The ubiquity of conservative translations

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Abstract

We study the notion of conservative translation between logics introduced by Feitosa and D'Ottaviano [5]. We show that classical propositional logic (**CPC**) is universal in the sense that every finitary logic over a countable set of formulas can be conservatively translated into **CPC**. The translation is computable if the logic is decidable. More generally, we show that one can take instead of **CPC** a broad class of logics (extensions of a certain fragment of full Lambek calculus **FL**) including most nonclassical logics studied in the literature, hence in a sense, (almost) any two reasonable logical systems can be conservatively translated into each other. We also provide some counterexamples, in particular the paraconsistent logic **LP** is not universal.

1 Introduction

There have been several proposals of a general concept of a translation or interpretation between abstract logical systems, see e.g. [1, 8] for overviews. A minimalist approach was taken by da Silva, D'Ottaviano and Sette [11]: a logic is given by any Tarski-style consequence operator, and then a translation of one logic in another is an arbitrary mapping of formulas to formulas preserving the consequence relation. Feitosa and D'Ottaviano [5] consider the stronger notion of *conservative translations*, which preserve the consequence relation in both directions. This avoids uninteresting examples of translations such as mapping all formulas to a fixed tautology.

This notion of a conservative translation is still very general (perhaps too general): for instance, translations are not required to respect the structure of formulas in any way, to be computable, or to preserve any properties of the logic. For this reason, it is natural to expect that there should exist a conservative translation between more or less any two reasonable logical systems. Nevertheless, no result to such effect appears in the literature. Instead, there are several papers devoted to proofs (often non-constructive) of the existence of conservative translations between particular pairs of logics: [2, 3, 4].

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The main purpose of this paper is to demonstrate that for a quite large class of logics, it is indeed possible to construct a conservative translation between any two of them. First, we prove that an arbitrary finitary logic L in countably many formulas can be conservatively translated into classical propositional logic (**CPC**), in either the single-conclusion or multiple-conclusion setting. Our translation is constructed by an explicit inductive definition, and it is computable whenever L is decidable. Moreover, the translation has the additional property of being most general in the sense that every other translation of L to **CPC** is equivalent to its substitution instance.

Let us define a logic L to be *universal* if every finitary logic in countably many formulas can be conservatively translated into L, so that the result above can be restated by saying that **CPC** is universal. We generalize this result by showing that every logic between the $\to, \leftarrow, \wedge$ fragment of the full Lambek calculus **FL** (see [6]) and the corresponding fragment of **CPC** is universal, and similarly, any logic between **BCK** (the implication fragment of $\mathbf{FL_{ew}}$) and $\mathbf{CPC}{\restriction_{\to}}$ is universal. This establishes the universality of most of nonclassical logics studied in the literature, as they typically extend (a suitable fragment of) **FL** in one way or another: this includes e.g. intuitionistic and intermediate logics, various modal, substructural, fuzzy, or relevant logics, both propositional and first-order.

As an additional example, we show that Kleene's logic with truth constants is universal, whereas the paraconsistent logic **LP**—based on the same algebra but with a different choice of designated truth values—is *not* universal. We also completely characterize universal fragments of **CPC**: a fragment $\mathbf{CPC}{\restriction_B}$ is universal if and only if implication is definable from B. (In particular, we obtain a couple of nontrivial examples of logics into which **CPC** cannot be conservatively translated, namely **LP** and the fragments $\mathbf{CPC}{\restriction_{\leftrightarrow,\neg}}$, $\mathbf{CPC}{\restriction_{\wedge,\vee,\bot,\top}}$.)

The paper is organized as follows. In Section 2 we give basic definitions, and we construct conservative translations into classical logic. In Section 3 we investigate the class of universal logics, as detailed above. Section 4 consists of concluding remarks.

2 Translation to classical logic

In this section, we are going to construct conservative translations of (almost) arbitrary logics into classical logic (Theorem 2.4). First we review the relevant definitions to fix the notation.

Definition 2.1 A pair $L = \langle F, \vdash \rangle$ is a *logic* over a set of *formulas* F if $\vdash \subseteq \mathcal{P}(F) \times F$ satisfies

(i) $\varphi \vdash \varphi$,

(ii) $\Gamma \vdash \varphi$ implies $\Gamma, \Gamma' \vdash \varphi$,

(iii) if $\Gamma \vdash \varphi$ and $\Delta \vdash \psi$ for all $\psi \in \Gamma$, then $\Delta \vdash \varphi$,

for every $\varphi \in F$ and $\Gamma, \Gamma', \Delta \subseteq F$. A logic is *finitary* if

(iv) $\Gamma \vdash \varphi$ implies $\Gamma' \vdash \varphi$ for some finite $\Gamma' \subseteq \Gamma$.

When discussing algorithmic issues, we will tacitly assume that F is encoded as a recursively enumerable subset of ω.

A logic $L = \langle F, \vdash \rangle$ is a *propositional logic* if F is the set of formulas built inductively from a set of variables and a set of finitary connectives (i.e., F is a free algebra in a particular signature), and $\vdash$ is structural (substitution-invariant):

(v) $\Gamma \vdash \varphi$ implies $\sigma(\Gamma) \vdash \sigma(\varphi)$ for every substitution σ,

where as usual, a substitution is a homomorphism of free algebras. Let $\mathbf{CPC} = \langle F_{\mathbf{CPC}}, \vdash_{\mathbf{CPC}} \rangle$ denote the usual consequence relation of classical propositional logic in countably infinitely many variables.

