Generalized linear differential equations in a Banach space: Continuous dependence on a parameter

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Abstract

This paper deals with integral equations in a Banach space X of the form

$$x(t) = \widetilde{x} + \int_a^t \mathrm{d}[A]\, x + f(t) - f(a), \quad t \in [a,b],$$

where $-\infty < a < b < \infty$, $\widetilde{x} \in X$, $f : [a,b] \to X$ is regulated on $[a,b]$, and $A(t)$ is for each $t \in [a,b]$ a linear bounded operator on X, while the mapping $A\colon [a,b] \to L(X)$ has a bounded variation on $[a,b]$. Such equations are called generalized linear differential equations. Our aim is to present new results on the continuous dependence of solutions of such equations on a parameter. Furthermore, an application of these results to dynamic equations on time scales is given.

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1 Introduction

The theory of generalized differential equations in Banach spaces enables the investigation of continuous and discrete systems, including the equations on time scales and the functional differential equations with impulses, from the common standpoint. This fact can be observed in several papers related to special kinds of equations, such as e.g. those by Imaz and Vorel [14], Oliva and Vorel [25], Federson and Schwabik [7], Schwabik [27] or Slavík [33]. This paper is devoted to generalized linear differential equations of the form

$$x(t) = \widetilde{x} + \int_a^t \mathrm{d}[A]\, x + f(t) - f(a), \quad t \in [a,b], \tag{1.1}$$

in a Banach space X. A complete theory for the case when $X = \mathbb{R}^m$ can be found, for instance, in the monographs by Schwabik [27] or Schwabik, Tvrdý and Vejvoda [32]. See also the pioneering paper by Hildebrandt [12]. Concerning integral equations in a general Banach space, it is worth to highlight the monograph by Hönig [13] having as a background the interior (Dushnik) integral. On the other hand, dealing with the Kurzweil-Stieltjes integral, the contributions by Schwabik in [29] and [30] are essential for this paper.

In the special case when $X = \mathbb{R}^m$ and (1.1) reduces to ordinary differential equation, fundamental results on the continuous dependence of solutions on a parameter based on the averaging principle have been delivered by Krasnoselskii and Krejn [16], Kurzweil and Vorel [18], Kurzweil [19], Opial [26] and Kiguradze [15]. In particular, the problem of continuous dependence gave an inspiration to Kurzweil to introduce the notion of generalized differential equation in the papers [19] and [20]. For linear ordinary differential equations, the most general result seems to be that given by Opial. An interesting observation is contained in the fundamental paper by Artstein [2]. A different approach can be found in the papers [21]–[23] by Meng Gang and Zhang Meirong dealing also with measure differential analogues of Sturm-Liouville equations and, in particular, describing the weak and weak* continuous dependence of related Dirichlet or Neumann eigenvalues on a potential.

After Kurzweil, the problem of continuous dependence on a parameter for generalized differential equations has been treated by several authors, see e.g. Schwabik [27], Ashordia [3], Fraňková [8], Tvrdý [35], [36], Halas [9], Halas and Tvrdý [11]. Up to now, to our knowledge, only Federson and Schwabik [7] (cf. also Appendix to [1]) dealt with the case of a general Banach space X. Our aim is to prove new results valid also for infinite dimensional spaces. In particular, in Sections 3 and 4 we give sufficient conditions ensuring that the sequence $\{x_n\}$ of solutions of the generalized linear differential equations

$$x_n(t) = \widetilde{x}_n + \int_a^t \mathrm{d}[A_n]\, x_n + f_n(t) - f_n(a), \quad t \in [a,b], \quad n \in \mathbb{N},$$

tends to the solution x of (1.1). The crucial assumptions of Section 3 are the uniform boundedness of the variations $\operatorname{var}_a^b A_n$ of A_n and uniform convergence of A_n to A. In Section 4, we present the extension of the classical result by Opial to the case $X \neq \mathbb{R}^m$, where we do not require the uniform boundedness of $\operatorname{var}_a^b A_n$ and the uniform convergence is replaced by a properly stronger concept. Finally in Section 5, we apply the obtained results to dynamic equations on time scales.

2 Preliminaries

Throughout these notes X is a Banach space and $L(X)$ is the Banach space of bounded linear operators on X. By $\|\cdot\|_X$ we denote the norm in X. Similarly, $\|\cdot\|_{L(X)}$ denotes the usual operator norm in $L(X)$.

Assume that $-\infty < a < b < \infty$ and $[a,b]$ denotes the corresponding closed interval. A set $D = \{\alpha_0, \alpha_1, \ldots, \alpha_{\nu(D)}\} \subset [a,b]$ with $\nu(D) \in \mathbb{N}$ is said to be a division of $[a,b]$ if $a = \alpha_0 < \alpha_1 < \ldots < \alpha_{\nu(D)} = b$. The set of all divisions of $[a,b]$ is denoted by $\mathcal{D}[a,b]$.

A function $f : [a,b] \to X$ is called a finite step function on $[a,b]$ if there exists a division $D = \{\alpha_0, \alpha_1, \ldots, \alpha_{\nu(D)}\}$ of $[a,b]$ such that f is constant on every open interval (α_{j-1}, α_j), $j = 1, 2, \ldots, \nu(D)$.

For an arbitrary function $f : [a,b] \to X$ we set $\|f\|_\infty = \sup_{t \in [a,b]} \|f(t)\|_X$ and

$$\operatorname{var}_a^b f = \sup_{D \in \mathcal{D}[a,b]} \sum_{j=1}^{\nu(D)} \|f(\alpha_j) - f(\alpha_{j-1})\|_X$$

is the variation of f over $[a,b]$. If $\operatorname{var}_a^b f < \infty$ we say that f is a function *of bounded variation* on $[a,b]$. $BV([a,b],X)$ denotes the set of functions $f : [a,b] \to X$ of bounded variation on $[a,b]$. equipped with the norm $\|f\|_{BV} = \|f(a)\|_X + \operatorname{var}_a^b f$.

Given $f : [a,b] \to X$, the function f is called *regulated* on $[a,b]$ if, for each $t \in [a,b)$ there is $f(t+) \in X$ such that $\lim_{s \to t+} \|f(s) - f(t+)\|_X = 0$ and for each $t \in (a,b]$ there is $f(t-) \in X$ such that $\lim_{s \to t-} \|f(s) - f(t-)\|_X = 0$. By $G([a,b],X)$ we denote the set of all regulated functions $f : [a,b] \to X$. For $t \in [a,b)$, $s \in (a,b]$ we put $\Delta^+ f(t) = f(t+) - f(t)$ and $\Delta^- f(s) = f(s) - f(s-)$. Recall that $BV([a,b],X) \subset G([a,b],X)$ cf. e.g. [29, 1.5]. Moreover, it is known that regulated function are uniform limits of finite step functions (see [13, Theorem I.3.1]) and that they can have at most a countable number of points of discontinuity (see [13, Corollary 3.2.b]).

In what follows, by an integral we mean the Kurzweil-Stieltjes integral. Let us recall its definition. As usual, a *partition* of $[a,b]$ is a tagged system, i.e., a couple $P = (D, \xi)$ where $D = \{\alpha_0, \alpha_1, \ldots, \alpha_{\nu(D)}\} \in \mathcal{D}[a,b]$, $\xi = (\xi_1, \ldots, \xi_{\nu(D)}) \in [a,b]^{\nu(D)}$ and $\alpha_{j-1} \le \xi_j \le \alpha_j$ holds for $j = 1, 2, \ldots, \nu(D)$. Furthermore, any positive function $\delta : [a,b] \to (0, \infty)$ is called a *gauge* on $[a,b]$. Given a gauge δ, the partition P is called δ*-fine* if $[\alpha_{j-1}, \alpha_j] \subset (\xi_j - \delta(\xi_j), \xi_j + \delta(\xi_j))$ holds for all $j = 1, 2, \ldots, \nu(D)$. We remark that for an arbitrary gauge δ on $[a,b]$ there always exists a δ-fine partition of $[a,b]$. It is stated by the Cousin lemma (see e.g. [27, Lemma 1.4]).

For given functions $F\colon[a,b]\to L(X)$ and $g\colon[a,b]\to X$ and a partition $P=(D,\xi)$ of $[a,b]$, where $D=\{\alpha_0,\alpha_1,\dots,\alpha_{\nu(D)}\}$, $\xi=(\xi_1,\dots,\xi_{\nu(D)})$, we define

$$S(\mathrm{d}F,g,P)=\sum_{j=1}^{\nu(D)}[F(\alpha_j)-F(\alpha_{j-1})]\,g(\xi_j)\,.$$

We say that $I\in X$ is the Kurzweil-Stieltjes integral (or shortly KS-integral) of g with respect to F on $[a,b]$ and denote $I=\displaystyle\int_a^b \mathrm{d}[F]\,g$ if for every $\varepsilon>0$ there exists a gauge δ on $[a,b]$ such that

$$\Big\|S(\mathrm{d}F,g,P)-I\Big\|_X<\varepsilon \quad \text{for all} \quad \delta-\text{fine partitions} \quad P \text{ of } \ [a,b]\,.$$

Analogously, we define the integral $\displaystyle\int_a^b F\,\mathrm{d}[g]$ using sums of the form

$$S(F,\mathrm{d}g,P)=\sum_{j=1}^{\nu(D)}F(\xi_j)\,[g(\alpha_j)-g(\alpha_{j-1})]\,.$$

For the reader's convenience some of the further properties of the KS-integral needed later are summarized in the following proposition.

