Bases of admissible rules of Łukasiewicz logic

Emil Jeřábek*

Institute of Mathematics of the Academy of Sciences
Žitná 25, 115 67 Praha 1, Czech Republic, email: jerabek@math.cas.cz

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Abstract

We construct explicit bases of single-conclusion and multiple-conclusion admissible rules of propositional Łukasiewicz logic, and we prove that every formula has an admissibly saturated approximation. We also show that Łukasiewicz logic has no finite basis of admissible rules.

1 Introduction

Investigation of nonclassical logics usually revolves around provability of formulas. When we generalize the problem from formulas to inference rules, there arises an important distinction between *derivable* and *admissible* rules, introduced by Lorenzen [15]. A rule

$$\varphi_1, \ldots, \varphi_n \;/\; \psi$$

is derivable if it belongs to the consequence relation of the logic (defined semantically, or by a proof system using a set of axioms and rules); and it is admissible if the set of theorems of the logic is closed under the rule. These two notions coincide for the standard consequence relation of classical logic, but nonclassical logics often admit rules which are not derivable. (A logic whose admissible rules are all derivable is called *structurally complete*.) For example, all superintuitionistic (si) logics admit the Kreisel–Putnam rule

$$\neg p \to q \vee r \;/\; (\neg p \to q) \vee (\neg p \to r),$$

whereas many of these logics (such as *IPC* itself) do not derive this rule.

Research into admissible rules was stimulated by a question of H. Friedman [5], asking whether admissibility of rules in *IPC* is decidable. The problem was extensively investigated in a series of papers by Rybakov, who has shown that admissibility is decidable for a large class of modal and si logics, found semantic criteria for admissibility, and obtained other results on various aspects of admissibility. His results on admissible rules in transitive modal and

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si logics are summarized in the monograph [18]. He also applied his method to tense logics [19, 20, 21]. Ghilardi [7, 8] discovered the connection of admissibility to projective formulas and unification, which provided another criteria for admissibility in certain modal and si logics, and new decision procedures for admissibility in some modal and si systems. Ghilardi's results were utilized by Iemhoff [9, 10, 11] to construct an explicit basis of admissible rules for *IPC* and some other si logics, and to develop Kripke semantics for admissible rules. These results were extended to modal logics by Jeřábek [12]. We note that decidability of admissibility is by no means automatic. An artificial decidable modal logic with undecidable admissibility problem was constructed by Chagrov [1], and natural examples of bimodal logics with undecidable admissibility (or even unification) problem were found by Wolter and Zakharyaschev [23]. In terms of computational complexity, admissibility in basic transitive logics is *coNE*-complete by Jeřábek [13], whereas derivability in these logics is *PSPACE*-complete.

In contrast to the situation in modal and superintuitionistic logics, only very little is known about admissibility in other nonclassical logics. Here we are particularly interested in substructural and fuzzy logics (cf. [6]). Structural completeness of various substructural logics was investigated by Olson et al. [17] and by Cintula and Metcalfe [3]. Dzik [4] studied unification in n-contractive extensions of Hájek's Basic Logic (**BL**).

In this paper we study admissible rules of Łukasiewicz logic (**Ł**). We have shown in [14] that admissibility in **Ł** is decidable (in *PSPACE*). Here we expand the methods of [14] to construct a simple explicit basis of **Ł**-admissible rules, with both a single-conclusion and a multiple-conclusion version. To this end we provide a semantic characterization of admissibly saturated formulas in **Ł** (i.e., formulas that are not premises of any nonderivable admissible rule), and show that every formula has a finite approximation by admissibly saturated formulas. We use a syntactic conservativity argument to construct a single-conclusion basis from a multiple-conclusion basis. We also show that our basis is independent; since it is infinite, it follows that **Ł** does not have a finite basis of admissible rules.

2 Preliminaries

The language of Łukasiewicz logic (**Ł**) consists of *propositional formulas* built from variables p_n, $n \in \omega$, using connectives $\to$ and $\bot$. A *substitution* is a mapping of propositional formulas to propositional formulas which commutes with all connectives. A formula φ is *derivable* from a set of formulas Γ, written as $\Gamma \vdash_{\mathbf{Ł}} \varphi$, if there exists a finite sequence of formulas $\varphi_1, \dots, \varphi_n$ such that $\varphi_n = \varphi$, and each φ_i is a member of Γ, an instance of one of the axioms

$$\begin{gathered}\varphi \to (\psi \to \varphi),\\ (\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi)),\\ ((\varphi \to \psi) \to \psi) \to ((\psi \to \varphi) \to \varphi),\\ \bot \to \varphi,\end{gathered}$$

or it is derived from some φ_j, φ_k, $j, k < i$ by an instance of the rule of modus ponens

$$\varphi, \varphi \to \psi \;/\; \psi.$$

We can introduce other connectives as abbreviations

$$\begin{aligned}
\neg\varphi &\equiv \varphi \to \bot, \\
\varphi \cdot \psi &\equiv \neg(\varphi \to \neg\psi), \\
\varphi \oplus \psi &\equiv \neg\varphi \to \psi, \\
\varphi \vee \psi &\equiv (\varphi \to \psi) \to \psi, \\
\varphi \wedge \psi &\equiv \varphi \cdot (\varphi \to \psi), \\
\varphi \leftrightarrow \psi &\equiv (\varphi \to \psi) \cdot (\psi \to \varphi), \\
\top &\equiv \neg\bot,
\end{aligned}$$

and we write $\varphi^n = \varphi \cdot \ldots \cdot \varphi$, $n\varphi = \varphi \oplus \cdots \oplus \varphi$ with n occurrences of φ (if $n = 0$, we put $\varphi^0 = \top$, $0\varphi = \bot$).

A *single-conclusion rule* is an expression of the form $\Gamma \ / \ \varphi$, where Γ is a finite set of formulas, and φ is a formula. We will usually omit set-builder braces when giving Γ by a list of formulas, and we will write Γ, Δ for $\Gamma \cup \Delta$. A rule $\Gamma \ / \ \varphi$ is **Ł**-*derivable* if $\Gamma \vdash_{\mathbf{Ł}} \varphi$. An **Ł**-*unifier* of a formula φ is a substitution σ such that $\vdash_{\mathbf{Ł}} \sigma\varphi$. A rule $\Gamma \ / \ \varphi$ is **Ł**-*admissible*, written as $\Gamma \mathrel{|\!\!\sim}_{\mathbf{Ł}} \varphi$, if every common unifier of Γ is also a unifier of φ. A (finitary structural) *consequence relation* is a set R of single-conclusion rules such that

(i) $\varphi \ / \ \varphi \in R$,

(ii) weakening: if $\Gamma \ / \ \varphi \in R$, then $\Gamma, \Gamma' \ / \ \varphi \in R$,

(iii) cut: if $\Gamma \ / \ \varphi \in R$ and $\Gamma, \varphi \ / \ \psi \in R$, then $\Gamma \ / \ \psi \in R$,

(iv) if $\Gamma \ / \ \varphi \in R$, then $\sigma\Gamma \ / \ \sigma\varphi \in R$,

for all sets of formulas Γ, Γ', all formulas φ, ψ, and all substitutions σ. We will usually write $R \vdash \varrho$ instead of $\varrho \in R$. The set of **Ł**-derivable rules, denoted by **Ł**, and the set of **Ł**-admissible rules, denoted by $\mathrel{|\!\!\sim}^1_{\mathbf{Ł}}$, are consequence relations. If R is a consequence relation, and X is a set of rules, then $R + X$ denotes the smallest consequence relation which includes both R and X. The set X is called a *basis* of $R + X$ over R. We take $R = \mathbf{Ł}$ if unspecified; in particular, a basis of single-conclusion **Ł**-admissible rules is a set X such that $\mathrel{|\!\!\sim}^1_{\mathbf{Ł}} = \mathbf{Ł} + X$.