Definition 2.2 A *translation* from a logic $L_0 = \langle F_0, \vdash_0 \rangle$ to a logic $L_1 = \langle F_1, \vdash_1 \rangle$ is a function $f\colon F_0 \to F_1$ such that

$$\Gamma \vdash_0 \varphi \;\Rightarrow\; f(\Gamma) \vdash_1 f(\varphi)$$

for every $\Gamma \subseteq F_0$, $\varphi \in F_0$. We will write this as $f\colon L_0 \to L_1$. The translation f is *conservative*, written as $f\colon L_0 \to_c L_1$, if

$$\Gamma \vdash_0 \varphi \;\Leftrightarrow\; f(\Gamma) \vdash_1 f(\varphi).$$

We write $L_0 \leq_c L_1$ if there exists a conservative translation $f\colon L_0 \to_c L_1$.

The translations to classical logic we construct have an additional property which might be of independent interest, hence we give it a name:

Definition 2.3 Let $L_0 = \langle F_0, \vdash_0 \rangle$ be a logic, and $L_1 = \langle F_1, \vdash_1 \rangle$ a propositional logic. A translation $f\colon L_0 \to L_1$ is *most general* if for every translation $g\colon L_0 \to L_1$, there exists a substitution σ such that $g(\varphi) \dashv\vdash_1 \sigma(f(\varphi))$ for every $\varphi \in F_0$.

Notice that if $L_0 \leq_c L_1$ and L_1 is finitary, then L_0 is also finitary.

The main result of this section is:

Theorem 2.4 *For every finitary logic $L = \langle F, \vdash \rangle$ over a countable set of formulas F, there exists a conservative most general translation $f\colon L \to_c$ **CPC**.*

If $\vdash$ is decidable, then f is computable. In general, f is Turing equivalent to (*the finitary fragment of*) $\vdash$.

We will prove Theorem 2.4 below as a corollary to its multiple-conclusion version. Apart from being more general, the construction of the translations in the multiple-conclusion case is more transparent and displays better the underlying symmetry, we thus find it preferable to giving a direct proof for the single-conclusion case, which feels a bit ad hoc.

Definition 2.5 A pair $L = \langle F, \vdash \rangle$ is a *multiple-conclusion logic* [10] if $\vdash\, \subseteq \mathcal{P}(F) \times \mathcal{P}(F)$ satisfies

(i) $\varphi \vdash \varphi$,

(ii) $\Gamma \vdash \Delta$ implies $\Gamma, \Gamma' \vdash \Delta, \Delta'$,

(iii) if $\Gamma, \Pi \vdash \Lambda, \Delta$ for every Π, Λ such that $\Pi \cup \Lambda = \Xi$, then $\Gamma \vdash \Delta$,

for every $\varphi \in F$ and $\Gamma, \Gamma', \Delta, \Delta', \Xi \subseteq F$. A multiple-conclusion logic is *finitary* if

(iv) $\Gamma \vdash \Delta$ implies $\Gamma' \vdash \Delta'$ for some finite $\Gamma' \subseteq \Gamma$, $\Delta' \subseteq \Delta$.

Note that if L is finitary, condition (iii) can be equivalently simplified to

(iii′) if $\Gamma, \varphi \vdash \Delta$ and $\Gamma \vdash \varphi, \Delta$, then $\Gamma \vdash \Delta$.

L is *consistent* if $\varnothing \nvdash \varnothing$. Let $\mathbf{CPC}_m = \langle F_{\mathbf{CPC}}, \vDash_{\mathbf{CPC}} \rangle$ denote the maximal structural multiple-conclusion consequence relation for classical propositional logic: $\Gamma \vDash_{\mathbf{CPC}} \Delta$ iff there is no 0–1 assignment v such that $v(\varphi) = 1$ for all $\varphi \in \Gamma$ and $v(\psi) = 0$ for all $\psi \in \Delta$. (In other words, $\Gamma \vDash_{\mathbf{CPC}} \Delta$ iff there are finite subsets $\Gamma' \subseteq \Gamma$, $\Delta' \subseteq \Delta$ such that $\vdash_{\mathbf{CPC}} \bigwedge \Gamma' \to \bigvee \Delta'$.)

We generalize the notions of translations, conservative translations, propositional logics, and most general translations to the multiple-conclusion setting in the obvious way.

Theorem 2.6 *For every finitary consistent multiple-conclusion logic $L = \langle F, \vdash \rangle$ over a countable set of formulas F, there exists a conservative most general translation $f\colon L \to_c \mathbf{CPC}_m$.*

If $\vdash$ is decidable, then f is computable. In general, f is Turing equivalent to $\vdash$.

Proof: Let $F = \{\alpha_n : n \in \omega\}$ be a (not necessarily injective) enumeration. We will define a sequence of formulas $f(\alpha_n) := \beta_n \in F_{\mathbf{CPC}}$ by induction on n. We denote by p_n the nth propositional variable of $\mathbf{CPC}$. We abbreviate $\alpha_X = \{\alpha_i : i \in X\}$, and similarly for β_X; moreover, we are going to use the identity $n = \{i \in \omega : i < n\}$.

Assume by the induction hypothesis that β_i have been already defined for all $i < n$ in such a way that

$$\alpha_X \vdash \alpha_Y \;\Rightarrow\; \beta_X \vDash_{\mathbf{CPC}} \beta_Y \tag{1}$$

for every $X, Y \subseteq n$. Define β_n by

$$\begin{aligned}
\gamma_n &:= \bigvee_{\substack{X,Y\subseteq n\\ \alpha_X \vdash \alpha_n, \alpha_Y}} \Big(\bigwedge \beta_X \wedge \neg \bigvee \beta_Y\Big),\\
\delta_n &:= \bigwedge_{\substack{X,Y\subseteq n\\ \alpha_X, \alpha_n \vdash \alpha_Y}} \Big(\bigwedge \beta_X \to \bigvee \beta_Y\Big),\\
\beta_n &:= \gamma_n \vee p_n \wedge \delta_n.
\end{aligned}$$

Notice that $\vDash_{\mathbf{CPC}} \gamma_n \to \delta_n$: if $\alpha_X \vdash \alpha_n, \alpha_Y$ and $\alpha_W, \alpha_n \vdash \alpha_Z$, then $\alpha_X, \alpha_W \vdash \alpha_Y, \alpha_Z$, hence

$$\vDash_{\mathbf{CPC}} \bigwedge \beta_X \wedge \bigwedge \beta_W \to \bigvee \beta_Y \vee \bigvee \beta_Z$$

by (1), i.e.,

$$\vDash_{\mathbf{CPC}} \Big(\bigwedge \beta_X \wedge \neg \bigvee \beta_Y\Big) \to \Big(\bigwedge \beta_W \to \bigvee \beta_Z\Big).$$

It follows that it does not matter whether we read the definition of β_n as $\gamma_n \vee (p_n \wedge \delta_n)$ or $(\gamma_n \vee p_n) \wedge \delta_n$.