2.1 . Proposition. *Let $F\colon[a,b]\to L(X)$ and $g\colon[a,b]\to X$.*

(i) *If $F\in BV([a,b],L(X))$ and $g\in G([a,b],X)$, then $\int_a^b \mathrm{d}[F]\,g$ exists and*

$$\Big\|\int_a^b \mathrm{d}[F]\,g\Big\|_X\le\int_a^b \mathrm{d}[\mathrm{var}_a^t F]\,\|g\|_X\le(\mathrm{var}_a^b F)\,\|g\|_\infty\,. \tag{2.1}$$

(ii) *If $F\in G([a,b],L(X))$ and $g\in BV([a,b],X)$, then $\int_a^b \mathrm{d}[F]\,g$ exists and*

$$\Big\|\int_a^b \mathrm{d}[F]\,g\Big\|_X\le 2\,\|F\|_\infty\,\|g\|_{BV}.$$

(iii) *If $F\in BV([a,b],L(X))$ and $g\in G([a,b],X)$ then both the integrals $\int_a^b F\,\mathrm{d}[g]$ and $\int_a^b \mathrm{d}[F]\,g$ exist, the sum $\sum_{a\le\tau<b}\Delta^+F(\tau)\,\Delta^+g(\tau)-\sum_{a<\tau\le b}\Delta^-F(\tau)\,\Delta^-g(\tau)$ converges in X and*

$$\int_a^b F\,\mathrm{d}[g]+\int_a^b \mathrm{d}[F]\,g$$
$$=F(b)\,g(b)-F(a)\,g(a)-\sum_{a\le t<b}\Delta^+F(t)\,\Delta^+g(t)+\sum_{a<t\le b}\Delta^-F(t)\,\Delta^-g(t)\,.$$

(iv) *If* $F\in BV([a,b],L(X))$ *and* g *is bounded on* $[a,b]$ *are such that the integral* $\displaystyle\int_a^b \mathrm{d}[F]\,g$ *exists, then both the integrals* $\displaystyle\int_a^b H(t)\,\mathrm{d}_t\Big[\int_a^t \mathrm{d}[F]\,g\Big]$ *and* $\displaystyle\int_a^b H\,\mathrm{d}[F]\,g$ *exist and the equality*

$$\int_a^b H(t)\,\mathrm{d}_t\Big[\int_a^t \mathrm{d}[F]\,g\Big] = \int_a^b H\,\mathrm{d}[F]\,g$$

holds for each $H\in G([a,b],L(X))$.

Proof. Let $F\in BV([a,b],L(X))$ and $g\in G([a,b],X)$. Then the integral $\int_a^b \mathrm{d}[F]\,g$ exists by e.g. [28, Proposition 15]. The estimate (2.1) follows directly from the definition of the KS-integral, as

$$\|S(\mathrm{d}F,g,P)\|_X \le \sum_{j=1}^{\nu(D)} \big(\mathrm{var}_{\alpha_{j-1}}^{\alpha_j} F\big)\,\|g(\xi_j)\|_X \le \big(\mathrm{var}_a^b F\big)\,\|g\|_\infty$$

for all partitions $P=(D,\xi)$, $D=\{\alpha_0,\alpha_1,\dots,\alpha_{\nu(D)}\}$, $\xi=(\xi_1,\xi_2,\dots,\xi_{\nu(D)})$. This proves the assertion (i). The assertion (ii) holds by [24, Lemma 2.2], (iii) follows from [24, Corollary 3.6] and (iv) from [24, Theorem 3.8]. □

In addition, we need the following convergence result.

2.2. Theorem. *Let* $g,\,g_n\in G([a,b],X)$, $F,F_n\in BV([a,b],L(X))$ *for* $n\in\mathbb{N}$. *Assume that*

$$\lim_{n\to\infty}\|g_n-g\|_\infty=0,\qquad \lim_{n\to\infty}\|F_n-F\|_\infty=0\quad\text{and}\quad \varphi^*:=\sup_{n\in\mathbb{N}}\mathrm{var}_a^b F_n<\infty.$$

Then

$$\lim_{n\to\infty}\Big(\sup_{t\in[a,b]}\Big\|\int_a^t \mathrm{d}[F_n]\,g_n-\int_a^t \mathrm{d}[F]\,g\Big\|_X\Big)=0\,. \tag{2.2}$$

Proof. Let $\varepsilon>0$ be given. By [13, Theorem I.3.1], we can choose a finite step function $\widetilde{g}:[a,b]\to X$ such that $\|g-\widetilde{g}\|_\infty<\varepsilon$. Furthermore, let $n_0\in\mathbb{N}$ be such that

$$\|g_n-g\|_\infty<\varepsilon\quad\text{and}\quad\|F_n-F\|_\infty<\varepsilon\quad\text{for } n\ge n_0.$$

For a fixed $t\in[a,b]$, by Proposition 2.1 (i) and (ii), we obtain for $n\ge n_0$

$$\begin{aligned}&\Big\|\int_a^t \mathrm{d}[F_n]\,g_n-\int_a^t \mathrm{d}[F]\,g\Big\|_X\\ &\quad\le\Big\|\int_a^t \mathrm{d}[F_n]\,(g_n-\widetilde{g})\Big\|_X+\Big\|\int_a^t \mathrm{d}[F_n-F]\,\widetilde{g}\Big\|_X+\Big\|\int_a^t \mathrm{d}[F]\,(\widetilde{g}-g)\Big\|_X\\ &\quad\le\big(2\,\varphi^*+2\,\|\widetilde{g}\|_{BV}+\mathrm{var}_a^b F\big)\,\varepsilon=K\,\varepsilon,\end{aligned}$$

where $K=\big(2\,\varphi^*+2\,\|\widetilde{g}\|_{BV}+\mathrm{var}_a^b F\big)\in(0,\infty)$ does not depend on n. This proves (2.2). □

2.3. Remark. In the case that X is a Hilbert space, Theorem 2.2 has been already given by Krejčí and Laurençot [17, Proposition 3.1] or Brokate and Krejčí [6, Proposition 1.10].

3 Uniformly bounded variations

Given $A \in BV([a,b],L(X))$, $f \in G([a,b],X)$ and $\widetilde{x} \in X$, consider the integral equation

$$x(t) = \widetilde{x} + \int_a^t \mathrm{d}[A]\,x + f(t) - f(a)\,, \quad t \in [a,b]\,. \tag{3.1}$$

A function $x\colon [a,b] \to X$ is called a solution of (3.1) on $[a,b]$ if the integral $\int_a^b \mathrm{d}[A]\,x$ exists and x satisfies the equality (3.1) for each $t \in [a,b]$.

For our purposes the following property is crucial

$$\big[I - \Delta^- A(t)\big]^{-1} \in L(X) \quad \text{for all} \quad t \in (a,b]\,. \tag{3.2}$$

In particular, taking into account the closing remark in [29] we can see that the following result is a particular case of [29, Proposition 2.10].

3.1 . Proposition. *Let $A \in BV([a,b],L(X))$ satisfy (3.2) Then, for every $\widetilde{x} \in X$ and every $f \in G([a,b],X)$, the equation (3.1) possesses a unique solution x on $[a,b]$ and $x \in G([a,b],X)$.*

In addition, the following two important auxiliary assertions are true:

3.2. Lemma. *Let $A \in BV([a,b],L(X))$ satisfy (3.2), $f \in G([a,b],X)$ and $\widetilde{x} \in X$ and let x be the corresponding solution of (3.1) on $[a,b]$. Then*

$$\operatorname{var}_a^b(x-f) \le (\operatorname{var}_a^b A)\,\|x\|_\infty < \infty\,, \tag{3.3}$$

$$0 < c_A := \sup_{t \in (a,b]} \big\|[I - \Delta^- A(t)]^{-1}\big\|_{L(X)} < \infty \tag{3.4}$$

and

$$\|x(t)\|_X \le c_A\,(\|\widetilde{x}\|_X + \|f(a)\|_X + \|f\|_\infty)\exp\,(c_A \operatorname{var}_a^t A) \quad \textit{for } t \in [a,b]\,. \tag{3.5}$$

Proof. i) Let $D = \{\alpha_0, \alpha_1, \dots, \alpha_{\nu(D)}\}$ be a division of $[a,b]$. Then

$$\sum_{j=1}^{\nu(D)} \Big\|x(\alpha_j) - f(\alpha_j) - x(\alpha_{j-1}) + f(\alpha_{j-1})\Big\|_X$$
$$= \sum_{j=1}^{\nu(D)} \Big\|\int_{\alpha_{j-1}}^{\alpha_j} \mathrm{d}[A]\,x\Big\|_X \le \sum_{j=1}^{\nu(D)} \big[(\operatorname{var}_{\alpha_{j-1}}^{\alpha_j} A)\,\|x\|_\infty\big] = (\operatorname{var}_a^b A)\,\|x\|_\infty < \infty,$$

i.e. (3.3) is true.
ii) For $t \in (a,b]$ such that $\|\Delta^- A(t)\|_{L(X)} < \frac{1}{2}$ we have

$$\big\|[I - \Delta^- A(t)]^{-1}\big\|_{L(X)} \le \frac{1}{1 - \|\Delta^- A(t)\|_{L(X)}} < 2$$

(cf. e.g. [34, Lemma 4.1-C]). Therefore, $0 \le c_A < \infty$ due to the fact that the set

$$\{t \in [a,b];\, \|\Delta^- A(t)\|_{L(X)} \ge \tfrac{1}{2}\}$$

has at most finitely many elements. As $c_A = 0$ is impossible, this proves (3.4).
iii) Now, let x be a solution of (3.1). Put $B(a) = A(a)$ and $B(t) = A(t-)$ for $t \in (a,b]$. Then, by [29, Corollary 2.6] and [29, Proposition 2.7], we get $A - B \in BV([a,b], L(X))$, $\mathrm{var}_a^b B \le \mathrm{var}_a^b A$, $A(t) - B(t) = \Delta^- A(t)$, and $\int_a^t \mathrm{d}[A-B]\,x = \Delta^- A(t)\,x(t)$ for $t \in (a,b]$. Consequently

$$[I - \Delta^- A(t)]\,x(t) = \widetilde{x} + \int_a^t \mathrm{d}[B]\,x + f(t) - f(a) \quad \text{for } t \in (a,b]$$

and (cf. Proposition 2.1 (i))

$$\|x(t)\|_X \le K_1 + K_2 \int_a^t \mathrm{d}[h]\,\|x\|_X \quad \text{for } \ t \in [a,b],$$

where

$$K_1 = c_A\,(\|\widetilde{x}\|_X + \|f(a)\|_X + \|f\|_\infty)\,, \quad K_2 = c_A \quad \text{and} \quad h(t) = \mathrm{var}_a^t\,B.$$