The concepts above can be generalized to rules with more (or less) than one formula in the conclusion (cf. e.g. [22]). A *multiple-conclusion rule* (or simply a *rule*) is an expression of the form $\Gamma \ / \ \Delta$, where Γ and Δ are finite sets of formulas. We will omit braces from Γ and Δ as in the case of single-conclusion rules, however we will usually retain explicit set-theoretic notation when Δ is empty: $\Gamma \ / \ \varnothing$. A rule $\Gamma \ / \ \Delta$ is **Ł**-*derivable* if $\Gamma \vdash_{\mathbf{Ł}} \varphi$ for some $\varphi \in \Delta$, and it is **Ł**-*admissible* if every common unifier of Γ also unifies some formula from Δ. A *multiple-conclusion consequence relation* is a set R of rules such that

(i) $\varphi \ / \ \varphi \in R$,

(ii) weakening: if $\Gamma \ / \ \Delta \in R$, then $\Gamma, \Gamma' \ / \ \Delta, \Delta' \in R$,

(iii) cut: if $\Gamma \ / \ \varphi, \Delta \in R$ and $\Gamma, \varphi \ / \ \Delta \in R$, then $\Gamma \ / \ \Delta \in R$,

(iv) if $\Gamma \,/\, \Delta \in R$, then $\sigma\Gamma \,/\, \sigma\Delta \in R$,

for all sets of formulas Γ, Γ', Δ, Δ', all formulas φ, and all substitutions σ. The set of $\mathbf{Ł}$-admissible multiple-conclusion rules, denoted by $\mathrel{|\!\!\sim}_{\mathbf{Ł}}$, is a multiple-conclusion consequence relation. The set of $\mathbf{Ł}$-derivable rules is also a multiple-conclusion consequence relation; more generally, if R is any single-conclusion consequence relation, then we can identify R with the multiple-conclusion consequence relation

$$R' = \{\Gamma \,/\, \Delta \mid \exists \varphi \in \Delta\, (\Gamma \,/\, \varphi \in R)\}.$$

Conversely, if R is a multiple-conclusion consequence relation, then its *single-conclusion fragment* R^1, consisting of all single-conclusion rules which belong to R, is a single-conclusion consequence relation. The $R + X$ notation, and bases, are introduced as in the single-conclusion case.

We now turn to the semantics of Łukasiewicz logic. An *MV-algebra* is a structure $\langle A, \oplus, \neg, 0\rangle$ which satisfies the identities

$$\begin{aligned}
(x \oplus y) \oplus z &= x \oplus (y \oplus z),\\
x \oplus 0 &= x,\\
x \oplus y &= y \oplus x,\\
\neg\neg x &= x,\\
x \oplus \neg 0 &= \neg 0,\\
\neg(\neg x \oplus y) \oplus y &= \neg(\neg y \oplus x) \oplus x.
\end{aligned}$$

We can define other operations on an MV-algebra by

$$\begin{aligned}
x \to y &= \neg x \oplus y,\\
x \cdot y &= \neg(\neg x \oplus \neg y),\\
x \vee y &= (x \to y) \to y,\\
x \wedge y &= x \cdot (x \to y),\\
x \leftrightarrow y &= (x \to y) \cdot (y \to x),\\
1 &= \neg 0.
\end{aligned}$$

The operations $\wedge, \vee$ turn A into a distributive lattice with bounds $0, 1$, which induces a partial order $\leq$ on A. We can identify propositional formulas with terms in the language of MV-algebras in a natural way. A *valuation* in an MV-algebra A is a homomorphism v from the term algebra to A. If φ is a formula in the first k variables, $a \in A^k$, and v is the valuation such that $v(p_i) = a_i$, we also write $\varphi(a) = v(\varphi)$. A valuation v satisfies a formula φ if $v(\varphi) = 1$, and it satisfies a rule $\Gamma \,/\, \Delta$ if $v(\varphi) \neq 1$ for some $\varphi \in \Gamma$, or $v(\varphi) = 1$ for some $\varphi \in \Delta$. A rule $\Gamma \,/\, \Delta$ is *valid* in an MV-algebra A, written as $A \vDash \Gamma \,/\, \Delta$, if the rule is satisfied by every valuation in A. In other words, $A \vDash \Gamma \,/\, \Delta$ if and only if the open first-order formula

$$\bigwedge_{\varphi \in \Gamma} (\varphi = 1) \to \bigvee_{\varphi \in \Delta} (\varphi = 1)$$

is valid in A.

Łukasiewicz logic is algebraizable, and the variety of MV-algebras is its equivalent algebraic semantics. It follows that a rule $\Gamma \ / \ \Delta$ is $\mathbf{Ł}$-derivable iff it is valid in all MV-algebras. The *standard MV-algebra* $[0,1]_{\mathbf{Ł}}$ is the algebra $\langle [0,1], \oplus, \neg, 0\rangle$, where

$$x \oplus y = \min\{x+y, 1\},$$
$$\neg x = 1 - x.$$

Notice that the rational interval $[0,1]_{\mathbb{Q}} = [0,1] \cap \mathbb{Q}$ is a subalgebra of $[0,1]_{\mathbf{Ł}}$. Both $[0,1]_{\mathbf{Ł}}$ and $[0,1]_{\mathbb{Q}}$ generate the variety of MV-algebras, even as a quasivariety, hence a single-conclusion rule is $\mathbf{Ł}$-derivable iff it is valid in $[0,1]_{\mathbf{Ł}}$ iff it is valid in $[0,1]_{\mathbb{Q}}$ (Chang [2]).