We claim that (1) holds for $X, Y \subseteq n+1$. If $X, Y \subseteq n$, this follows from the induction hypothesis. If $n \in X \cap Y$, then trivially $\beta_X \vDash_{\mathbf{CPC}} \beta_Y$. Assume that $\alpha_X \vdash \alpha_n, \alpha_Y$, where $X, Y \subseteq n$. Then

$$\bigwedge \beta_X \wedge \neg \bigvee \beta_Y \vDash_{\mathbf{CPC}} \gamma_n \vDash_{\mathbf{CPC}} \beta_n,$$

hence by reasoning in **CPC**,

$$\beta_X \vDash_{\mathbf{CPC}} \beta_Y, \beta_n.$$

The case $\alpha_X, \alpha_n \vdash \alpha_Y$ is handled similarly using the definition of δ_n.

Thus, f is well defined, and by (1) and finitarity of L, it is a translation of L to $\mathbf{CPC}_m$. In order to show that f is conservative, assume that $\alpha_W \nvdash \alpha_Z$, we need to prove $\beta_W \nvDash_{\mathbf{CPC}} \beta_Z$. Obviously, $W \cap Z = \varnothing$. By the cut rule (i.e., Definition 2.5 (iii), applied with $\Xi = F$), we may assume that $W \cup Z = \omega$. Let v be the valuation such that

$$v(p_n) = \begin{cases} 1, & n \in W, \\ 0, & n \in Z. \end{cases}$$

We will show $v(\beta_n) = v(p_n)$ by induction on n, which implies $\beta_W \nvDash_{\mathbf{CPC}} \beta_Z$.

Assume that $n \in W$. If $X, Y \subseteq n$ are such that $\alpha_X, \alpha_n \vdash \alpha_Y$, then we cannot have simultaneously $X \subseteq W$ and $Y \subseteq Z$. If $i \in X \smallsetminus W$, then $v(\beta_i) = 0$ by the induction hypothesis; similarly, if $i \in Y \smallsetminus Z$, then $v(\beta_i) = 1$. Thus, $v(\bigwedge \beta_X) = 0$ or $v(\bigvee \beta_Y) = 1$. Since X, Y were arbitrary, we obtain $v(\beta_n) = v(\delta_n) = 1$.

If $n \in Z$, we obtain $v(\beta_n) = v(\gamma_n) = 0$ by a similar argument.

Clearly, the explicit recursive definition of f can be realized by an algorithm with an oracle for $\vdash$. On the other hand, since f is a conservative translation into the decidable logic $\mathbf{CPC}_m$, the relation $\vdash$ is Turing reducible to f (or its graph, if we insist on oracles' being sets rather than functions).

It remains to show that f is a most general translation of L to $\mathbf{CPC}_m$. Let $g\colon L \to \mathbf{CPC}_m$, and let σ be the substitution defined by $\sigma(p_n) = g(\alpha_n)$. We have to establish

$$\vDash_{\mathbf{CPC}} g(\alpha_n) \leftrightarrow \sigma(\beta_n) \tag{2}$$

for every n, and we proceed by induction on n. The definition of β_n implies that (2) is equivalent to

$$\sigma(\gamma_n) \vDash_{\mathbf{CPC}} g(\alpha_n) \vDash_{\mathbf{CPC}} \sigma(\delta_n).$$

Using the definitions of γ_n, δ_n and the induction hypothesis, this is equivalent to

$$\bigvee_{\substack{X,Y \subseteq n \\ \alpha_X \vdash \alpha_n, \alpha_Y}} \Big(\bigwedge g(\alpha_X) \wedge \neg \bigvee g(\alpha_Y)\Big) \vDash_{\mathbf{CPC}} g(\alpha_n) \vDash_{\mathbf{CPC}} \bigwedge_{\substack{X,Y \subseteq n \\ \alpha_X, \alpha_n \vdash \alpha_Y}} \Big(\bigwedge g(\alpha_X) \to \bigvee g(\alpha_Y)\Big),$$

which in turn follows from the fact that g is a translation: for example, if $X, Y \subseteq n$ are such that $\alpha_X \vdash \alpha_n, \alpha_Y$, then $g(\alpha_X) \vDash_{\mathbf{CPC}} g(\alpha_n), g(\alpha_Y)$, hence $\bigwedge g(\alpha_X) \wedge \neg \bigvee g(\alpha_Y) \vDash_{\mathbf{CPC}} g(\alpha_n)$. □

Proof (*of Theorem* 2.4)*:* Let $L = \langle F, \vdash \rangle$ be a finitary logic over countable F, and define its conservative multiple-conclusion extension $L^m = \langle F, \vdash^m \rangle$ by

$$\Gamma \vdash^m \Delta \quad \text{iff} \quad \exists \psi \in \Delta \, \Gamma \vdash \psi.$$

Let $f \colon L^m \to_c \mathbf{CPC}^m$ be the conservative minimal translation from Theorem 2.6. Since $\vdash_{\mathbf{CPC}}$ is the single-conclusion fragment of $\vDash_{\mathbf{CPC}}$, $f \colon L \to_c \mathbf{CPC}$. Moreover, if $g \colon L \to \mathbf{CPC}$, then $g \colon L^m \to \mathbf{CPC}_m$, hence g is $\mathbf{CPC}$-equivalent to $\sigma \circ f$ for some substitution σ. $\square$

Remark 2.7 Even if $L = \mathbf{CPC}$, the translation f from Theorem 2.4 is not (equivalent to) the identity, since it has the additional property that $f(\Gamma) \vdash_{\mathbf{CPC}} \bigvee_{i<n} f(\varphi_i)$ implies $f(\Gamma) \vdash_{\mathbf{CPC}} f(\varphi_i)$ for some $i < n$.