The function h is nondecreasing and, since B is left-continuous on $(a,b]$, h is also left-continuous on $(a,b]$. Therefore we can use the generalized Gronwall inequality (see e.g. [32, Lemma I.4.30] or [27, Corollary 1.43]) to get the estimate (3.5). □

3.3. Lemma. *Let* $A,\ A_n \in BV([a,b], L(X)), n \in \mathbb{N}$, *be such that* (3.2) *and*

$$\lim_{n\to\infty} \|A_n - A\|_\infty = 0 \tag{3.6}$$

are satisfied. Then

$$\big[I - \Delta^- A_n(t)\big]^{-1} \in L(X) \tag{3.7}$$

for all $t \in (a,b]$ *and all* $n \in \mathbb{N}$ *sufficiently large. Moreover, there is* $\mu^* \in (0,\infty)$ *such that*

$$c_{A_n} := \sup_{t\,\in\,(a,b]} \big\|[I - \Delta^- A_n(t)]^{-1}\big\|_{L(X)} \le \mu^* \tag{3.8}$$

for all $n \in \mathbb{N}$ *sufficiently large.*

Proof. Notice that, since $A \in BV([a,b], L(X))$, the set $D := \{t \in (a,b];\, \|\Delta^- A(t)\|_{L(X)} \ge \frac{1}{4}\}$ has at most a finite number of elements. Let c_A be defined as in (3.4). Then, as by (3.6) $\lim\limits_{n\to\infty} \|\Delta^- A_n - \Delta^- A\|_\infty = 0$, there is $n_0 \in \mathbb{N}$ such that

$$\|\Delta^- A_n(t) - \Delta^- A(t)\|_{L(X)} < \tfrac{1}{4}\,\min\{1, \tfrac{1}{c_A}\} \quad \text{for } t \in [a,b] \quad \text{and } n \ge n_0. \tag{3.9}$$

Thus, $\|\Delta^- A_n(t)\|_{L(X)} \le \|\Delta^- A(t)\|_{L(X)} + \|\Delta^- A_n(t) - \Delta^- A(t)\|_{L(X)} < \frac{1}{2}$ for $t \in [a,b] \setminus D$ and $n \ge n_0$. By [34, Lemma 4.1-C], this implies that

$$[I - \Delta^- A_n(t)] \text{ is invertible and } \|[I - \Delta^- A_n(t)]^{-1}\|_{L(X)} < 2 \text{ for } t \in [a,b] \setminus D \text{ and } n \ge n_0.$$

Furthermore, due to (3.2), the relation

$$I - \Delta^- A_n(t) = [I - \Delta^- A(t)]\,\big[I - [I - \Delta^- A(t)]^{-1}\,(\Delta^- A_n(t) - \Delta^- A(t))\big] \tag{3.10}$$

holds for all $t\in[a,b]$ and $n\in\mathbb{N}$. Denote $T_n(t):=[I-\Delta^- A(t)]^{-1}(\Delta^- A_n(t)-\Delta^- A(t))$ for $n\in\mathbb{N}$ and $t\in[a,b]$. Then (3.10) means that, $I-\Delta^- A_n(t)$ is invertible if and only if $I-T_n(t)$ is invertible.

Now, let $t\in D$ and $n\ge n_0$ be given. Then, due to (3.4) and (3.9), we have $\|T_n(t)\|_{L(X)}<\frac{1}{4}$. Consequently, by [34, Lemma 4.1-C], $I-T_n(t)$ and therefore also $[I-\Delta^- A_n(t)]$ are invertible. Moreover, taking into account (3.4) and (3.10), we can see that $\|[I-\Delta^- A_n(t)]^{-1}\|_{L(X)}<2\,c_A$.

To summarize, there exists $n_0\in\mathbb{N}$ such that

$$[I-\Delta^- A_n(t)] \text{ is invertible and } \|[I-\Delta^- A_n(t)]^{-1}\|_{L(X)}\le\mu^*=2\,\max\{1,c_A\}$$

for all $t\in(a,b]$ and $n\ge n_0$. This completes the proof. □

The main result of this section is the following Theorem, which generalizes in a linear case the recent results by Federson and Schwabik [7]) and covers the results known for generalized linear differential equations in the case $X=\mathbb{R}^m$. Unlike [3], to prove it we do not utilize the variation-of-constants formula. Therefore it is not necessary to assume the additional condition $[I+\Delta^+ A(t)]^{-1}\in L(X)$ for $t\in[a,b]$.

3.4. Theorem. *Let $A,\ A_n\in BV([a,b],L(X))$, $f,\ f_n\in G([a,b],X)$, $\widetilde{x},\ \widetilde{x}_n\in X$ for $n\in\mathbb{N}$. Furthermore, let A satisfy (3.2), (3.6),*

$$\alpha^*:=\sup_{n\in\mathbb{N}}\Big(\operatorname{var}_a^b A_n\Big)<\infty\,, \tag{3.11}$$

$$\lim_{n\to\infty}\|f_n-f\|_\infty=0\,, \tag{3.12}$$

$$\lim_{n\to\infty}\|\widetilde{x}_n-\widetilde{x}\|_X=0\,. \tag{3.13}$$

Then equation (3.1) has a unique solution x on $[a,b]$. Furthermore, for each $n\in\mathbb{N}$ large enough there is a unique solution x_n on $[a,b]$ to the equation

$$x_n(t)=\widetilde{x}_n+\int_a^t \mathrm{d}[A_n]\,x_n+f_n(t)-f_n(a)\,,\quad t\in[a,b] \tag{3.14}$$

and

$$\lim_{n\to\infty}\|x_n-x\|_\infty=0\,. \tag{3.15}$$

Proof. Due to (3.2) equation (3.1) has a unique solution x on $[a,b]$. Furthermore, by Lemma 3.2, there is $n_0\in\mathbb{N}$ such that (3.7) is true for $n\ge n_0$. Hence, for each $n\ge n_0$, equation (3.14) possesses a unique solution x_n on $[a,b]$. Set

$$w_n=(x_n-f_n)-(x-f) \tag{3.16}$$

Then

$$w_n(t)=\widetilde{w}_n+\int_a^t \mathrm{d}[A_n]\,w_n+h_n(t)-h_n(a)\quad\text{for}\quad n\in\mathbb{N}\quad\text{and}\quad t\in[a,b],$$

where $\widetilde{w}_n=(\widetilde{x}_n-f_n(a))-(\widetilde{x}-f(a))$ and

$$h_n(t)=\int_a^t \mathrm{d}[A_n-A]\,(x-f)+\Big(\int_a^t \mathrm{d}[A_n]\,f_n-\int_a^t \mathrm{d}[A]\,f\Big).$$

First, notice that according to (3.12) and (3.13) we have

$$\lim_{n\to\infty} \|\widetilde{w}_n\|_X = 0. \tag{3.17}$$

Furthermore, in view of Theorem 2.2, we have

$$\lim_{n\to\infty} \Big\| \int_a^t \mathrm{d}[A_n]\, f_n - \int_a^t \mathrm{d}[A]\, f \Big\|_X = 0.$$

Moreover, since $(x-f)\in BV([a,b],X)$ by (3.3), we get by Proposition 2.1 (ii)

$$\Big\| \int_a^t \mathrm{d}[A_n - A]\,(x-f) \Big\|_X \le 2\, \|A_n - A\|_\infty\, \|x-f\|_{BV} \quad \text{for all} \quad t\in[a,b].$$

Having in mind (3.6), we can see that the relation

$$\lim_{n\to\infty} \Big\| \int_a^t \mathrm{d}[A_n - A]\,(x-f) \Big\|_X = 0$$

holds. To summarize,

$$\lim_{n\to\infty} \|h_n\|_\infty = 0. \tag{3.18}$$

By (3.11) and by Lemmas 3.2 and 3.3 we have

$$\|w_n(t)\|_X \le \mu^*\, (\|\widetilde{w}_n\|_X + \|h_n\|_\infty)\, \exp\,(\mu^*\, \alpha^*) \quad \text{for} \quad t\in[a,b].$$

Consequently, using (3.17) and (3.18) we deduce that $\lim_{n\to\infty} \|w_n\|_X = 0$. Finally, by (3.12) and (3.16), we conclude that (3.15) is true. □

We will close this section by a comparison of Theorem 3.4 with two similar available results: Proposition 8.3 in [1] (see also [27, Theorem 8.2] where $\dim X < \infty$) and Theorem 8.8 from [27]. We will use the usual notation

$$B_r := \{x \in X : \|x\|_X \le r\} \quad \text{for} \quad r>0.$$

The former result can be for the linear case reformulated as follows.

3.5. Theorem. *Let $A, A_n : [a,b] \to L(X)$, $f, f_n : [a,b] \to X$, $\widetilde{x}_n, \widetilde{x} \in X$ for $n\in\mathbb{N}$ and let $r>1$. Further, let (3.13) and*

$$\lim_{n\to\infty} \|A_n(t) - A(t)\|_{L(X)} = 0, \quad \lim_{n\to\infty} \|f_n(t) - f(t)\|_X = 0 \quad \textit{for} \quad t\in[a,b] \tag{3.19}$$

hold. Moreover, let a nondecreasing function $h : [a,b] \to \mathbb{R}$ and a continuous increasing function $\omega : [0,\infty) \to \mathbb{R}$ be given such that $\omega(0) = 0$,

$$\|[A(t_2) - A(t_1)]\, x + f(t_2) - f(t_1)\|_X \le |h(t_2) - h(t_1)|, \tag{3.20}$$

$$\|[A_n(t_2) - A_n(t_1)]\, x + f_n(t_2) - f_n(t_1)\|_X \le |h(t_2) - h(t_1)|, \tag{3.21}$$

$$\|[A(t_2) - A(t_1)]\, (y-x)\|_X \le \omega(\|y-x\|_X)\, |h(t_2) - h(t_1)|, \tag{3.22}$$

$$\|[A_n(t_2) - A_n(t_1)]\, (y-x)\|_X \le \omega(\|y-x\|_X)\, |h(t_2) - h(t_1)| \tag{3.23}$$

$$\textit{for } t_1, t_2 \in [a,b],\ x, y \in B_r \textit{ and } n\in\mathbb{N}.$$

Finally, let x_n be solutions of (3.14) for $n\in\mathbb{N}$ and let $x\colon[a,b]\to X$ be such that

$$\lim_{n\to\infty}\|x_n(t)-x(t)\|_X=0 \quad \text{and} \quad x(t)\in B_r \text{ for } t\in[a,b].$$

Then $x\in BV([a,b],X)$ and it is a solution of (3.1) on $[a,b]$.