A *free MV-algebra* over a set X of generators is an MV-algebra $F \supseteq X$ such that every mapping from X to an MV-algebra A can be uniquely extended to a homomorphism from F to A. As another corollary to algebraizability of $\mathbf{Ł}$, free MV-algebras can be described as Lindenbaum–Tarski algebras of $\mathbf{Ł}$: F consists of equivalence classes of formulas using elements of X as propositional variables modulo the equivalence relation $\varphi \sim \psi$ iff $\vdash_{\mathbf{Ł}} \varphi \leftrightarrow \psi$, with operations defined in the natural way. Note that valuations in F correspond to substitutions whose range consists of formulas using variables from X, and a formula φ is satisfied under a valuation given by such a substitution σ if and only if $\vdash_{\mathbf{Ł}} \sigma\varphi$. We obtain the following characterization of admissibility: a rule $\Gamma \ / \ \Delta$ is $\mathbf{Ł}$-admissible iff it is valid in all free MV-algebras iff it is valid in all free MV-algebras over finite sets of generators. In the case of Łukasiewicz logic, we can in fact do better:

Theorem 2.1 ([14]) *$\mathbf{Ł}$ is 1-reducible with respect to admissible rules. That is, for every $\mathbf{Ł}$-inadmissible rule $\Gamma \ / \ \Delta$, there exists a substitution σ using formulas in only one variable such that $\vdash_{\mathbf{Ł}} \sigma \bigwedge \Gamma$ and for every $\delta \in \Delta$, $\nvdash_{\mathbf{Ł}} \sigma\delta$.*

In algebraic terms, a rule $\Gamma \ / \ \Delta$ is $\mathbf{Ł}$-admissible if and only if it is valid in the free MV-algebra over one generator. □

A description of free MV-algebras over finite sets of generators was given by McNaughton. Let $n \in \omega$. A function $f\colon [0,1]^n \to [0,1]$ is called *piecewise linear with integer coefficients*, if there are finitely many functions $L_j\colon [0,1]^n \to [0,1]$ such that for every $x \in [0,1]^n$ there exists j such that $f(x) = L_j(x)$, and each $L_j(x_0, \dots, x_{n-1})$ is of the form $\sum_{i<n} a_i x_i + b$ for some $\vec{a}, b \in \mathbb{Z}$. Let F_n be the MV-algebra of continuous piecewise linear functions $f\colon [0,1]^n \to [0,1]$ with integer coefficients, with operations defined pointwise (i.e., F_n is a subalgebra of the Cartesian power $[0,1]_{\mathbf{Ł}}^{[0,1]^n}$).

Theorem 2.2 (McNaughton [16]) *F_n is the free n-generated MV-algebra. The projection functions $\pi_i(x_0, \dots, x_{n-1}) = x_i$ for $i < n$ are its free generators.* □

If $f = \langle f_0, \dots, f_{k-1}\rangle$ is a k-tuple of functions $f_i \in F_n$, we will identify f with the corresponding function $f\colon [0,1]^n \to [0,1]^k$. A (convex rational) *polytope* is a set of the form $C = \{x \in \mathbb{R}^n \mid \forall i < k\, L_i(x) \geq 0\}$, where L_i are linear functions with integer (or rational, it makes no difference) coefficients. Bounded (i.e., contained in some $[-r,r]^n$) polytopes are exactly the convex hulls of finite sets of rational points. For each $f \in F_n$, there exists a finite set of polytopes $\{C_i \mid i < k\}$ such that $\bigcup_{i<k} C_i = [0,1]^n$, and f is linear on each C_i [16].

Since F_n is isomorphic to the Lindenbaum–Tarski algebra of **Ł** in n variables, elements $f \in F_n$ represent formulas in n variables up to **Ł**-provable equivalence. We will therefore identify formulas in n variables with elements of F_n. In particular, we will sometimes define formulas by describing their McNaughton function instead of an explicit representation using connectives of **Ł**. In the same way, homomorphisms $\sigma\colon F_n \to F_m$ will be identified with substitutions mapping formulas in n variables to formulas in m variables. Notice that such substitutions can be uniquely described by n-tuples of formulas giving $\sigma(p_i)$ for each $i < n$. In the algebraic setting, this corresponds to representation of a homomorphism $\sigma\colon F_n \to F_m$ by an n-tuple $f \in F_m^n$. Explicitly, the correspondence is given by $f = \langle \sigma(\pi_i) \mid i < n \rangle$, and $\sigma(g) = g \circ f$ for $g \in F_n$ (where we consider f as a function $[0,1]^m \to [0,1]^n$, by our above-mentioned convention). We say that σ is the substitution *induced* by f.

In view of Theorem 2.1, we will often work with F_1 and its powers F_1^m, hence it is useful to introduce notation for their elements. If $t_0 < t_1 < \cdots < t_k$ and $x_0, \dots, x_k \in \mathbb{R}^m$, then we denote by $f = L(t_0, x_0; t_1, x_1; \dots; t_k, x_k)$ the continuous piecewise linear function $f\colon [t_0, t_k] \to \mathbb{R}^m$ such that $f(t_i) = x_i$, and f is linear on each interval $[t_i, t_{i+1}]$. Also, if $L\colon [0,1]^n \to \mathbb{R}$ is a linear function with integer coefficients, then $L^= \in F_n$ is the function $f(x) = \min\{1, \max\{0, L(x)\}\}$.

We will also need some concepts and results from [14]. (We warn the reader that some of the notation in [14] is slightly different than here, since there we defined F_n to consist of functions $f\colon [0,1]^n_{\mathbb{Q}} \to [0,1]_{\mathbb{Q}}$ rather than $f\colon [0,1]^n \to [0,1]$.) If $X \subseteq \mathbb{R}^n$, let $C(X)$ and $A(X)$ denote the convex hull and affine hull of X, respectively. That is, $C(X)$ is the smallest convex subset of $\mathbb{R}^n$ that includes X, which can be explicitly expressed as

$$C(X) = \Big\{ \sum_{i<k} \alpha_i x_i \;\Big|\; k \in \omega, \alpha_i \in \mathbb{R}, \alpha_i \geq 0, x_i \in X, \sum_{i<k} \alpha_i = 1 \Big\},$$

and $A(X)$ is the smallest affine subspace of $\mathbb{R}^n$ that includes X, which we can express as

$$A(X) = \Big\{ \sum_{i<k} \alpha_i x_i \;\Big|\; k \in \omega, \alpha_i \in \mathbb{R}, x_i \in X, \sum_{i<k} \alpha_i = 1 \Big\}.$$

Notice that we count the empty set as an affine subspace. We say that $X \subseteq \mathbb{R}^n$ is *anchored*, if $A(X) \cap \mathbb{Z}^n \neq \varnothing$. We have the following characterization:

Lemma 2.3 ([14]) *The following are equivalent for any $X \subseteq \mathbb{Q}^n$.*

(i) X is anchored.

(ii) For every $u \in \mathbb{Z}^n$ and $a \in \mathbb{Q}$, if $u^\mathsf{T} x = a$ for all $x \in X$, then $a \in \mathbb{Z}$.

□

(Here we view $x \in \mathbb{Q}^n$ as column vectors, hence $u^\mathsf{T} x$ is the inner product of u and x, where $^\mathsf{T}$ denotes the matrix transpose operator.) The next lemma essentially describes elements of F_1^n up to a change of parameter.

Lemma 2.4 ([14]) *Let $x_0, \dots, x_k \in [0,1]^n_{\mathbb{Q}}$.*

(i) *If there are rationals $0 = t_0 < t_1 < \dots < t_k = 1$ such that $L(t_0, x_0; t_1, x_1; \dots; t_k, x_k) \in F^n_1$, then $x_0, x_k \in \{0,1\}^n$, and $\{x_i, x_{i+1}\}$ is anchored for each $i < k$.*

(ii) *If $x_0, x_k \in \{0,1\}^n$, and $\{x_i, x_{i+1}\}$ is anchored for each $i < k$, then there exist rationals $0 < t_0 < t_1 \dots < t_k < 1$ such that $L(0, x_0; t_0, x_0; t_1, x_1; \dots; t_k, x_k; 1, x_k) \in F^n_1$.*

□

(Lemma 4.10 in [14] does not explicitly state that we can take $0 < t_0 < t_k < 1$, but this is obvious from its proof. Then the extra end-segments $L(0, x_0; t_0, x_0)$ and $L(t_k, x_k; 1, x_k)$ have integer coefficients as they are constant functions with values $x_0, x_k \in \{0,1\}^n$.)