3 Universal logics

From now on, we only consider single-conclusion logics.

Definition 3.1 A logic L_0 is *universal* if $L \leq_c L_0$ for every finitary logic L over countably many formulas.

(In all cases where we establish universality below, it is possible to construct an $f \colon L \to_c L_0$ Turing equivalent to $\vdash_L$, as in the case of $\mathbf{CPC}$. However, we decided not to include this condition in the definition.)

In the previous section, we proved that $\mathbf{CPC}$ is a universal logic. The main result of this section, Theorem 3.10, is a generalization of our construction to a large class of nonclassical logics in place of $\mathbf{CPC}$.

Before we get to the main result, we discuss some examples showing that the question of which logics $\mathbf{CPC}$ can be conservatively translated into is considerably more subtle than which logics can be conservatively translated into $\mathbf{CPC}$.

Corollary 3.2

(*i*) *If L_0 is universal and $L_0 \leq_c L_1$, then L_1 is universal.*

(*ii*) *L_0 is universal iff* $\mathbf{CPC} \leq_c L_0$.

Example 3.3 Let $A_3 = \langle \{0, *, 1\}, \wedge, \vee, 0, 1, \neg \rangle$ be the 3-element bounded lattice endowed with an operation $\neg 0 = 1$, $\neg 1 = 0$, $\neg * = *$. Recall that *Kleene's* 3*-valued logic* $\mathbf{K}$ (with truth constants) is the propositional logic using connectives $\wedge, \vee, \neg, \top, \bot$ whose consequence relation is defined by A_3 where 1 is the only designated value, and the paraconsistent *logic of paradox* $\mathbf{LP}$ is defined similarly but with both $1, *$ taken as designated. We have:

(i) $\mathbf{K}$ is universal.

(ii) $\mathbf{LP}$ is not universal.

(Notice that $\mathbf{K}$ without truth constants is trivially not universal, as it has no tautologies.)

Proof: (i): Let $f(\varphi)$ be a conjunctive normal form of φ, obeying the convention that no variable and its negation can appear simultaneously in a clause, and we use $\top, \bot$ for empty conjunctions and disjunctions, respectively. We claim that $f\colon \mathbf{CPC} \to_c \mathbf{K}$. Since $\mathbf{K} \subseteq \mathbf{CPC}$ and $\varphi \dashv\vdash_{\mathbf{CPC}} f(\varphi)$, we clearly have

$$f(\Gamma) \vdash_{\mathbf{K}} f(\varphi) \;\Rightarrow\; \Gamma \vdash_{\mathbf{CPC}} \varphi.$$

In order to show the converse implication, it suffices to prove that

$$\Gamma \vdash_{\mathbf{CPC}} \varphi \;\Rightarrow\; \Gamma \vdash_{\mathbf{K}} \varphi$$

holds whenever $\Gamma \cup \{\varphi\}$ is a set of clauses. Let v be a valuation in A_3 such that $v(\Gamma) = 1$ and $v(\varphi) \neq 1$. We modify v to make it a Boolean valuation v' as follows. If $l \in \varphi$ is a literal such that $v(l) = *$, we put $v'(l) = 0$; we can do this for all such l simultaneously since φ does not contain both l and $\neg l$. If p is a variable such that $v(p) = *$ and neither p nor $\neg p$ appears in φ, we pick $v'(p) \in \{0, 1\}$ arbitrarily. After this modification, v' is a classical valuation such that $v'(\varphi) = 0$, and since all literals with value 1 kept their value, we still have $v'(\Gamma) = 1$.

(ii): Assume for contradiction $f\colon \mathbf{CPC} \to_c \mathbf{LP}$. Let $\{v_i : i < n\}$ be the list of all valuations in A_3 such that $v_i(f(\bot)) = 0$ and $v_i(p_j) = *$ for every variable p_j not occurring in $f(\bot)$. Put $\varphi_i = p_i$ for $i < n$, $\varphi_n = \neg \bigwedge_{i<n} p_i$. We have $\varphi_0, \dots, \varphi_n \vdash_{\mathbf{CPC}} \bot$, which implies $f(\varphi_0), \dots, f(\varphi_n) \vdash_{\mathbf{LP}} f(\bot)$. Since $v_i(f(\bot)) = 0$, we must have $v_i(f(\varphi_{j_i})) = 0$ for some $j_i \leq n$. Put $J = \{j_i : i < n\}$. We claim that

$$\{f(\varphi_j) : j \in J\} \vdash_{\mathbf{LP}} f(\bot).$$

Indeed, if $v(f(\bot)) = 0$, there exists an i such that v and v_i coincide on variables occurring in $f(\bot)$. We have $v_i(f(\varphi_j)) = 0$ for some $j \in J$. If $\prec$ is the partial order induced by $* \prec 0$, $* \prec 1$, then functions definable in A_3 are $\prec$-monotone, and $v_i \preceq v$, hence $v(f(\varphi_j)) = 0$. Thus, by the conservativity of f,

$$\{\varphi_j : j \in J\} \vdash_{\mathbf{CPC}} \bot \text{ for some } |J| \leq n.$$

This contradicts the definition of $\varphi_0, \dots, \varphi_n$. $\square$

In order to get some insight which logics can or cannot be expected to be universal depending on their available list of connectives, we characterize universal fragments of classical logic below.

