma}_K) \right\|_{0,K}^2 + \|\boldsymbol{\sigma}_K\|_{0,K}^2 \right) \equiv \eta_K^2(\boldsymbol{\sigma}_K)$$

- No constants
- Completely computable
- Guaranteed upper bound – up to $\operatorname{osc}_K(f)$
- Elementwise local
- Robust for $\kappa^2 \in (0, \infty)$

# Thank you for your attention

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