Finally, we will use the lemma below.

Lemma 2.5 ([14]) *Let X be an anchored subset of $\mathbb{Q}^n$, and $x_0, \dots, x_k \in \mathbb{Q}^n$. Then there exists $w \in C(X) \cap \mathbb{Q}^n$ such that $\{x_i, w\}$ is anchored for each $i \le k$.* □

The reader may find it generally helpful to be familiar with Section 4 of [14].

3 Admissibly saturated formulas

Definition 3.1 A formula φ is *admissibly saturated* in a logic L if for every finite set Δ of formulas, $\varphi \mathrel{|\!\!\sim}_L \Delta$ implies $\varphi \vdash_L \psi$ for some $\psi \in \Delta$.

In this section, we will semantically characterize admissibly saturated formulas in **Ł**, and we will show that every formula can be approximated (see below for the definition) by admissibly saturated formulas.

We first observe that since formulas φ, ψ such that $\varphi \dashv\vdash \psi$ are indistinguishable with respect to admissibility, we only need to care about the "truth sets" of our formulas rather than their full McNaughton functions:

Definition 3.2 If $\varphi \in F_n$, let $t(\varphi) = \{x \in [0,1]^n \mid \varphi(x) = 1\}$.

In this notation, we can reformulate the completeness of $[0,1]_{\text{Ł}}$ for derivable rules as follows:

Corollary 3.3 *Let $\varphi, \psi \in F_n$. Then $\varphi \vdash_{\text{Ł}} \psi$ if and only if $t(\varphi) \subseteq t(\psi)$.* □

Lemma 3.4 *If $X \subseteq [0,1]^n$ is a finite union of polytopes, there exists $\varphi \in F_n$ such that $t(\varphi) = X$.*

Proof: Let $X = \bigcup_i C_i$, and $C_i = \{x \mid \forall j\, L_{i,j}(x) \ge 0\}$, where $L_{i,j}$ are linear functions with integer coefficients. Then $X = t(\varphi)$, where $\varphi = \bigvee_i \bigwedge_j (L_{i,j} + 1)^=$. □

Theorem 3.5 *A formula $\varphi \in F_n$ is admissibly saturated in* $\mathbf{Ł}$ *if and only if*

(i) $t(\varphi) \cap \{0,1\}^n \neq \varnothing$,

(ii) $t(\varphi)$ *is connected, and*

(iii) $t(\varphi)$ *is a finite union of anchored polytopes.*

Proof: Right-to-left: assume that φ satisfies (i)–(iii), and let Δ be a finite set of formulas such that $\varphi \nvdash_{\mathbf{Ł}} \psi$ for each $\psi \in \Delta$. The conditions (i)–(iii) remain valid if we reconsider φ as a member of F_m for any $m \geq n$, hence we may assume $\Delta \subseteq F_n$ without loss of generality. We enumerate $\Delta = \{\psi_i \mid i < k\}$, and for each $i < k$ we choose $x_i \in [0,1]^n_{\mathbb{Q}}$ such that $\varphi(x_i) = 1 > \psi_i(x_i)$ using Corollary 3.3. We fix $w \in t(\varphi) \cap \{0,1\}^n$, and write $t(\varphi) = \bigcup_j C_j$, where each C_j is an anchored polytope.

Let $i < k$. As $t(\varphi)$ is connected, there exists a sequence $\{j_u \mid u \leq r_i\}$ such that $w \in C_{j_0}$, $x_i \in C_{j_{r_i}}$, and $C_{j_u} \cap C_{j_{u+1}} \neq \varnothing$ for each $u < r_i$. Put $x_{i,0} = w$, $x_{i,2r_i+2} = x_i$, and choose $x_{i,2u+2} \in C_{j_u} \cap C_{j_{u+1}}$ for each $u < r_i$. We can assume $x_{i,2u+2} \in [0,1]^n_{\mathbb{Q}}$ as C_j have integer coefficients. Since C_{j_u} is anchored and convex, we can find $x_{i,2u+1} \in C_{j_u} \cap [0,1]^n_{\mathbb{Q}}$ such that $\{x_{i,2u}, x_{i,2u+1}\}$ and $\{x_{i,2u+1}, x_{i,2u+2}\}$ are anchored by Lemma 2.5.

Let $\{y_i \mid i \leq s\}$ be the enumeration of the sequence

$$\begin{aligned} w = x_{0,0}, x_{0,1}, \dots, x_{0,2r_0+2}, x_{0,2r_0+1}, \dots, x_{0,1}, x_{0,0} = x_{1,0}, x_{1,1}, \dots, \\ \dots, x_{k-2,0} = x_{k-1,0}, x_{k-1,1} \dots, x_{k-1,2r_{k-1}+2}, x_{k-1,2r_{k-1}+1}, \dots, x_{k-1,1}, x_{k-1,0} = w. \end{aligned}$$

(If $k = 0$, we understand it as $s = 0$, $y_0 = w$.) By Lemma 2.4, there exist rationals $0 < t_0 < \cdots < t_s < 1$ such that $f = L(0, w; t_0, y_0; \dots; t_s, y_s; 1, w) \in F^n_1$. Let $\sigma\colon F_n \to F_1$ be the substitution induced by f. For each $i < s$ there exists j such that $y_i, y_{i+1} \in C_j$, hence $\operatorname{rng}(f \restriction [t_i, t_{i+1}]) = C(y_i, y_{i+1}) \subseteq C_j$, where rng denotes the range of a function. It follows that $\operatorname{rng}(f) \subseteq t(\varphi)$, thus $\vdash_{\mathbf{Ł}} \sigma\varphi$. On the other hand, for any $\psi \in \Delta$ there exists i such that $\psi(y_i) < 1$, hence $\nvdash_{\mathbf{Ł}} \sigma\psi$. Consequently $\varphi \not\mathrel{|\!\!\!\sim}_{\mathbf{Ł}} \Delta$.

Left-to-right: if $t(\varphi) \cap \{0,1\}^n = \varnothing$, then φ is classically unsatisfiable and therefore not $\mathbf{Ł}$-unifiable, thus $\varphi \mathrel{|\!\!\!\sim}_{\mathbf{Ł}} \varnothing$, and φ is not admissibly saturated.