Definition 3.4 If L is a propositional logic, and B a set of connectives definable in L (i.e., L-formulas), we denote by $L{\restriction_B}$ the fragment of L using only formulas built from B.

A *clone* on a set X is a set of finitary operations on X which is closed under composition and contains all projections. If B is a set of operations on X, then we denote by $[B]$ the clone generated by B. Notice that if B is a set of Boolean functions, then definable functions in $\mathbf{CPC}{\restriction_B}$ are exactly the functions from $[B]$, hence clones on $\{0, 1\}$ are in 1–1 correspondence with fragments of $\mathbf{CPC}$ considered up to term equivalence.

The lattice of clones on $\{0, 1\}$ was completely described by Post [9] (see also Lau [7] for a modern exposition). We will in particular need to refer to the following clones:

- The clone P_0 of all 0-preserving functions (i.e., $f(0,\dots,0)=0$).
- The clone D of all self-dual functions (i.e., $f(\neg x_1,\dots,\neg x_n)=\neg f(x_1,\dots,x_n)$).
- The clone A of all affine functions ($f(x_1,\dots,x_n)=\sum_{i\in I} x_i + c$, where $c\in\{0,1\}$, $I\subseteq\{1,\dots,n\}$, and $+$ denotes addition modulo 2).
- The clone M of all monotone functions.
- The clone T_1^∞ of functions bounded below by a variable (there exists i such that $x_i\le f(x_1,\dots,x_n)$ for every $\vec{x}\in\{0,1\}^n$).

The following lemma follows immediately from inspection of Post's lattice, though we invite the reader to give a direct proof:

Lemma 3.5 *If B is a set of Boolean functions, then $\to\notin[B]$ if and only if B is included in P_0, D, A, or M.* □

Theorem 3.6 *Let B be a set of Boolean functions. The fragment $\mathbf{CPC}{\restriction_B}$ is universal if and only if $\to$ is definable from B.*

Proof: Left-to-right: if $\to\notin[B]$, then B is included in one of the clones mentioned in Lemma 3.5. If $B\subseteq P_0$ or $B\subseteq D$, then $\top\notin[B]$; in other words, $\mathbf{CPC}{\restriction_B}$ has no tautologies, and therefore cannot be universal.

Let $B\subseteq A$. We claim that if $\varphi,\psi\in A$, then $\varphi\le\psi$ only if $\varphi=0$ or $\psi=1$ or $\varphi=\psi$. Write $\varphi(\vec{x})=\sum_{i\in I}x_i+c$, $\psi(\vec{x})=\sum_{i\in J}x_i+d$. If $\psi\ne1$, there is a Boolean valuation v such that $v(\psi)=0$. If $I\nsubseteq J$, we can change the valuation of any x_i such that $i\in I\smallsetminus J$ to make $v(\varphi)=1$, contradicting $\varphi\le\psi$. Thus, $\psi=1$ or $I\subseteq J$. Since $\varphi\le\psi$ implies $\neg\psi\le\neg\varphi$, the same argument gives $\varphi=0$ or $J\subseteq I$. Finally, if $I=J$, then $\psi=\varphi$ or $\psi=\neg\varphi$; in the latter case, $\varphi\le\psi$ can only hold if φ and ψ are constant 0 and 1, respectively.

In particular, there is no strictly increasing chain of length more than 3 of affine functions ordered by entailment, hence we cannot conservatively translate $\mathbf{CPC}$ (even with just 2 variables) into $\mathbf{CPC}{\restriction_B}$.

If $B\subseteq M$, we will show $\mathbf{CPC}{\restriction_B}\le_c\mathbf{LP}$, hence $\mathbf{CPC}{\restriction_B}$ is not universal by Example 3.3. Since $M=[\wedge,\vee,\top,\bot]$, we may assume $B=\{\wedge,\vee,\top,\bot\}$. Let σ be the substitution such that $\sigma(p)=p\wedge\neg p$. We claim

$$\Gamma\vdash_{\mathbf{CPC}\restriction_B}\varphi\ \Leftrightarrow\ \sigma(\Gamma)\vdash_{\mathbf{LP}}\sigma(\varphi).$$

Notice that σ, being a substitution, is a bounded lattice homomorphism of the respective free algebras. Let A_2 denote the 2-element bounded lattice. The mapping $\pi\colon A_3\to A_2$ such that $\pi(1)=\pi(*)=1$, $\pi(0)=0$, is also a bounded lattice homomorphism, and it preserves (in both directions) the sets of designated elements.

If v is a valuation in A_3 such that $v(\sigma(\Gamma))\ge *$, $v(\sigma(\varphi))=0$, then $v'=\pi\circ v\circ\sigma$ is a valuation in A_2 such that $v'(\Gamma)=1$ and $v'(\varphi)=0$, hence $\Gamma\nvdash_{\mathbf{CPC}}\varphi$.

Conversely, if v' is a valuation in A_2 such that $v'(\Gamma) = 1$ and $v'(\varphi) = 0$, let v be the valuation in A_3 induced by

$$v(p_i) = \begin{cases} * & v'(p_i) = 1, \\ 0 & v'(p_i) = 0. \end{cases}$$

Then $\pi \circ v \circ \sigma = v'$, hence $v(\sigma(\Gamma)) \geq *$, $v(\sigma(\varphi)) = 0$.