Similarly, when restricted to the linear case, Theorem 8.8 from [27] reduces to

3.6 . Theorem. *Let $X=\mathbb{R}^m$ for some $m\in\mathbb{N}$. Let $A,\ A_n:[a,b]\to L(X)$, $f,\ f_n:[a,b]\to X$, $\widetilde{x}_n,\widetilde{x}\in X$ for $n\in\mathbb{N}$ and $r>1$. Let (3.2), (3.13), (3.19), (3.20) and (3.22) hold, where $h{:}[a,b]\to\mathbb{R}$ is nondecreasing and continuous on $[a,b]$ and $\omega:[0,\infty)\to\mathbb{R}$ is continuous increasing and such that $\omega(0)=0$. Further, assume*

$$\|[A_n(t_2)-A_n(t_1)]\,x+f_n(t_2)-f_n(t_1)\|_X\le|h_n(t_2)-h_n(t_1)|, \tag{3.24}$$

$$\|[A_n(t_2)-A_n(t_1)]\,(y-x)\|_X\le\omega(\|y-x\|_X)\,|h_n(t_2)-h_n(t_1)|, \tag{3.25}$$

$$\text{for } t_1,t_2\in[a,b],\ x,y\in B_r \text{ and } n\in\mathbb{N},$$

where $h_n:[a,b]\to\mathbb{R}$, $n\in\mathbb{N}$, are nondecreasing and continuous from the left on $(a,b]$ and such that

$$\limsup_{n\to\infty}\,[h_n(t_2)-h_n(t_1)]\le h(t_2)-h(t_1) \quad \text{if } a\le t_1\le t_2\le b. \tag{3.26}$$

If x is the solution of (3.1) then, for any $n\in\mathbb{N}$ sufficiently large, equation (3.14) has a unique solution x_n on $[a,b]$ and (3.15) holds.

3.7 . Remark. Notice that the proof of Theorem 3.6 as given in [27] cannot be extended to the case of a general Banach space X since it relies on the Helly's Choice Theorem.

3.8 . Proposition. *Let $A,A_n:[a,b]\to L(X)$, $f,f_n:[a,b]\to X$, $\widetilde{x}_n,\widetilde{x}\in X$ for $n\in\mathbb{N}$ and let $h:[a,b]\to\mathbb{R}$ and $\omega\colon[0,\infty)\to\mathbb{R}$ be as in Theorem 3.5. Furthermore, let (3.19), (3.20)–(3.23) hold.*

Then $A_n,A\in BV([a,b],L(X))$, $f_n,f\in BV([a,b],X)$ for $n\in\mathbb{N}$ and the relations (3.6), (3.11) and (3.12) are satisfied.

Proof. First of all, note that, inserting $x=0$ into (3.20)–(3.23), we get

$$\left\{\begin{aligned} \|f_n(t_2)-f_n(t_1)\|_X &\le|h(t_2)-h(t_1)|,\\ \|f(t_2)-f(t_1)\|_X &\le|h(t_2)-h(t_1)| \end{aligned}\right. \qquad \text{for } t_1,t_2\in[a,b],\ n\in\mathbb{N} \tag{3.27}$$

and

$$\left\{\begin{aligned} \|A_n(t_2)-A_n(t_1)\|_{L(X)} &\le\omega(1)\,|h(t_2)-h(t_1)|,\\ \|A(t_2)-A(t_1)\|_{L(X)} &\le\omega(1)\,|h(t_2)-h(t_1)| \end{aligned}\right. \qquad \text{for } t_1,t_2\in[a,b],\ n\in\mathbb{N}. \tag{3.28}$$

i) The relation (3.11) follows immediately from (3.28). In particular, $A_n\in BV([a,b],L(X))$ for $n\in\mathbb{N}$. Similarly, $A\in BV([a,b],L(X))$ and, by (3.27), $f,f_n\in BV([a,b],X)$ for $n\in\mathbb{N}$.

ii) Notice that (3.28) and (3.19) imply that

$$\begin{cases} \|A_n(t-)-A_n(s)\|_{L(X)} & \le \omega(1)\,|h(t-)-h(s)|, \\ \|A(t-)-A(s)\|_{L(X)} & \le \omega(1)\,|h(t-)-h(s)| \end{cases} \quad \text{for } \ t\in(a,b],\ s\in[a,b],\ n\in\mathbb{N} \tag{3.29}$$

and

$$\begin{cases} \|A_n(t+)-A_n(s)\|_{L(X)} & \le \omega(1)\,|h(t+)-h(s)|, \\ \|A(t+)-A(s)\|_{L(X)} & \le \omega(1)\,|h(t+)-h(s)| \end{cases} \quad \text{for } \ t\in[a,b),\ s\in[a,b],\ n\in\mathbb{N}. \tag{3.30}$$

iii) Let $\varepsilon>0$ and $t\in(a,b]$ be given and let us choose $s_0\in(a,t)$ and $n_0\in\mathbb{N}$ so that

$$|h(t-)-h(s_0)|<\frac{\varepsilon}{3\,\omega(1)} \quad \text{and} \quad \|A_n(s_0)-A(s_0)\|_{L(X)}<\frac{\varepsilon}{3} \quad \text{for } n\ge n_0. \tag{3.31}$$

Then, by (3.29) and (3.31),

$$\begin{aligned} &\|A_n(t-)-A(t-)\|_{L(X)} \\ &\quad \le \|A_n(t-)-A_n(s_0)\|_{L(X)}+\|A_n(s_0)-A(s_0)\|_{L(X)}+\|A(s_0)-A(t-)\|_{L(X)} \\ &\quad < \omega(1)\,|h(t-)-h(s_0)|+\frac{\varepsilon}{3}+\omega(1)\,|h(t-)-h(s_0)|<\varepsilon. \end{aligned}$$

This means that

$$\lim_{n\to\infty} A_n(t-)=A(t-) \quad \text{for } \ t\in(a,b]. \tag{3.32}$$

Similarly, using (3.30) we get

$$\lim_{n\to\infty} A_n(t+)=A(t+) \quad \text{for } \ t\in[a,b). \tag{3.33}$$

iv) Now, suppose that (3.6) is not valid. Then there is $\widetilde{\varepsilon}>0$ such that for any $\ell\in\mathbb{N}$ there exist $m_\ell\ge\ell$ and $t_\ell\in[a,b]$ such that

$$\|A_{m_\ell}(t_\ell)-A(t_\ell)\|_{L(X)}\ge\widetilde{\varepsilon}. \tag{3.34}$$

We may assume that $m_{\ell+1}>m_\ell$ for any $\ell\in\mathbb{N}$ and

$$\lim_{\ell\to\infty} t_\ell=t_0\in[a,b]. \tag{3.35}$$

Let $t_0\in(a,b]$ and assume that the set of those $\ell\in\mathbb{N}$ for which $t_\ell\in(a,t_0)$ has infinitely many elements, i.e. there is a sequence $\{\ell_k\}\subset\mathbb{N}$ such that $t_{\ell_k}\in(a,t_0)$ for all $k\in\mathbb{N}$ and $\lim_{k\to\infty} t_{\ell_k}=t_0$. Denote $s_k=t_{\ell_k}$ and $B_k=A_{m_{\ell_k}}$ for $k\in\mathbb{N}$. Then, in view of (3.34), we have

$$s_k\in(a,t_0) \quad \text{for } k\in\mathbb{N}, \quad \lim_{k\to\infty} s_k=t_0 \tag{3.36}$$

and

$$\|B_k(s_k)-A(s_k)\|_{L(X)}\ge\widetilde{\varepsilon} \quad \text{for } k\in\mathbb{N}. \tag{3.37}$$

By (3.29), we get

$$\|A(t_0-)-A(s_k)\|_{L(X)} \le \omega(1)\,[h(t_0-)-h(k_n)],$$

and

$$\|B_k(t_0-)-B_k(s_k)\|_{L(X)} \le \omega(1)\,[h(t_0-)-h(k_n)]$$

for $k \in \mathbb{N}$. Therefore, by (3.32) and since $\lim_{k\to\infty}(h(t_0-)-h(s_k))=0$ due to (3.36), we can choose $k_0 \in \mathbb{N}$ so that

$$\|B_{k_0}(t_0-)-A(t_0-)\|_{L(X)} < \frac{\widetilde{\varepsilon}}{3}, \quad \|A(t_0-)-A(s_{k_0})\|_{L(X)} \le \omega(1)\,[h(t_0-)-h(s_{k_0})] < \frac{\widetilde{\varepsilon}}{3}$$

and

$$\|B_{k_0}(t_0-)-B_{k_0}(s_{k_0})\|_{L(X)} < \frac{\widetilde{\varepsilon}}{3}\,.$$

As a consequence, we get finally by (3.37)

$$\begin{aligned}\widetilde{\varepsilon} &\le \|B_{k_0}(s_{k_0})-A(s_{k_0})\|_{L(X)}\\ &\le \|B_{k_0}(s_{k_0})-B_{k_0}(t_0-)\|_{L(X)}+\|B_{k_0}(t_0-)-A(t_0-)\|_{L(X)}+\|A(t_0-)-A(s_{k_0})\|_{L(X)} < \widetilde{\varepsilon},\end{aligned}$$

a contradiction.