Assume that $t(\varphi)$ is disconnected, and fix open sets U_0, U_1 such that $X_i = t(\varphi) \cap U_i$ are nonempty, disjoint, and $X_0 \cup X_1 = t(\varphi)$. As $t(\varphi)$ is a finite union of polytopes, and any polytope is connected, X_i are also finite unions of polytopes. Let ψ_0 and ψ_1 be formulas such that $X_i = t(\psi_i)$ by Lemma 3.4. Let $\sigma\colon F_n \to F_1$ be a substitution induced by $f \in F^n_1$ such that $\vdash_{\mathbf{Ł}} \sigma\varphi$. Then $\operatorname{rng}(f) \subseteq t(\varphi)$, and $\operatorname{rng}(f)$, being a continuous image of a connected space, is connected, hence $\operatorname{rng}(f) \subseteq X_i$ for some $i = 0, 1$, i.e., $\vdash_{\mathbf{Ł}} \sigma\psi_i$. Thus, $\varphi \mathrel{|\!\!\!\sim}_{\mathbf{Ł}} \psi_0, \psi_1$ by Theorem 2.1, but $\varphi \nvdash_{\mathbf{Ł}} \psi_i$ as $X_i \subsetneq t(\varphi)$, hence φ is not admissibly saturated.

Write $t(\varphi) = \bigcup_{i<k} C_i$ where C_i are polytopes, and assume that k is smallest possible. In particular for every $i < k$, $C_i \nsubseteq \bigcup_{j \neq i} C_j$. Assume that C_i is not anchored, and using Lemma 3.4 find a formula ψ such that $t(\psi) = \bigcup_{j \neq i} C_j$. Clearly $\varphi \nvdash_{\mathbf{Ł}} \psi$. We claim $\varphi \mathrel{|\!\!\!\sim}_{\mathbf{Ł}} \psi$, hence φ is not admissibly saturated. Let $\sigma\colon F_n \to F_1$ be a substitution induced by $f \in F^n_1$ such that $\vdash_{\mathbf{Ł}} \sigma\varphi$, i.e., $\operatorname{rng}(f) \subseteq t(\varphi)$. We can write $f = L(t_0, x_0; t_1, x_1; \dots; t_s, x_s)$ for some rationals $0 = t_0 < t_1 < \cdots < t_s = 1$ and $x_0, \dots, x_s \in [0,1]^n_{\mathbb{Q}}$. The preimage $f^{-1}(C_j) \cap [t_u, t_{u+1}]$

(if nonempty) is an interval with rational endpoints for each u and j, hence we can refine the sequence of t_u's to ensure that for each $u < s$ there exists $j < k$ such that $\mathrm{rng}(f \restriction [t_u, t_{u+1}]) = C(x_u, x_{u+1}) \subseteq C_j$. However, $\{x_u, x_{u+1}\}$ is anchored by Lemma 2.4, hence C_j is anchored, too. In particular, $j \neq i$, i.e., $\mathrm{rng}(f) \subseteq t(\psi)$, and $\vdash_{\text{Ł}} \sigma\psi$. □

Definition 3.6 An *admissibly saturated approximation* of a formula φ in a logic L is a set A_φ such that

(i) A_φ is a finite set of admissibly saturated formulas,

(ii) $\varphi \mathrel{|\!\!\sim}_L A_\varphi$, and

(iii) $\psi \vdash_L \varphi$ for each $\psi \in A_\varphi$.

Notice that admissibly saturated approximations can be used to reduce admissibility to derivability (we assume for simplicity that the logic has a well-behaved conjunction connective):

Observation 3.7 $\Gamma \mathrel{|\!\!\sim}_L \Delta$ *if and only if for each* $\psi \in A_{\bigwedge\Gamma}$ *there exists* $\delta \in \Delta$ *such that* $\psi \vdash_L \delta$.

Proof: Left-to-right: if $\psi \in A_{\bigwedge\Gamma}$ and $\Gamma \mathrel{|\!\!\sim}_L \Delta$, then $\psi \mathrel{|\!\!\sim}_L \Delta$. As ψ is admissibly saturated, there exists $\delta \in \Delta$ such that $\psi \vdash_L \delta$.

Right-to-left: we have $\Gamma \mathrel{|\!\!\sim}_L A_{\bigwedge\Gamma}$, and $\psi \mathrel{|\!\!\sim}_L \Delta$ for every $\psi \in A_{\bigwedge\Gamma}$, hence $\Gamma \mathrel{|\!\!\sim}_L \Delta$. □

Theorem 3.8 *Every formula has an admissibly saturated approximation in* **Ł**.

Proof: Let $\varphi \in F_n$, and write $t(\varphi) = \bigcup_{i<k} C_i$ for some polytopes C_i. Let

$$I = \{i < k \mid C_i \text{ is anchored}\},$$
$$X = \bigcup_{i \in I} C_i.$$

Since polytopes are connected, all connected components of X are of the form $\bigcup_{i \in I'} C_i$ for some $I' \subseteq I$. We can therefore write I as a disjoint union $I = \dot{\bigcup}_{j<l} I_j$ so that the sets $X_j = \bigcup_{i \in I_j} C_i$ are the connected components of X. Let

$$J = \{j < l \mid X_j \cap \{0,1\}^n \neq \varnothing\},$$

and using Lemma 3.4 we find formulas ψ and ψ_j such that $t(\psi) = X$ and $t(\psi_j) = X_j$. We claim that $A_\varphi = \{\psi_j \mid j \in J\}$ is an admissibly saturated approximation of φ.

Clearly $\psi_j \vdash_{\text{Ł}} \varphi$ by Corollary 3.3, and the formulas $\psi_j \in A_\varphi$ are admissibly saturated by Theorem 3.5. It remains to show $\varphi \mathrel{|\!\!\sim}_{\text{Ł}} A_\varphi$. Let $\sigma\colon F_n \to F_1$ be a substitution such that $\vdash_{\text{Ł}} \sigma\varphi$, induced by $f = L(t_0, x_0; t_1, x_1; \dots; t_s, x_s) \in F_1^n$. We have $\mathrm{rng}(f) \subseteq t(\varphi)$. The same reasoning as in the proof of Theorem 3.5 shows that actually $\mathrm{rng}(f) \subseteq X$. Being a continuous image of a connected space, $\mathrm{rng}(f)$ is connected, hence $\mathrm{rng}(f) \subseteq X_j$ for some $j < l$. As $f(0) \in \mathrm{rng}(f) \cap \{0,1\}^n$, we have $j \in J$, hence $\vdash_{\text{Ł}} \sigma\psi_j$ and $\psi_j \in A_\varphi$. □

Remark 3.9 Following Iemhoff [10], a *maximal admissible consequence* of a formula φ in a logic L is a formula $\overline{\varphi}$ such that

$$\varphi \mathrel{|\!\sim}_L \psi \Leftrightarrow \overline{\varphi} \vdash_L \psi$$

for every formula ψ. Since $\bigvee_i \varphi_i \vdash_{\mathbf{Ł}} \bigvee_i \varphi_i^n$, it is easy to see from Theorem 3.8 that every formula has a maximal admissible consequence in $\mathbf{Ł}$, namely $\overline{\varphi} = \bigvee A_\varphi$.

Remark 3.10 Admissibly saturated formulas are related to *projective formulas*, which have proved valuable for investigation of admissible rules in many modal and superintuitionistic logics. Recall that a formula φ is projective in a logic L if there exists an L-unifier σ of φ such that

$$\varphi \vdash_L \psi \leftrightarrow \sigma\psi$$

for every formula ψ. It is easy to see that every projective formula is admissibly saturated, and thus a projective approximation of a formula is also its admissibly saturated approximation. Moreover, an admissibly saturated formula can have a projective approximation only if it is itself projective. Thus for any fixed logic L, all formulas have projective approximations if and only if all formulas have admissibly saturated approximations and all admissibly saturated formulas are projective. It is not clear whether admissibly saturated formulas of $\mathbf{Ł}$ are projective.