Right-to-left: we construct $f\colon \mathbf{CPC} \to \mathbf{CPC}{\restriction_{\to}}$ as follows. First, we rename all propositional variables in the style of Hilbert's hotel so that we obtain a spare variable q which does not occur in any formulas. Then, for each formula φ not containing q, let $f(\varphi)$ be an implicational formula equivalent to $\varphi \vee q$; it exists as $[\to] = T_1^\infty$. (For a more explicit construction, we can use the functional completeness of $\{\to, \bot\}$ to write $\varphi(\vec{p}) \leftrightarrow \psi(\vec{p}, \bot)$ for some $\psi \in [\to]$, and then put $f(\varphi) = (\psi(\vec{p}, q) \to q) \to q$.) It is easy to see that $f\colon \mathbf{CPC} \to_c \mathbf{CPC}{\restriction_{\to}}$. $\square$

Remark 3.7 Note in particular that the affine fragments $\mathbf{CPC}{\restriction_{\leftrightarrow}}$ or $\mathbf{CPC}{\restriction_{\leftrightarrow,\neg}}$ are strongly regularly finitely algebraizable, i.e., as nice as it can get from the general point of view. On the other hand, Kleene's logic is not even equivalential. This shows that universality does not have much to do with abstract algebraic properties of the logic. Consequently, if we want to establish universality of a class of logics, we cannot rely only on their general properties, at some point we have to resort to working with particular systems. We at least try to pick as weak a base system as possible so that our result covers a broad class of logics including most systems studied in the literature.

Definition 3.8 A *residuated lattice* is a structure $\langle L, \wedge, \vee, \cdot, \to, \leftarrow, 1\rangle$ where $\langle L, \wedge, \vee\rangle$ is a lattice, $\langle L, \cdot, 1\rangle$ is a monoid, and

$$b \leq a \to c \iff a \cdot b \leq c \iff a \leq c \leftarrow b$$

for every $a, b, c \in L$. (In particular, $a \cdot (a \to b) \leq b$, $(b \leftarrow a) \cdot a \leq b$). An *FL-algebra* is a residuated lattice L with a distinguished point $0 \in L$. The *full Lambek calculus* **FL** is the propositional logic using connectives $\wedge, \vee, \cdot, \to, \leftarrow, 1, 0$ such that $\vdash_{\mathbf{FL}}$ is complete with respect to the class of logical matrices whose underlying algebras are FL-algebras L, with $\{x \in L : x \geq 1\}$ taken as the set of designated elements. $\mathbf{FL_e}$ is complete with respect to commutative FL-algebras ($x \cdot y = y \cdot x$), and $\mathbf{FL_{ew}}$ with respect to 0-bounded integral ($0 \leq x \leq 1$) commutative FL-algebras. In a sequent calculus formulation of **FL**, **e** corresponds to the exchange rule, and **w** to the weakening rule. For more information about **FL** and its extensions or fragments, we refer the reader to [6].

If $\Gamma = \langle \varphi_1, \ldots, \varphi_k\rangle$ is a sequence of formulas, we define

$$\Gamma \to \psi := \varphi_1 \to (\varphi_2 \to (\varphi_3 \to \cdots (\varphi_k \to \psi) \cdots)),$$
$$\psi \leftarrow \Gamma := (\cdots ((\psi \leftarrow \varphi_1) \leftarrow \varphi_2) \cdots \leftarrow \varphi_{k-1}) \leftarrow \varphi_k.$$

If $k = 0$, it is understood that $\Gamma \to \psi = \psi \leftarrow \Gamma = \psi$. We also put $\prod \Gamma = \varphi_1 \cdot \varphi_2 \cdot \ldots \cdot \varphi_k$ ($\prod \Gamma = 1$ if $k = 0$), and $\varphi^k = \underbrace{\varphi \cdot \varphi \cdot \ldots \cdot \varphi}_{k \text{ times}}$. Notice that in **FL**, $\Gamma \to \psi$ is equivalent to $\prod \Gamma^{-1} \to \psi$, and $\psi \leftarrow \Gamma$ is equivalent to $\psi \leftarrow \prod \Gamma^{-1}$, where Γ^{-1} denotes the reversal of the sequence Γ.

Remark 3.9 Let L be a finitary logic over countably many formulas $F = \{\alpha_n : n < \omega\}$. From the proofs of Theorems 2.4 and 2.6 we know that there is a conservative translation $f\colon L \to_c \mathbf{CPC}$ such that $f(\alpha_n) = \beta_n$ is inductively defined to be equivalent to the formula

$$\bigwedge_{\substack{X \subseteq n > k \\ \alpha_X, \alpha_n \vdash_L \alpha_k}} (\beta_X \to \beta_k) \wedge \Bigl(p_n \vee \bigvee_{\substack{Z \subseteq n \\ \alpha_Z \vdash_L \alpha_n}} \bigwedge \beta_Z\Bigr). \tag{3}$$

Theorem 3.10 *A logic is universal whenever it conservatively extends a logic L_0 such that*

(*i*) $\mathbf{FL}\restriction_{\to,\leftarrow,\wedge} \subseteq L_0 \subseteq \mathbf{CPC}\restriction_{\to,\leftarrow,\wedge}$, *or*

(*ii*) $\mathbf{FL_e}\restriction_{\to,\wedge} \subseteq L_0 \subseteq \mathbf{CPC}\restriction_{\to,\wedge}$, *or*

(*iii*) $\mathbf{FL_{ew}}\restriction_{\to} = \mathbf{BCK} \subseteq L_0 \subseteq \mathbf{CPC}\restriction_{\to}$.

Proof: (i): Put $\pi(p,q) = (p \to q) \to q$. Using the notation from Definition 3.8, Remark 3.9, and from the proof of Theorem 2.6, we put $f(\alpha_n) = \beta_n$, where we define inductively

$$\beta_n := (q \to q) \wedge \bigwedge_{\alpha_X, \alpha_Y, \alpha_n \vdash_L \alpha_k} ((\beta_Y \to \beta_k) \leftarrow \beta_X) \wedge \Bigl(\Bigl(\bigwedge_{\alpha_Z \vdash_L \alpha_n} (\pi(p_n, q) \leftarrow \beta_Z)\Bigr) \to \pi(p_n, q)\Bigr).$$

The first big conjunction in β_n is taken over all $k < n$ and all *repetition-free* disjoint sequences X and Y consisting of elements $i < n$ such that $\alpha_X, \alpha_Y, \alpha_n \vdash_L \alpha_k$, and similarly for the second conjunction. If there are no $Z \subseteq n$ such that $\alpha_Z \vdash_L \alpha_n$, then the last conjunct of β_n is understood to be just $\pi(p_n, q)$.