If $t_0 \in [a,b)$ and the set of those $\ell \in \mathbb{N}$ for which $t_\ell \in (a,t_0)$ has only finitely many elements, then there is a sequence $\{\ell_k\} \subset \mathbb{N}$ such that $t_{\ell_k} \in (t_0,b)$ for all $k \in \mathbb{N}$ and $\lim_{k\to\infty} t_{\ell_k} = t_0$. As before, let $s_k = t_{\ell_k}$ and $B_k = A_{m_{\ell_k}}$ for $k \in \mathbb{N}$ and notice that $s_k \in (t_0,b)$ for $k \in \mathbb{N}$, $\lim_{k\to\infty} s_k = t_0$ and (3.37) are true. Arguing similarly as before we get that there is $k_0 \in \mathbb{N}$ such that

$$\begin{aligned}\widetilde{\varepsilon} &\le \|B_{k_0}(s_{k_0})-A(s_{k_0})\|_{L(X)}\\ &\le \|B_{k_0}(s_{k_0})-B_{k_0}(t_0+)\|_{L(X)}+\|B_{k_0}(t_0+)-A(t_0+)\|_{L(X)}+\|A(t_0+)-A(s_{k_0})\|_{L(X)} < \widetilde{\varepsilon},\end{aligned}$$

a contradiction. Thus, (3.6) is satisfied.

To obtain (3.12) we would use the inequalities in (3.27) and follow the steps (ii)-(iv). □

3.9 . Proposition. *Let $A, A_n : [a,b] \to L(X)$, $f, f_n : [a,b] \to X$, $\widetilde{x}_n, \widetilde{x} \in X$ for $n \in \mathbb{N}$ and let $h, h_n : [a,b] \to \mathbb{R}$ and $\omega : [0,\infty) \to \mathbb{R}$ be as in Theorem 3.6. Furthermore, let (3.19), (3.20), (3.22) and (3.24)–(3.26) hold.*

Then $A_n, A \in BV([a,b], L(X))$, $f_n, f \in BV([a,b], X)$ for $n \in \mathbb{N}$ and the relations (3.6), (3.11) and (3.12) are satisfied.

Proof. As in the proof of Proposition 3.8, using (3.20), (3.22), (3.24) and (3.25), we get

$$\left\{\begin{aligned}\|A_n(t_2)-A_n(t_1)\|_{L(X)} &\le \omega(1)\,|h_n(t_2)-h_n(t_1)|,\\ \|A(t_2)-A(t_1)\|_{L(X)} &\le \omega(1)\,|h(t_2)-h(t_1)|\end{aligned}\right. \quad \text{for } t_1,t_2 \in [a,b],\ n \in \mathbb{N}, \tag{3.38}$$

and

$$\begin{cases} \|f_n(t_2)-f_n(t_1)\|_X & \le |h_n(t_2)-h_n(t_1)|, \\ \|f(t_2)-f(t_1)\|_X & \le |h(t_2)-h(t_1)| \end{cases} \qquad \text{for } \ t_1,t_2\in[a,b], \ n\in\mathbb{N}. \tag{3.39}$$

By (3.26) there is $n_0\in\mathbb{N}$ such that $h_n(b)-h_n(a)\le h(b)-h(a)+1$ for $n\ge n_0$. Hence, in view of (3.38), for any $n\in\mathbb{N}$ we have

$$\operatorname{var}_a^b A_n \le \max\Big(\big\{\operatorname{var}_a^b A_n\,; n\le n_0\big\}\cup\big\{h(b)-h(a)+1\big\}\Big)<\infty.$$

This proves (3.11).

Suppose that (3.6) does not hold. Then there is $\widetilde{\varepsilon}>0$ such that for any $\ell\in\mathbb{N}$ there exist $m_\ell\ge\ell$ and $t_\ell\in[a,b]$ such that $m_{\ell+1}>m_\ell$ for $\ell\in\mathbb{N}$ and the relations (3.34) and (3.35) are true.

Let $t_0\in(a,b)$ and let an arbitrary $\varepsilon>0$ be given. Since h is continuous, we may choose $\eta>0$ in such a way that $t_0-\eta$, $t_0+\eta\in[a,b]$ and

$$h(t_0{+}\eta)-h(t_0{-}\eta)<\frac{\varepsilon}{\omega(1)}. \tag{3.40}$$

Furthermore, by (3.19) there is $\ell_1\in\mathbb{N}$ such that

$$\|A_{m_\ell}(t_0)-A(t_0)\|_{L(X)}<\varepsilon \quad \text{for all } \ \ell\ge\ell_1 \tag{3.41}$$

and by (3.26), (3.38) and (3.40) there is $\ell_2\in\mathbb{N}$, $\ell_2\ge\ell_1$, such that

$$\begin{cases} \|A_{m_\ell}(\tau_2)-A_{m_\ell}(\tau_1)\|_{L(X)}\le\omega(1)\,[h(t_0{+}\eta)-h(t_0{-}\eta)]+\varepsilon<2\,\varepsilon \\ \qquad\qquad \text{whenever } \ \tau_1,\tau_2\in(t_0{-}\eta,t_0{+}\eta) \text{ and } \ \ell\ge\ell_2\,. \end{cases} \tag{3.42}$$

The relations (3.19) and (3.42) imply immediately that

$$\begin{cases} \|A(\tau_2)-A(\tau_1)\|_{L(X)}=\lim\limits_{\ell\to\infty}\|A_{m_\ell}(\tau_2)-A_{m_\ell}(\tau_1)\|_{L(X)}\le 2\,\varepsilon \\ \qquad\qquad \text{whenever } \ \tau_1,\tau_2\in(t_0{-}\eta,t_0{+}\eta)\,. \end{cases} \tag{3.43}$$

Finally, let $\ell_3\in\mathbb{N}$ be such that $\ell_3\ge\ell_2$ and

$$|t_\ell-t_0|<\eta \quad \text{for all } \ \ell\ge\ell_3, \tag{3.44}$$

then in virtue of the relations (3.35), (3.40)–(3.44) we have

$$\begin{aligned} &\|A_{m_\ell}(t_\ell)-A(t_\ell)\|_{L(X)} \\ &\quad\le\|A_{m_\ell}(t_\ell)-A_{m_\ell}(t_0)\|_{L(X)}+\|A_{m_\ell}(t_0)-A(t_0)\|_{L(X)}+\|A(t_0)-A(t_\ell)\|_{L(X)}\le 5\,\varepsilon\,. \end{aligned}$$

Hence, choosing $\varepsilon<\frac{1}{5}\,\widetilde{\varepsilon}$ and making use of (3.34), we get $\widetilde{\varepsilon}>\|A_{m_\ell}(t_\ell)-A(t_\ell)\|_{L(X)}\ge\widetilde{\varepsilon}$, a contradiction. This proves that (3.6) is satisfied. The modification of the proof in the cases $t_0=a$ or $t_0=b$ is obvious.

Finally, by the same argument, using (3.39), we obtain (3.12). □

4 Variations bounded with a weight

The main result of this section deals with the homogeneous generalized linear differential equation

$$x(t) = \widetilde{x} + \int_a^t \mathrm{d}[A]\, x, \quad t \in [a,b], \tag{4.1}$$

where, as before, $A \in BV([a,b], L(X))$ and $\widetilde{x} \in X$. Further, it extends the result cf.[26, Theorem 1] obtained by Z. Opial for the case $X = \mathbb{R}^m$ for some $m \in \mathbb{N}$, and A, A_n absolutely continuous on $[a,b]$.

As in the previous section we will assume that the fundamental existence assumption (3.2) is satisfied. To our aim, we need the following estimate well known in the case $\dim X < \infty$.

4.1. Lemma. *If* $g \in BV([a,b], X)$, *then* $\displaystyle\sum_{t\in[a,b)} \|\Delta^+ g(t)\|_X + \sum_{t\in(a,b]} \|\Delta^- g(t)\|_X \le \operatorname{var}_a^b g\,.$

Proof. Let $\{s_k \in X; k \in \mathbb{N}\}$ be the set of points of discontinuity of g in (a,b), so we can write

$$\sum_{t\in[a,b)} \|\Delta^+ g(t)\|_X + \sum_{t\in(a,b]} \|\Delta^- g(t)\|_X = \lim_{n\to\infty} S_n\,,$$

where

$$S_n = \|\Delta^+ g(a)\|_X + \|\Delta^- g(b)\|_X + \sum_{k=1}^n \big[\|\Delta^- g(s_k)\|_X + \|\Delta^+ g(s_k)\|_X\big] \text{ for } n \in \mathbb{N}\,.$$

Let $\varepsilon > 0$ and $n \in \mathbb{N}$ be given and let $\{t_1, t_2, \dots, t_n\} \subset (a,b)$ be such that

$$\{t_1, t_2, \dots, t_n\} = \{s_1, s_2, \dots, s_n\} \quad \text{and} \quad a < t_1 < t_2 < \dots < t_n < b\,.$$

Then $S_n = \|\Delta^+ g(a)\|_X + \|\Delta^- g(b)\|_X + \displaystyle\sum_{k=1}^n \big[\|\Delta^- g(t_k)\|_X + \|\Delta^+ g(t_k)\|_X\big]$. Furthermore, for each $k = 1, 2, \dots, n$, choose $\delta_k > 0$ in such a way that

$$\|g(t_k+\delta_k) - g(t_k+)\|_X < \frac{\varepsilon}{4\,(n+1)}, \quad \|g(t_k-\delta_k) - g(t_k-)\|_X < \frac{\varepsilon}{4\,(n+1)}$$

and $[t_k - \delta_k, t_k + \delta_k] \cap \{t_1, t_2, \dots, t_n\} = \{t_k\}$. Analogously, let $\delta_0 > 0$ be such that

$$\|g(a+\delta_0) - g(a+)\|_X < \frac{\varepsilon}{4}, \quad \|g(b-) - g(b-\delta_0)\|_X < \frac{\varepsilon}{4}, \quad a+\delta_0 < t_1 \text{ and } b-\delta_0 > t_n\,.$$