4 Bases of admissible rules

In this section, we will construct bases of multiple-conclusion and single-conclusion admissible rules of $\mathbf{Ł}$, and we will show that there are no finite bases.

We start by presenting the rules we are going to work with.

Definition 4.1 We introduce the rules

(WDP)	$p \vee \neg p \;/\; p, \neg p,$
(CC_n)	$\neg(q \vee \neg q)^n \;/\; \varnothing,$
(CC_n^1)	$\neg(q \vee \neg q)^n \;/\; \bot,$
(RCC_n)	$(q \vee \neg q)^n \to p, p \vee \neg p \;/\; p,$
(Con)	$\bot \;/\; \varnothing$

for $n \in \omega$. For any $k \geq 2$, let $\chi_k \in F_1$ be a formula such that $t(\chi_k) = \{\frac{1}{k}\}$. For definiteness, we may take $\chi_k = L(0, 0; \frac{1}{k}, 1; \frac{2}{k}, 0; 1, 0)$, which can be represented as

$$\chi_k(q) = \begin{cases} kq \wedge 2(\neg q)^{k/2}, & k \text{ even}, \\ kq \wedge \big((\neg q)^{k-1} \oplus (\neg q \cdot 2(\neg q)^{\lfloor k/2 \rfloor})\big), & k \text{ odd}. \end{cases}$$

We introduce the rules

(NA_k)	$p \vee \chi_k(q) \;/\; p,$

and we put $NA = \{NA_p \mid p \text{ is a prime}\}$.

(*WDP* denotes “weak disjunction property”. *CC* stands for “classical contradiction”, as it derives rules $\Gamma \,/\, \varnothing$ where Γ is inconsistent in classical logic, cf. [14]. *RCC* means “relativized *CC*”. *NA* stands for “not anchored”; as we will see below, the rule allows us to delete non-anchored polytopes from $t(\varphi)$. *Con* means “consistency”, as it is admissible in a logic L iff L is consistent.) The basic relations between these rules are summarized in the next lemma.

Lemma 4.2

(i) $\mathbf{Ł} + CC_n = \mathbf{Ł} + CC_n^1 + Con$.

(ii) $\mathbf{Ł} + WDP + CC_n^1 \vdash RCC_n$, $\mathbf{Ł} + RCC_n \vdash CC_n^1$.

(iii) $\mathbf{Ł}$ *admits* WDP, CC_n, RCC_n, *and* NA_k, $k \geq 2$.

(iv) $\mathbf{Ł} \vdash CC_0^1, CC_1^1, RCC_0, RCC_1$, *and* $\mathbf{Ł} + NA_2 \vdash CC_2^1, RCC_2$.

(v) *If* $k \mid l$, *then* $\mathbf{Ł} + NA_k \vdash NA_l$.

(vi) $\mathbf{Ł} + CC_3 \vdash CC_n$, *and* $\mathbf{Ł} + RCC_3 \vdash RCC_n$.

Proof: (i) and (ii) are straightforward.

(iii): We will show that the rules are valid in F_1.

WDP: let $f \in F_1$ be such that $f \vee \neg f = 1$. Then $\mathrm{rng}(f) \subseteq \{0,1\}$, and $\mathrm{rng}(f)$ is connected, hence $f = 0$ or $f = 1$.

CC_n: if $f \in F_1$, then $f(0) \in \{0,1\}$, hence $(\neg(f \vee \neg f)^n)(0) = 0$, and $\neg(f \vee \neg f)^n \neq 1$.

RCC_n follows by (ii).

NA_k: let $f, g \in F_1$ be such that $f \vee \chi_k(g) = 1$. This means that for each $t \in [0,1]$, $f(t) = 1$ or $g(t) = \frac{1}{k}$. Write $g = L(t_0, x_0; t_1, x_1; \dots; t_s, x_s)$. If $g(t) = \frac{1}{k}$ for two distinct $t \in [t_i, t_{i+1}]$, then $g \restriction [t_i, t_{i+1}]$ is the constant $\frac{1}{k}$ function, which is not a linear function with integer coefficients, contradicting $g \in F_1$. Thus $g(t) = \frac{1}{k}$ can only hold for finitely many $t \in [0,1]$. It follows that $f^{-1}(\{1\})$ contains all of $[0,1]$ except for finitely many points, and being a continuous preimage of $\{1\}$, it is closed, hence it equals $[0,1]$, and $f = 1$.

(iv): RCC_0 is trivial.

RCC_1: we assume $p \vee \neg p$ and $q \vee \neg q \to p$. The latter gives $\neg q \to p$ and $q \to p$, hence $\neg p \to \neg q$, thus $\neg p \to p$. Using $p \vee \neg p$ we get $p \vee p$, i.e., p.

RCC_2: it suffices to show

$$p \vee \neg p, (q \vee \neg q)^2 \to p \vdash_{\mathbf{Ł}} p \vee \chi_2(q).$$

Let v be a valuation in $[0,1]$ such that $v(p \vee \neg p) = v((q \vee \neg q)^2 \to p) = 1$, and put $x = v(p)$, $y = v(q)$. We have $x \vee \neg x = 1$, thus $x = 1$ or $x = 0$. In the latter case, $(y \vee \neg y)^2 = 0$, hence $y \vee \neg y \leq \frac{1}{2}$. This can only happen for $y = \frac{1}{2}$, hence $x \vee \chi_2(y) = 1$.

(v): If $l = km$, then

$$p \vee \chi_l(q) \vdash_{\mathbf{Ł}} p \vee \chi_k(mq).$$

Indeed, if $x \vee \chi_l(y) = 1$, then $x = 1$ or $y = \frac{1}{l}$. In the latter case $my = \frac{1}{k}$, hence in both cases $x \vee \chi_k(my) = 1$.