Since $\beta_n(q/\bot)$ is classically equivalent to (3), we obtain immediately

$$f(\Gamma) \vdash_{\mathbf{CPC}} f(\varphi) \;\Rightarrow\; \Gamma \vdash_L \varphi.$$

In order to show

$$\Gamma \vdash_L \varphi \;\Rightarrow\; f(\Gamma) \vdash_{\mathbf{FL}} f(\varphi),$$

it suffices to prove by induction on n that for every $k < n$ and every sequence Z of elements of n,

$$\alpha_Z \vdash_L \alpha_k \;\Rightarrow\; \vdash_{\mathbf{FL}} \beta_Z \to \beta_k. \tag{4}$$

The statement is vacuously true for $n = 0$. Assume that it holds for n, we will prove it for $n + 1$.

Claim 1 *Let $\langle L, \wedge, \vee, \cdot, \to, \leftarrow, 1\rangle$ be a residuated lattice, and $u \in L$.*

(*i*) $L_u := \{a \in L : au, ua \le a\}$ *is closed under* $\to, \leftarrow, \wedge$ (*as well as* $\cdot, \vee$, *but we will not need this*).

(*ii*) *If $u = q \to q$ for some $q \in L$, then $1 \le u$, $u^2 \le u$, and L_u contains u as well as all elements of the form $\pi(a, q)$.*

(iii) **FL** *proves* $\beta_i \cdot \beta_j \to \beta_i$, $\beta_j \cdot \beta_i \to \beta_i$, *and* $\beta_X \to (q \to q)$.

Proof: (i): Let $a, b \in L_u$.

We have $u(a \wedge b) \leq ua \leq a$ and $u(a \wedge b) \leq ub \leq b$, hence $u(a \wedge b) \leq a \wedge b$. The proof of $(a \wedge b)u \leq a \wedge b$ is symmetric.

Since $a(a \to b)u \leq bu \leq b$, we have $(a \to b)u \leq a \to b$. Similarly, $au(a \to b) \leq a(a \to b) \leq b$, hence $u(a \to b) \leq a \to b$.

The case of $a \leftarrow b$ is symmetric.

(ii): $1 \leq u$ is clear, and $u^2 \leq u$ (which implies $u \in L_u$) is a special case of

$$(a \to b)(b \to c) \leq a \to c. \tag{5}$$

Put $p = (a \to q) \to q$. We have $p(q \to q) \leq p$ from (5). Also, $(a \to q)(q \to q)p \leq (a \to q)p \leq q$, hence $(q \to q)p \leq p$.

(iii): Consider a valuation v in a residuated lattice L, and put $u = v(q) \to v(q)$. By (i) and (ii), $v(\beta_i) \in L_u$, and clearly $v(\beta_j) \leq u$, hence $v(\beta_i)v(\beta_j) \leq v(\beta_i)$ and $v(\beta_j)v(\beta_i) \leq v(\beta_i)$. Finally, $v(\prod \beta_{X^{-1}}) \leq u^{|X|} \leq u$ by (ii). $\square$ (Claim 1)

If follows from the Claim that it is enough to prove (4) for repetition-free sequences Z not containing k. The only interesting cases are those involving n: i.e., $Z = X \frown n \frown Y$ or $k = n$.

Assume that $\alpha_X, \alpha_n, \alpha_Y \vdash_L \alpha_k$. Then the definition of β_n ensures $\vdash_{\mathbf{FL}} \beta_n \to ((\beta_Y \to \beta_k) \leftarrow \beta_X)$, hence $\vdash_{\mathbf{FL}} \beta_n \cdot \prod \beta_{X^{-1}} \to (\beta_Y \to \beta_k)$, which in turn gives $\vdash_{\mathbf{FL}} \prod \beta_{X^{-1}} \to (\beta_n \to (\beta_Y \to \beta_k))$ and $\vdash_{\mathbf{FL}} \beta_X \to (\beta_n \to (\beta_Y \to \beta_k))$.

Assume that $\alpha_Z \vdash_L \alpha_n$. We have $\vdash_{\mathbf{FL}} \beta_Z \to (q \to q)$ by the Claim. Whenever $\alpha_X, \alpha_n, \alpha_Y \vdash_L \alpha_k$, we have $\alpha_X, \alpha_Z, \alpha_Y \vdash_L \alpha_k$ by cut, hence $\vdash_{\mathbf{FL}} \beta_X \to (\beta_Z \to (\beta_Y \to \beta_k))$ by the induction hypothesis. By a similar argument as above, this is equivalent to $\vdash_{\mathbf{FL}} \beta_Z \to ((\beta_Y \to \beta_k) \leftarrow \beta_X)$. Finally, that β_Z implies the last conjunct of β_n follows from $\vdash_{\mathbf{FL}} \beta_Z \to ((\pi(p_n, q) \leftarrow \beta_Z) \to \pi(p_n, q))$.

(ii) follows immediately from (i), as $(\varphi \leftarrow \psi) = (\psi \to \varphi)$ in $\mathbf{FL_e}$.

(iii): We define inductively

$$\begin{aligned}
r_0 &:= 0, \\
r_{n+1} &:= 1 + n2^n r_n, \\
\varepsilon_n &:= \Big(\prod_{\alpha_X, \alpha_n \vdash_L \alpha_k} (\beta_X^{r_n} \to \beta_k) \Big) \cdot \Big(\Big(\prod_{\alpha_Z \vdash_L \alpha_n} (\beta_Z \to p_n) \Big) \to p_n \Big), \\
\beta_n &:= (\varepsilon_n \to q) \to q,
\end{aligned}$$

where $X, Z \subseteq n$, $k < n$. (We can treat X, Z like sets, as we have exchange.) Notice that fusion only appears in β_n in premises of implications, hence β_n can be equivalently rewritten as a formula $f(\alpha_n)$ using only $\to$.