It follows that

$$\begin{aligned} S_n \le\ & \Big(\|g(a+) - g(a+\delta_0)\|_X + \|g(a+\delta_0) - g(a)\|_X\Big) \\ & + \sum_{k=1}^n \|g(t_k+) - g(t_k+\delta_k)\|_X + \sum_{k=1}^n \|g(t_k+\delta_k) - g(t_k)\|_X \\ & + \sum_{k=1}^n \|g(t_k-) - g(t_k-\delta_k)\|_X + \sum_{k=1}^n \|g(t_k-\delta_k) - g(t_k)\|_X \\ & + \Big(\|g(b) - g(b-\delta_0)\|_X + \|g(b-\delta_0) - g(b-)\|_X\Big) \end{aligned}$$

$$< \frac{\varepsilon}{4} + \|g(a+\delta_0) - g(a)\|_X + \frac{n\,\varepsilon}{4\,(n+1)} + \sum_{k=1}^{n} \|g(t_k+\delta_k) - g(t_k)\|_X$$
$$+ \frac{n\,\varepsilon}{4\,(n+1)} + \sum_{k=1}^{n} \|g(t_k) - g(t_k-\delta_k)\|_X + \|g(b) - g(b-\delta_0)\|_X + \frac{\varepsilon}{4}$$

holds for any $n \in \mathbb{N}$. To summarize, for any $n \in \mathbb{N}$ we have

$$S_n < \varepsilon + \Big(\|g(a+\delta_0) - g(a)\|_X + \sum_{k=1}^{n} \|g(t_k+\delta_k) - g(t_k)\|_X\Big)$$
$$+ \Big(\sum_{k=1}^{n} \|g(t_k) - g(t_k-\delta_k)\|_X + \|g(b) - g(b-\delta_0)\|_X\Big).$$

Therefore $S_n \le \varepsilon + (\operatorname{var}_a^b g)$ for each $n \in \mathbb{N}$ and $\varepsilon > 0$. Thus, $S_n \le \operatorname{var}_a^b g$ for all $n \in \mathbb{N}$, wherefrom the desired estimate immediately follows. □

4.2. Theorem. *Let A, $A_n \in BV([a,b], L(X))$ and $\widetilde{x}$, $\widetilde{x}_n \in X$ for $n \in \mathbb{N}$. Assume (3.2), (3.13) and*

$$\lim_{n\to\infty} \|A_n - A\|_\infty \big(1 + \operatorname{var}_a^b A_n\big) = 0\,. \tag{4.2}$$

Then (4.1) has a unique solution x on $[a,b]$. Moreover, for each $n \in \mathbb{N}$ sufficiently large, the equation

$$x_n(t) = \widetilde{x}_n + \int_a^t \mathrm{d}[A_n]\, x_n\,, \quad t \in [a,b] \tag{4.3}$$

has a unique solution x_n on $[a,b]$ and (3.15) holds.

Proof. First, notice that, since

$$\|A_n - A\|_\infty \le \|A_n - A\|_\infty \big(1 + \operatorname{var}_a^b A_n\big) \quad \text{for all} \quad n \in \mathbb{N},$$

(4.2) implies (3.6). Therefore, by Lemma 3.3, there is $n_0 \in \mathbb{N}$ such that (3.7) holds for each $t \in (a,b]$ and each $n \ge n_0$.

Assume $n \ge n_0$. Let x and x_n be the solutions on $[a,b]$ of (4.1) and (4.3), respectively. Then

$$x_n(t) - x(t) = \widetilde{x}_n - \widetilde{x} + \int_a^t \mathrm{d}[A]\,(x_n - x) + h_n(t) - h_n(a) \quad \text{for} \quad t \in [a,b]\,, \tag{4.4}$$

where

$$h_n(t) = \int_a^t \mathrm{d}[A_n - A]\, x_n \quad \text{for} \quad t \in [a,b]\,. \tag{4.5}$$

By Lemma 3.2 we have

$$\|x_n - x\|_\infty \le c_A\,(\|\widetilde{x}_n - \widetilde{x}\|_X + \|h_n\|_\infty)\,\exp\,(c_A \operatorname{var}_a^b A). \tag{4.6}$$

Thus, in view of the assumption (3.13), to prove the assertion of the theorem, we have to show that $\lim_{n\to\infty} \|h_n\|_\infty = 0$.

To this aim, we integrate by parts (cf. Proposition 2.1 (iii)) in the right-hand side of (4.5) and use Substitution Formula (cf. Proposition 2.1 (iv)). Then we get

$$h_n(t)=[A_n(t)-A(t)]\,x_n(t)-[A_n(a)-A(a)]\,\widetilde{x}_n-\int_a^t (A_n-A)\,\mathrm{d}[A_n]\,x_n-\Delta_a^t(A_n-A,x_n) \tag{4.7}$$

for $t\in[a,b]$, where

$$\Delta_a^t(A_n-A,x_n)=\sum_{a\le s<t}[\Delta^+(A_n(s)-A(s))\,\Delta^+x_n(s)]-\sum_{a<s\le t}[\Delta^-(A_n(s)-A(s))\,\Delta^-x_n(s)]\,. \tag{4.8}$$

Inserting the relations (cf. [29, Proposition 2.3])

$$\Delta^+x_n(t)=\Delta^+A_n(t)\,x_n(t)\quad\text{for } t\in[a,b)\quad\text{and}\quad \Delta^-x_n(t)=\Delta^-A_n(t)\,x_n(t)\quad\text{for } t\in(a,b]$$

into the right-hand side of (4.8) and using Lemma 4.1, we obtain the estimates

$$\|\Delta_a^t(A_n-A,x_n)\|_X\le 2\,\|A_n-A\|_\infty\,(\operatorname{var}_a^t A_n)\,\|x_n\|_\infty\quad\text{for } t\in[a,b]\,.$$

Hence $\|h_n(t)\|_X\le\|A_n-A\|_\infty\,\big(2+3\,(\operatorname{var}_a^t A_n)\big)\,\|x_n\|_\infty$, that is,

$$\|h_n\|_\infty\le\alpha_n\,\|x_n\|_\infty, \tag{4.9}$$

where $\alpha_n=\|A_n-A\|_\infty\Big(2+3\operatorname{var}_a^b A_n\Big)$. Note that, due to (4.2), we have

$$\lim_{n\to\infty}\alpha_n=0. \tag{4.10}$$

We can see that to show that $\lim_{n\to\infty}\|h_n\|_\infty=0$, it is sufficient to prove that the sequence $\{\|x_n\|_\infty\}$ is bounded. By (4.6) and (4.9) we have

$$\|x_n\|_\infty\le\|x_n-x\|_\infty+\|x\|_\infty\le c_A\left(\|\widetilde{x}_n-\widetilde{x}\|_X+\alpha_n\|x_n\|_\infty\right)\exp\left(c_A\operatorname{var}_a^b A\right)+\|x\|_\infty\,.$$

Hence $\left(1-c_A\,\alpha_n\,\exp\left(c_A\operatorname{var}_a^b A\right)\right)\|x_n\|_\infty\le c_A\,\|\widetilde{x}_n-\widetilde{x}\|_X\,\exp\left(c_A\operatorname{var}_a^b A\right)+\|x\|_\infty$ for $n\ge n_0$. By (3.13) and (4.10), there is $n_1\ge n_0$ such that $\|\widetilde{x}_n-\widetilde{x}\|_X<1$ and $c_A\,\alpha_n\,\exp\left(c_A\operatorname{var}_a^b A\right)<\frac{1}{2}$ for $n\ge n_1$. In particular, $\|x_n\|_\infty<2\left(c_A\,\exp\left(c_A\operatorname{var}_a^b A\right)+\|x\|_\infty\right)$ for $n\ge n_1$, i.e. the sequence $\{\|x_n\|_\infty\}$ is bounded and this completes the proof. □

4.3. Remark. In comparison with Theorem 3.4, the uniform boundedness of variation (3.11) was not needed in Theorem 4.2. On the other hand, if (3.11) is assumed, Theorem 4.2 reduces to Theorem 3.4.

If $X=\mathbb{R}^m$ for some $m\in\mathbb{N}$ and $f,\ f_n\in BV([a,b],\mathbb{R}^m)$ for $n\in\mathbb{N}$, then Theorem 4.2 can be, similarly as in the ODE's case, extended to the nonhomogeneous equations (3.1) and (3.14). Indeed, let us define the $(m+1)\times(m+1)-$matrix valued function $B\colon[a,b]\to L(\mathbb{R}^{m+1})$ by

$$B(t)=\begin{pmatrix}A(t) & f(t)\\ 0 & 0\end{pmatrix}\quad\text{for}\quad t\in[a,b]\quad\text{and}\quad \widetilde{y}=\begin{pmatrix}\widetilde{x}\\ 1\end{pmatrix}.$$

Similarly, let

$$B_n(t) = \begin{pmatrix} A_n(t) & f_n(t) \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \widetilde{y}_n = \begin{pmatrix} \widetilde{x}_n \\ 1 \end{pmatrix} \quad \text{for} \quad t \in [a,b] \quad \text{and} \quad n \in \mathbb{N}.$$

It is easy to check that equations (3.1) and (3.14) are respectively equivalent to the equations

$$y(t) = \widetilde{y} + \int_a^t \mathrm{d}[B]\, y \tag{4.11}$$

and

$$y_n(t) = \widetilde{y}_n + \int_a^t \mathrm{d}[B_n]\, y_n, \ \ n \in \mathbb{N} \tag{4.12}$$

in the following sense: if x is a solution to (3.1) and $y(t) = \begin{pmatrix} x(t) \\ 1 \end{pmatrix}$, then y is a solution to (4.11). Conversely, if y is a solution to (4.11) and x is formed by its first m components then x is a solution to (3.1), where $\widetilde{x} \in \mathbb{R}^m$ is formed by the first m components of $\widetilde{y}$. An analogous relationship holds also between equations (3.14) and (4.12), of course. Having this in mind, we can see that the following assertion is true.

4.4. Corollary. *Let $m \in \mathbb{N}$, $A, A_n \in BV([a,b], L(\mathbb{R}^m))$, $f, f_n \in BV([a,b], \mathbb{R}^m)$, and $\widetilde{x}, \widetilde{x}_n \in \mathbb{R}^m$ for $n \in \mathbb{N}$. Assume (3.2), (4.2), (3.13) and*

$$\lim_{n\to\infty} \Big(\|f_n - f\|_\infty \big(1 + \operatorname{var}_a^b f_n\big)\Big) = 0. \tag{4.13}$$

Then equation (3.1) has a unique solution x on $[a,b]$ and, for each $n \in \mathbb{N}$ large enough there is a unique solution x_n on $[a,b]$ to the equation (3.14) and (3.15) is true.