(vi): We will prove $\text{Ł} + CC_3 \vdash CC_n$, the case of RCC is similar. It suffices to show $\text{Ł} + CC_n \vdash CC_{n+1}$ for every $n \geq 3$. Consider the formula

$$\varphi = L\left(0,0;\tfrac{1}{3},\tfrac{2}{3};\tfrac{2}{3},\tfrac{1}{3};1,1\right) \in F_1.$$

It is easy to see that $\varphi([1/(n+1), 1-1/(n+1)]) \subseteq [1/n, 1-1/n]$. Since $\neg(x \vee \neg x)^n = 1$ if and only if $x \in [1/n, 1-1/n]$, we obtain

$$\neg(q \vee \neg q)^{n+1} \vdash_{\text{Ł}} \neg(\varphi(q) \vee \neg\varphi(q))^n. \qquad \square$$

Theorem 4.3 *WDP + CC_3 + NA is a basis of multiple-conclusion Ł-admissible rules.*

Proof: On the one hand, the given rules are admissible by Lemma 4.2. On the other hand, consider an admissible rule $\Gamma \;/\; \Delta$, and put $\varphi = \bigwedge\Gamma$. For each $\psi \in A_\varphi$, the rule $\psi \;/\; \Delta$ is derivable in Ł, hence it suffices to show that

$$\text{Ł} + \mathit{WDP} + CC_3 + \mathit{NA} \vdash \varphi \;/\; A_\varphi.$$

As in the proof of Theorem 3.8, we write $t(\varphi) = \bigcup_{i<k} C_i$, and put

$$I = \{i < k \mid C_i \text{ is anchored}\}, \qquad X = \bigcup_{i \in I} C_i, \qquad t(\psi) = X,$$

$$I = \dot{\bigcup_{j<l}} I_j, \qquad X_j = \bigcup_{i \in I_j} C_i, \qquad t(\psi_j) = X_j,$$

so that X_j are the connected components of X,

$$J = \{j < l \mid X_j \cap \{0,1\}^n \neq \varnothing\}.$$

Assume that C_i is not anchored, and let $t(\varphi') = \bigcup_{i' \neq i} C_{i'}$. By Lemma 2.3, there exists $u \in \mathbb{Z}^n$ and $a \in \mathbb{Q} \smallsetminus \mathbb{Z}$ such that $u^\intercal x = a$ for all $x \in C_i$. We can multiply u and a by a suitable integer to ensure $a = t/p$ for some prime number p. Since $a \notin \mathbb{Z}$, t is coprime to p, hence there exists s such that $ts \equiv 1 \pmod p$. We can multiply u and a by s, hence we can assume that $a \in \mathbb{Z} + 1/p$. Let $L(x) = u^\intercal x - \lfloor a \rfloor$, and $\vartheta(x) = L^= \in F_n$. We have $\vartheta(x) = 1/p$ for all $x \in C_i$, hence $\varphi \vdash_{\text{Ł}} \varphi' \vee \chi_p(\vartheta)$, thus $\text{Ł} + \mathit{NA}_p \vdash \varphi \;/\; \varphi'$. By repeating this construction for every $i < k$, $i \notin I$ in turn, we obtain

$$(*) \qquad \text{Ł} + \mathit{NA} \vdash \varphi \;/\; \psi.$$

The next task is to derive $\psi \;/\; \{\psi_j \mid j < l\}$. If $l = 1$, there is nothing to prove. If $l = 0$, i.e., $X = \varnothing$, we have $\psi \vdash_{\text{Ł}} \bot$, hence

$$\text{Ł} + CC_3 \supseteq \text{Ł} + \mathit{Con} \vdash \psi \;/\; \varnothing.$$

Assume $l > 1$, and put $\psi' = \bigvee_{j \neq 0} \psi_j$. We have $t(\psi_0) \cap t(\psi') = \varnothing$, thus $\psi_0, \psi' \vdash_{\text{Ł}} \bot$, hence there exists an m such that $\vdash_{\text{Ł}} \psi_0^m \cdot \psi'^m \to \bot$. Put $\alpha = \psi_0^m$. We have $\psi_0 \vdash_{\text{Ł}} \alpha$ and $\psi' \vdash_{\text{Ł}} \neg\alpha$, hence $\psi \vdash_{\mathbf{Ł}} \alpha \vee \neg\alpha$. Also $\psi, \alpha \vdash_{\mathbf{Ł}} \psi_0$ and $\psi, \neg\alpha \vdash_{\mathbf{Ł}} \psi'$, thus $\mathbf{Ł} + \mathit{WDP} \vdash \psi \mathbin{/} \psi_0, \psi'$. We can continue in a similar way with ψ'; by repeating this construction $(l-1)$-times, we obtain

$$(**) \qquad \mathbf{Ł} + \mathit{WDP} \vdash \psi \mathbin{/} \psi_0, \ldots, \psi_{l-1}.$$

Let $j < l$, $j \notin J$, and put $\beta = \bigwedge_{i<n}(p_i \vee \neg p_i)$. We have $t(\beta) = t(\beta \vee \neg\beta) = \{0,1\}^n$, hence $\psi_j, \beta \vee \neg\beta \vdash_{\mathbf{Ł}} \bot$. It follows that there exists m such that $\psi_j \vdash_{\mathbf{Ł}} \neg(\beta \vee \neg\beta)^m$, hence $\mathbf{Ł} + CC_m \vdash \psi_j \mathbin{/} \varnothing$. Using Lemma 4.2 (vi), we obtain

$$(***) \qquad \mathbf{Ł} + CC_3 \vdash \psi_j \mathbin{/} \varnothing.$$

If we put $(*)$, $(**)$, and $(***)$ together using cuts, we get $\mathbf{Ł} + \mathit{WDP} + CC_3 + \mathit{NA} \vdash \varphi \mathbin{/} \{\psi_j \mid j \in J\} = A_\varphi$. $\square$

In order to find a basis of single-conclusion $\mathbf{Ł}$-admissible rules, we have to axiomatize the single-conclusion fragment of $\mathbf{Ł} + \mathit{WDP} + CC_3 + \mathit{NA}$. We will prove a general result characterizing single-conclusion fragments of consequence relations involving the *WDP* rule. (The characterization actually holds for all logics extending $\mathbf{FL_{ew}}$ in place of $\mathbf{Ł}$ with the same proof, but we do not want to bother with defining $\mathbf{FL_{ew}}$ just for this reason as we have no further use for it.)

Theorem 4.4 *Let X be a set of single-conclusion rules. Then the following are equivalent.*

(i) $\mathbf{Ł} + X + \mathit{WDP}$ is conservative over $\mathbf{Ł} + X$ wrt single-conclusion rules.

(ii) For every $\Gamma \mathbin{/} \varphi \in X$, $\mathbf{Ł} + X$ derives $\Gamma \vee r, r \vee \neg r \mathbin{/} \varphi \vee r$, where r is a fresh variable.

Here, $\Gamma \vee r$ denotes $\{\gamma \vee r \mid \gamma \in \Gamma\}$.

Proof: On the one hand, $\Gamma \vee r, r \vee \neg r \mathbin{/} \varphi \vee r$ is clearly derivable in $\mathbf{Ł} + X + \mathit{WDP}$. On the other hand, assume that (ii) holds, we will show that for every rule $\Gamma \mathbin{/} \Delta$ derivable in $\mathbf{Ł} + X + \mathit{WDP}$, and every finite set of formulas $\Pi \cup \{\varphi\}$, we have

$$(*) \qquad \forall \delta \in \Delta\, (\mathbf{Ł} + X \vdash \Pi, \delta \mathbin{/} \varphi) \;\Rightarrow\; \mathbf{Ł} + X \vdash \Pi, \Gamma \mathbin{/} \varphi.$$

The result then follows by taking $\Delta = \{\varphi\}$, $\Pi = \varnothing$. We will show $(*)$ by induction on the length of the derivation of $\Gamma \mathbin{/} \Delta$ in $\mathbf{Ł} + X + \mathit{WDP}$.