Since $\beta_n(q/\bot)$ is classically equivalent to (3), we have

$$f(\Gamma) \vdash_{\mathbf{CPC}} f(\varphi) \;\Rightarrow\; \Gamma \vdash_L \varphi.$$

By induction on n, we will show that

$$\alpha_W \vdash_L \alpha_k \;\Rightarrow\; \vdash_{\mathbf{FL_{ew}}} \beta_W^{r_n} \to \beta_k \tag{6}$$

holds for every $k < n$ and every $W \subseteq n$. The statement is vacuously true for $n = 0$. Assume that it holds for n, we will prove it for $n+1$. Since we have weakening, it suffices to consider the cases $k = n$, $W \subseteq n$ and $k < n$, $W = X \cup \{n\}$, $X \subseteq n$.

Assume $\alpha_X, \alpha_n \vdash_L \alpha_k$. Using the definition and commutativity, we have $\vdash_{\mathbf{FL_{ew}}} \beta_X^{r_n} \to (\varepsilon_n \to \beta_k)$, which implies

$$\vdash_{\mathbf{FL_{ew}}} \beta_X^{r_n} \to ((\beta_k \to q) \to (\varepsilon_n \to q)).$$

The definition of β_k gives $\vdash_{\mathbf{FL_{ew}}} (\varepsilon_k \to q) \to (\beta_k \to q)$ using commutativity, hence

$$\vdash_{\mathbf{FL_{ew}}} \beta_X^{r_n} \to ((\varepsilon_k \to q) \to (\varepsilon_n \to q)).$$

This implies $\vdash_{\mathbf{FL_{ew}}} \beta_X^{r_n} \to (((\varepsilon_n \to q) \to q) \to ((\varepsilon_k \to q) \to q))$, i.e., $\vdash_{\mathbf{FL_{ew}}} \beta_X^{r_n} \to (\beta_n \to \beta_k)$. We obtain

$$\vdash_{\mathbf{FL_{ew}}} \beta_X^{r_{n+1}} \to (\beta_n^{r_{n+1}} \to \beta_k)$$

by weakening, using $r_n \leq r_{n+1}$.

Assume $\alpha_W \vdash_L \alpha_n$. We have $\vdash_{\mathbf{FL_{ew}}} \beta_W \to ((\beta_W \to p_n) \to p_n)$, hence

$$\vdash_{\mathbf{FL_{ew}}} \beta_W \to \Big(\Big(\prod_{\alpha_Z \vdash_L \alpha_n} (\beta_Z \to p_n)\Big) \to p_n\Big)$$

by weakening. Whenever $\alpha_X, \alpha_n \vdash_L \alpha_k$, we have $\alpha_X, \alpha_W \vdash_L \alpha_k$ by cut, hence

$$\vdash_{\mathbf{FL_{ew}}} \beta_W^{r_n} \to (\beta_X^{r_n} \to \beta_k)$$

by the induction hypothesis and weakening. Since there are at most $n2^n$ pairs $\langle X, k\rangle$ such that $X \subseteq n$, $k < n$, and $\alpha_X, \alpha_n \vdash_L \alpha_k$, we have

$$\vdash_{\mathbf{FL_{ew}}} \beta_W^{n2^n r_n} \to \prod_{\alpha_X, \alpha_n \vdash_L \alpha_k} (\beta_X^{r_n} \to \beta_k).$$

Putting the pieces together, we have $\vdash_{\mathbf{FL_{ew}}} \beta_W^{r_{n+1}} \to \varepsilon_n$, hence

$$\vdash_{\mathbf{FL_{ew}}} \beta_W^{r_{n+1}} \to \beta_n. \qquad \square$$

Remark 3.11 Every consistent substitution-invariant extension of **BCK** (in the same language) is contained in $\mathbf{CPC}{\restriction_{\to}}$. This is no longer true for $\mathbf{FL_e}{\restriction_{\to,\wedge}}$, nevertheless one can show that (i) and (ii) of Theorem 3.10 remain true when **CPC** is replaced with any consistent substitution-invariant extension of $\mathbf{FL}{\restriction_{\to,\leftarrow,\wedge}}$ or $\mathbf{FL_e}{\restriction_{\to,\wedge}}$, respectively.

4 Conclusion

Our results (Theorems 2.4 and 3.10 and Example 3.3) show that any countable finitary logic can be conservatively translated into (among others):

- Classical propositional logic.
- Intuitionistic, minimal and intermediate logics.
- Modal logics (classical or intuitionistic), including variants such as temporal or epistemic logics.
- Substructural logics, such as various extensions of **FL** or linear logic.
- Fuzzy and many-valued logics, such as **MTL**, **BL** and their extensions (e.g., Łukasiewicz logic).
- Relevant logics, such as **R**.
- Kleene's logic.
- First-order (or higher-order) extensions of the above logics.
- Implication fragments of many of the above logics.

This includes most of logical systems (fitting into the framework of Tarski-style consequence relations) studied in the literature on non-classical logic. We have also discovered some counterexamples, namely **CPC** cannot be conservatively translated into its monotone or affine fragments, or into the paraconsistent logic **LP**.

While there are still some loose ends left (most importantly, we were unable to determine whether the logic $\mathbf{BCI} = \mathbf{FL_e}{\restriction_\to}$ is universal, though it seems plausible), these results show that the mere existence of a conservative translation of one logic into another without further restrictions does not provide useful information on the relationship of the two logics, and a more refined criterion is needed to formalize the intuitive notion of translatability.

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