5 Linear dynamic equations on time scales

The theory of dynamic equations on time scales has recently received a considerable attention since it can treat in a unified way both continuous and discrete problems. In this section we apply the continuous dependence results obtained in Sections 3 and 4 to linear dynamic equations on time scales.

Let us recall some preliminary definitions and notations (e.g. [4]). A time scale $\mathbb{T}$ is a nonempty closed subset of $\mathbb{R}$. Given $a, b \in \mathbb{T}$, by $[a,b]_{\mathbb{T}}$ we denote the compact interval in $\mathbb{T}$, that is, $[a,b]_{\mathbb{T}} = [a,b] \cap \mathbb{T}$. For each $t \in \mathbb{T}$, consider

$$\rho(t) := \sup\, [a,t) \cap \mathbb{T}\,, \quad \sigma(t) := \inf\, (t,b] \cap \mathbb{T} \quad \text{and} \quad \widetilde{\sigma}(t) := \inf\, [t,b] \cap \mathbb{T}\,.$$

If $\sigma(t) = t$ we say that t is *right-dense*, while if $\rho(t) = t$ then t is called *left-dense*. A function $f : [a,b]_{\mathbb{T}} \to \mathbb{R}^m$ is *rd-continuous* in $[a,b]_{\mathbb{T}}$ if f is continuous at every right-dense point of $[a,b]_{\mathbb{T}}$ and there exists $\lim\limits_{s\to t-} f(s)$ for every left-dense point $t \in [a,b]_{\mathbb{T}}$.

Consider the linear dynamic equation

$$y^{\Delta}(t) = P(t)\, y(t) + h(t)\,, \quad y(a) = \widetilde{y}\,, \quad t \in [a,b]_{\mathbb{T}}, \tag{5.1}$$

where $\widetilde{y} \in \mathbb{R}^m$, $P\colon [a,b]_{\mathbb{T}} \to L(\mathbb{R}^m)$ and $h\colon [a,b]_{\mathbb{T}} \to \mathbb{R}^m$ are rd-continuous in $[a,b]_{\mathbb{T}}$ and y^Δ stands for the Δ-derivative of y. The initial value problem (5.1) can be rewritten as a time scale integral equation

$$y(t) = \widetilde{y} + \int_a^t \big[P(s)\,y(s) + h(s)\big]\,\Delta s\,, \quad t \in [a,b]_{\mathbb{T}}\,,$$

where the integral is the Riemann Δ-integral defined e.g. in [5]. Slavík proved in [33] that this Δ-integral corresponds to a special case of the Kurzweil-Stieltjes integral. In addition, in [33] the relationship between dynamic equations on time scale and generalized differential equations is described. For the reader's convenience, we summarize the needed results from [33] in the following proposition.

5.1 . Proposition.

(i) [33, Theorem 5] *Let $f\colon [a,b]_{\mathbb{T}} \to \mathbb{R}^m$ be a rd-continuous function. Define*

$$F_1(t) = \int_a^t f(s)\,\Delta s \quad \textit{for } t \in [a,b]_{\mathbb{T}} \quad \textit{and} \quad F_2(t) = \int_a^t f(\widetilde{\sigma}(s))\,\mathrm{d}[\widetilde{\sigma}(s)] \quad \textit{for } t \in [a,b]\,.$$

Then $F_2 = F_1 \circ \widetilde{\sigma}$ on $[a,b]$.

(ii) [33, Theorem 12] *If $y\colon [a,b]_{\mathbb{T}} \to \mathbb{R}^m$ is a solution of (5.1) then $x = y \circ \widetilde{\sigma}$ is a solution of (3.1), where*

$$A(t) = \int_a^t P(\widetilde{\sigma}(s))\,\mathrm{d}[\widetilde{\sigma}(s)] \quad \textit{and} \quad f(t) = \int_a^t h(\widetilde{\sigma}(s))\,\mathrm{d}[\widetilde{\sigma}(s)] \quad \textit{for} \;\; t \in [a,b]\,. \tag{5.2}$$

Symmetrically, if $x\colon [a,b] \to \mathbb{R}^m$ is a solution of (3.1), then the function $y\colon [a,b]_{\mathbb{T}} \to \mathbb{R}^m$ defined by $y(t) = x(t)$ for $t \in [a,b]_{\mathbb{T}}$ is a solution of (5.1).

5.2 . Remark. Note that $\widetilde{\sigma} : [a,b] \to [a,b]_{\mathbb{T}} \subset [a,b]$ is monotone and left continuous on $(a,b]$. In particular, $\mathrm{var}_a^b\,\widetilde{\sigma} \le b - a$. In view of this, it is easy to check that the functions $A\colon [a,b] \to L(\mathbb{R}^m)$ and $f\colon [a,b] \to \mathbb{R}^m$ given by (5.2) are well-defined, left-continuous and have bounded variations on $[a,b]$.

The following theorem is the first main result of this section.

5.3 . Theorem. *Let $m \in \mathbb{N}$ and let P, $P_n\colon [a,b]_{\mathbb{T}} \to L(\mathbb{R}^m)$, h, $h_n\colon [a,b]_{\mathbb{T}} \to \mathbb{R}^m$ for $n \in \mathbb{N}$ be rd-continuous functions in $[a,b]_{\mathbb{T}}$ and let $\widetilde{y}$, $\widetilde{y}_n \in \mathbb{R}^m$, $n \in \mathbb{N}$, be given. Assume that*

$$\lim_{n\to\infty} \|\widetilde{y}_n - \widetilde{y}\|_{\mathbb{R}^m} = 0 \tag{5.3}$$

and that there is $M \in (0,\infty)$ such that

$$\sup_{t\in[a,b]_{\mathbb{T}}} \|P_n(t)\|_{L(\mathbb{R}^m)} \le M \quad \textit{for } n \in \mathbb{N}, \tag{5.4}$$

and

$$\begin{cases} \lim_{n\to\infty} \sup_{t\in[a,b]_{\mathbb{T}}} \Big\| \int_a^t (P_n(s)-P(s))\,\Delta s \Big\|_{L(\mathbb{R}^m)} = 0\,, \\ \lim_{n\to\infty} \sup_{t\in[a,b]_{\mathbb{T}}} \Big\| \int_a^t (h_n(s)-h(s))\,\Delta s \Big\|_{\mathbb{R}^m} = 0\,. \end{cases} \tag{5.5}$$

Then initial value problem (5.1) has a solution y, initial value problems

$$y_n^{\Delta}(t) = P_n(t)\,y_n(t) + h_n(t)\,, \quad y_n(a) = \widetilde{y}_n\,, \quad t\in[a,b]_{\mathbb{T}} \tag{5.6}$$

have solutions y_n for all $n\in\mathbb{N}$, and

$$\lim_{n\to\infty} \sup_{t\in[a,b]_{\mathbb{T}}} \|y_n(t)-y(t)\|_{\mathbb{R}^m} = 0\,. \tag{5.7}$$

Proof. Let $A\in BV([a,b],L(\mathbb{R}^m))$ and $f\in BV([a,b],\mathbb{R}^m)$ be given by (5.2). Furthermore, define

$$A_n(t) = \int_a^t P_n(\widetilde{\sigma}(s))\,\mathrm{d}[\widetilde{\sigma}(s)] \text{ and } f_n(t) = \int_a^t h_n(\widetilde{\sigma}(s))\,\mathrm{d}[\widetilde{\sigma}(s)] \text{ for } t\in[a,b] \text{ and } n\in\mathbb{N}\,. \tag{5.8}$$

Since A and all A_n, $n\in\mathbb{N}$, are left-continuous, equation (3.1) has a solution $x\in BV([a,b],\mathbb{R}^m)$ and equations (3.14) have solutions $x_n\in BV([a,b],\mathbb{R}^m)$ for each $n\in\mathbb{N}$. Furthermore, by Proposition 5.1 (i), we have

$$\|A_n-A\|_\infty = \sup_{t\in[a,b]} \Big\| \int_a^{\widetilde{\sigma}(t)} (P_n(s)-P(s))\,\Delta s \Big\|_{L(\mathbb{R}^m)} \le \sup_{t\in[a,b]_{\mathbb{T}}} \Big\| \int_a^t (P_n(s)-P(s))\,\Delta s \Big\|_{L(\mathbb{R}^m)}$$

for each $n\in\mathbb{N}$, that is,

$$\|A_n-A\|_\infty \le \sup_{t\in[a,b]_{\mathbb{T}}} \Big\| \int_a^t (P_n(s)-P(s))\,\Delta s \Big\|_{L(\mathbb{R}^m)}\,. \tag{5.9}$$

Analogously,

$$\|f_n-f\|_\infty \le \sup_{t\in[a,b]_{\mathbb{T}}} \Big\| \int_a^t (h_n(s)-h(s))\,\Delta s \Big\|_{\mathbb{R}^m}\,. \tag{5.10}$$

This, with respect to (5.5), means that the assumptions (3.6) and (3.12) of Theorem 3.4 are satisfied. Furthermore, if $a\le c<d\le b$, then

$$\begin{aligned} \|A_n(d)-A_n(c)\|_{L(\mathbb{R}^m)} &= \Big\| \int_c^d P_n(\widetilde{\sigma}(s))\,\mathrm{d}[\widetilde{\sigma}(s)] \Big\|_{L(\mathbb{R}^m)} \\ &\le \|P_n\circ\widetilde{\sigma}\|_\infty \left(\mathrm{var}_c^d\,\widetilde{\sigma}\right) = \Big(\sup_{t\in[a,b]_{\mathbb{T}}} \|P_n(t)\|_{L(\mathbb{R}^m)}\Big) \left(\mathrm{var}_a^b\,\widetilde{\sigma}\right) \end{aligned}$$

holds for each $n\in\mathbb{N}$, wherefrom, by (5.4) and Remark 5.2, the estimate

$$\mathrm{var}_a^b\,A_n \le \Big(\sup_{t\in[a,b]_{\mathbb{T}}} \|P_n(t)\|_{L(\mathbb{R}^m)}\Big) \left(\mathrm{var}_a^b\,\widetilde{\sigma}\right) \le M\,(b-a) \quad \text{for all } n\in\mathbb{N} \tag{5.11}$$

follows. Hence, the assumption (3.11) of Theorem 3.4 is satisfied, as well. Consequently, we can use Theorem 3.4 to prove that (3.15) holds.