If $\Gamma \mathbin{/} \Delta$ is derivable in $\mathbf{Ł} + X$, then $(*)$ follows by a cut. The induction step for weakening is trivial. Assume that $\Gamma \mathbin{/} \Delta$ was derived by a cut

$$\frac{\Gamma \mathbin{/} \Delta, \alpha \quad \Gamma, \alpha \mathbin{/} \Delta}{\Gamma \mathbin{/} \Delta}.$$

If $\mathbf{Ł} + X \vdash \Pi, \delta \mathbin{/} \varphi$ for each $\delta \in \Delta$, then $\mathbf{Ł} + X \vdash \Pi, \Gamma, \alpha \mathbin{/} \varphi$ by the induction hypothesis. Thus $\mathbf{Ł} + X \vdash \Pi, \Gamma, \delta \mathbin{/} \varphi$ for each $\delta \in \Delta \cup \{\alpha\}$, hence $\mathbf{Ł} + X \vdash \Pi, \Gamma \mathbin{/} \varphi$ by the induction hypothesis.

Claim 1 *If $\mathbf{Ł} + X \vdash \Gamma \mathbin{/} \varphi$, then $\mathbf{Ł} + X \vdash \Gamma \vee \alpha, \alpha \vee \neg\alpha \mathbin{/} \varphi \vee \alpha$.*

Proof: By induction on the length of derivation. If $\Gamma \mathbin{/} \varphi$ is an axiom of $\mathbf{Ł}$ or an instance of modus ponens, then $\Gamma \vee \alpha \vdash_{\mathbf{Ł}} \varphi \vee \alpha$. If $\Gamma \mathbin{/} \varphi$ is an instance of a rule from X, the result holds by assumption. The induction steps for cut or weakening follow by an application of cut or weakening, respectively. $\square$ (Claim 1)

We return to the proof of $(*)$. The only remaining case is when $\Gamma \;/\; \Delta$ is an instance of *WDP*. We assume that $\text{Ł} + X$ derives $\Pi, \alpha \;/\; \varphi$ and $\Pi, \neg\alpha \;/\; \varphi$. Using the Claim, we have $\Pi \vee \alpha, \alpha \vee \neg\alpha \;/\; \varphi \vee \alpha$, hence also $\Pi, \alpha \vee \neg\alpha \;/\; \varphi \vee \alpha$. Symmetrically, we have $\Pi, \alpha \vee \neg\alpha \;/\; \varphi \vee \neg\alpha$. As $\varphi \vee \alpha, \varphi \vee \neg\alpha \vdash_{\text{Ł}} \varphi$, we obtain $\Pi, \alpha \vee \neg\alpha \;/\; \varphi$. □

Theorem 4.5 *$RCC_3 + NA$ is a basis of single-conclusion* Ł*-admissible rules.*

Proof: By Theorem 4.3, it suffices to show that $\text{Ł} + WDP + CC_3 + NA$ is conservative over $\text{Ł} + RCC_3 + NA$ wrt single-conclusion rules. Consider a derivation of $\Gamma \;/\; \varphi$ in $\text{Ł} + WDP + CC_3 + NA$. We include $\bot$ in the conclusion of every rule in the proof, fix instances of axioms from $\text{Ł} + WDP + NA$ using weakening, and derive $\Gamma \;/\; \varphi$ from $\Gamma \;/\; \varphi, \bot$ and the Ł-derivable rule $\bot \;/\; \varphi$ by a cut. We obtain a proof of $\Gamma \;/\; \varphi$ in $\text{Ł} + WDP + CC_3^1 + NA$, which is contained in $\text{Ł} + WDP + RCC_3 + NA$. It thus suffices to show that $\text{Ł} + WDP + RCC_3 + NA$ is conservative over $\text{Ł} + RCC_3 + NA$ wrt single-conclusion rules. We demonstrate it using Theorem 4.4.

Clearly, $\text{Ł} + NA_k$ derives $p \vee r \vee \chi_k(q) \;/\; p \vee r$, and a fortiori $p \vee \chi_k(q) \vee r, r \vee \neg r \;/\; p \vee r$. Since $p \vee \neg p \vee r, r \vee \neg r \vdash_{\text{Ł}} p \vee r \vee \neg(p \vee r)$, the rule

$$((q \vee \neg q)^3 \to p) \vee r, p \vee \neg p \vee r, r \vee \neg r \;/\; p \vee r$$

is derivable from the instance

$$(q \vee \neg q)^3 \to p \vee r, (p \vee r) \vee \neg(p \vee r) \;/\; p \vee r$$

of RCC_3. □

Definition 4.6 A basis of (multiple-conclusion or single-conclusion) admissible rules of a logic L is *independent* if none of its proper subsets is a basis.

Theorem 4.7 *The bases $WDP + CC_3 + NA$ and $RCC_3 + NA$ are independent.*

Proof: We have to show that none of the rules is derivable from the others over Ł.

WDP: The 4-element Boolean algebra satisfies *NA* and CC_n (even RCC_n), but does not satisfy *WDP*.

CC_3 and RCC_3: Let A be the MV-algebra $\{f \restriction [\frac{1}{3}, \frac{2}{3}] \mid f \in F_1\}$. The same argument as in Lemma 4.2 shows that $A \vDash NA$ and $A \vDash WDP$. On the other hand, $A \nvDash CC_3^1$: we have $\neg(f \vee \neg f)^3 = 1$ for $f = L(\frac{1}{3}, \frac{1}{3}; \frac{2}{3}, \frac{2}{3}) \in A$.

NA_p: Let A_p be the MV-algebra of continuous piecewise linear functions $f\colon [0,1] \to [0,1]$ with coefficients from $\frac{1}{p}\mathbb{Z}$ such that $f(0) \in \{0,1\}$. We have $A \vDash WDP$ and $A \vDash CC_n$ (hence also $A \vDash RCC_n$) as in Lemma 4.2. Also $A \vDash NA_k$ for $k \neq 1, p$ by the same argument, as the constant $\frac{1}{k}$ function does not have coefficients in $\frac{1}{p}\mathbb{Z}$. On the other hand, $A_p \nvDash NA_p$: we have $f \vee \chi_p(g) = 1$ and $f \neq 1$, where $f = L(0,1; \frac{1}{2}, 1; 1, 0)$ and $g = L(0,0; \frac{1}{p}, \frac{1}{p}; 1, \frac{1}{p})$. □

For the sake of completeness, we also mention that we cannot simplify the basis of single-conclusion Ł-admissible rules to $CC_3^1 + NA$. Let A be the MV-algebra of functions $f\colon [0,1] \to [0,1]$ such that f is piecewise linear with integer coefficients, and continuous except for a finite set where it is right-continuous. Then $A \vDash NA$ and $A \vDash CC_n$, but $A \nvDash RCC_3$.

If **Ł** has a finite basis of multiple-conclusion or single-conclusion admissible rules, then a finite subset of $WDP + CC_3 + NA$ or $RCC_3 + NA$, respectively, is also a basis. However, this cannot happen, as these two bases are infinite and independent. Thus:

Corollary 4.8 **Ł** *does not have a finite basis of admissible rules* (*in either the multiple-conclusion or single-conclusion setting*). ☐

On the other hand, the bases $WDP + CC_3 + NA$ and $RCC_3 + NA$ use only finitely many (in fact, two) propositional variables.

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