By Proposition 5.1 (ii), the functions y, $y_n : [a,b]_{\mathbb{T}} \to \mathbb{R}^m$, $n \in \mathbb{N}$, obtained as the restriction of x and x_n to $[a,b]_{\mathbb{T}}$, respectively, are solutions to (5.1) and (5.6). Therefore, thanks to (3.15), (5.7) is also true, which completes the proof. □

5.4. Remark. Two results on the continuous dependence of solutions to nonlinear dynamic equations have been recently delivered by A. Slavík, cf. [33, Theorems 14 and 16]. To prove them, it was sufficient to apply Proposition 5.1 and Theorems 8.2 and 8.7 from [27]. So, with respect to our Propositions 3.8 and 3.9, we can see that the above Theorem 5.3 provides for the linear case more general result than both Theorem 14 and Theorem 16 in [33].

Making use of Corollary 4.4 we obtain the following assertion.

5.5 . Theorem. *Let $m \in \mathbb{N}$ and let P, P_n: $[a,b]_{\mathbb{T}} \to L(\mathbb{R}^m)$, h, h_n: $[a,b]_{\mathbb{T}} \to \mathbb{R}^m$ for $n \in \mathbb{N}$ be rd-continuous functions in $[a,b]_{\mathbb{T}}$ and let $\widetilde{y}$, $\widetilde{y}_n \in \mathbb{R}^m$, $n \in \mathbb{N}$, be given. Assume that* (5.3) *holds and*

$$\begin{cases} \lim\limits_{n\to\infty} \sup\limits_{t\in[a,b]_{\mathbb{T}}} \left\| \int_a^t (P_n(s) - P(s))\,\Delta s \right\|_{L(\mathbb{R}^m)} \big[1 + \sup\limits_{t\in[a,b]_{\mathbb{T}}} \|P_n(t)\|_{L(\mathbb{R}^m)}\big] = 0\,, \\ \lim\limits_{n\to\infty} \sup\limits_{t\in[a,b]_{\mathbb{T}}} \left\| \int_a^t (h_n(s) - h(s))\,\Delta s \right\|_{L(\mathbb{R}^m)} \big[1 + \sup\limits_{t\in[a,b]_{\mathbb{T}}} \|h_n(t)\|_{\mathbb{R}^m}\big] = 0\,. \end{cases} \tag{5.12}$$

Then equation (5.1) *has a solution y, equations* (5.6) *have solutions y_n for all $n \in \mathbb{N}$ and* (5.7) *holds.*

Proof. Let A_n, A, f_n, f be defined by (5.2) and (5.8). Recall that as $A \in BV([a,b], L(\mathbb{R}^m))$, $A_n \in BV([a,b], L(\mathbb{R}^m))$ for $n \in \mathbb{N}$ and A, A_n, $n \in \mathbb{N}$, are left-continuous on $(a,b]$ (cf. Remark 5.2), equation (3.1) has a solution $x \in BV([a,b], \mathbb{R}^m)$ and equations (3.14) have solutions $x_n \in BV([a,b], \mathbb{R}^m)$ for each $n \in \mathbb{N}$. Similarly as in the proof of Theorem 5.3, the estimates (5.9) and (5.10) are true. In addition,

$$\operatorname{var}_a^b A_n \le \Big(\sup_{t\in[a,b]_{\mathbb{T}}} \|P_n(t)\|_{L(\mathbb{R}^m)} \Big) (b-a) \quad \text{and} \quad \operatorname{var}_a^b f_n \le \Big(\sup_{t\in[a,b]_{\mathbb{T}}} \|h_n(t)\|_{\mathbb{R}^m} \Big) (b-a)\,.$$

These estimates, together with (5.12), imply that

$$\lim_{n\to\infty} (\|A_n - A\|_\infty)\,[1 + \operatorname{var}_a^b A_n] = 0 \quad \text{and} \quad \lim_{n\to\infty} \|f_n - f\|_\infty\,[1 + \operatorname{var}_a^b f_n] = 0\,.$$

Applying Theorem 4.4 we arrive again at (3.15) and thus we may complete the proof of the theorem using the same argument as in the close of the proof of Theorem 5.3. □

6 Closing remarks

1.- *Continuity in the weak topologies.* Assume that $P_n,\ P:[a,b]\to L(\mathbb{R}^m)$, $n\in\mathbb{N}$, are Lebesque integrable on $[a,b]$ and $\widetilde{x}\in\mathbb{R}^m$. Let x be a solution to $x'=P(t)\,x$, $x(a)=\widetilde{x}$, and let x_n, $n\in\mathbb{N}$, be solutions to $x_n'=P_n(t)\,x_n$, $x_n(a)=\widetilde{x}$. It follows from [26, Theorem 1]) that if

$$\text{(i)}\quad \|P_n\|_1\le p^*<\infty \quad\text{for}\quad n\in\mathbb{N}, \qquad \text{(ii)}\quad \int_a^t P_n\,\mathrm{d}s \rightrightarrows \int_a^t P\,\mathrm{d}s,$$

then the sequence $\{x_n\}$ tends uniformly to x on $[a,b]$ ($x_n \rightrightarrows x$ on $[a,b]$). Furthermore, by M.R. Zhang and G. Meng (cf. [21, Lemma 2], the sequence $\{P_n\}$ of Lebesgue integrable functions tends weakly in the space L_1 to P if and only if both the above conditions (i) and (ii) are satisfied. Thus in the case of linear ODE's, the weak convergence of the coefficient matrices P_n to P implies the uniform convergence of the corresponding solutions. (For the direct proof, see also [21, Theorem 6].)

Theorem 1.2 in [22] indicates that for generalized linear second order differential equations, weak* convergence could have a similar meaning. In particular, the authors show that if μ_n tends to μ in the weak* topology of the dual space $NBV([a,b],\mathbb{R})$ to $C([a,b],R)$ (we write $\mu_n \rightharpoonup^* \mu$), then the relation $x_n \rightrightarrows x$ on $[a,b]$ holds for solutions (x_n,y_n), $n\in\mathbb{N}$ and (x,y) of the systems

$$\begin{cases} x_n(t)=\widetilde{x}+\displaystyle\int_a^t y_n\,\mathrm{d}t, \\ y_n(t)=\displaystyle\int_a^t \mathrm{d}[\mu_n]\,x_n, \end{cases} \qquad \begin{cases} x(t)=\widetilde{x}+\displaystyle\int_a^t y\,\mathrm{d}t, \\ y(t)=\displaystyle\int_a^t \mathrm{d}[\mu]\,x. \end{cases}$$

Moreover, $y_n(b)\to y(b)$. Let us recall that, as usual, $C([a,b],\mathbb{R})$ stands for the space of continuous scalar functions on $[a,b]$ and $NBV([a,b],\mathbb{R})$ is the space of functions $\mu\in BV([a,b],\mathbb{R})$ such that $\mu(a)=0$ and $\mu(t-)=\mu(t)$ for $t\in(a,b]$. Furthermore, it is known that $\mu_n \rightharpoonup^* \mu$ in $NBV([a,b],\mathbb{R})$ if and only if

$$\lim_{n\to\infty}\mu_n(b)=\mu(b), \quad \operatorname{var}_a^b\mu_n\le\alpha^*<\infty \quad\text{for } n\in\mathbb{N} \ \text{ and } \ \lim_{n\to\infty}\int_a^b|\mu_n-\mu|\,\mathrm{d}t=0.$$

However, for systems of the dimension ≥ 2, result analogous to that mentioned above for ODE's is not true, as shown by the following example: Put

$$A_n(t)=\quad Pt+I\begin{cases} nt & \text{if } 0\le t\le\frac{1}{n}, \\ 1 & \text{if } \frac{1}{n}\le t\le 1, \end{cases} \quad \text{for}\ \ n\in\mathbb{N}\ \ \text{and}\ \ A(t)=Pt+I\begin{cases} 0 & \text{if } t=0, \\ 1 & \text{if } t\in(0,1], \end{cases}$$

where I stands for the identity $m\times m$-matrix. Then $\lim\limits_{n\to\infty}A_n(t)=A(t)$ holds for all $t\in[0,1]$. This convergence is locally uniform on $(0,1]$. Moreover, it is easy to see that $A_n \rightharpoonup^* A$ in $NBV([a,b],\mathbb{R})$, the solutions x_n, $n\in\mathbb{N}$ to (4.3) are given by

$$x_n(t)=\begin{cases} \exp(P\,t+n\,I\,t)\,\widetilde{x} & \text{if } 0<t\le\frac{1}{n}, \\ \exp(Pt+I)\,\widetilde{x} & \text{if } \frac{1}{n}\le t\le 1, \end{cases}$$

and

$$\lim_{n\to\infty} x_n(t) = x_0(t) \quad \text{for } t\in[0,1], \text{ where } x_0(t) = \begin{cases} \widetilde{x} & \text{if } t=0, \\ \exp(Pt+I)\,\widetilde{x} & \text{if } 0<t\le 1. \end{cases}$$

However, x_0 cannot be a solution to (4.1) as

$$\Delta^+ x_0(0) = (\exp(I) - I)\,\widetilde{x} \neq \widetilde{x} = \Delta^+ A(0)\,\widetilde{x}.$$

This is caused by the fact that the convergence $A_n \to A$ is not uniform on $[0,1]$.

2. - *Emphatic convergence.* If the condition (3.6) is violated, the situation is rather more complicated. When $\dim X < \infty$, then some results based on the notion of the emphatic convergence can be found e.g. in [20], [8], [27, Chapter 9], [9] (cf. also [10]). We suppose to treat the case when X is a general Banach space later.

3. - *Linear functional differential equations with impulses.* In view of the observations from Federson and Schwabik [7], we can see that the results of this paper can be applied also to linear functional differential equations with impulses. More details will be given later elsewhere.

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