Canonical rules

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Abstract

We develop canonical rules capable of axiomatizing all systems of multiple-conclusion rules over $K4$ or IPC, by extension of the method of canonical formulas by Zakharyaschev [37]. We use the framework to give an alternative proof of the known analysis of admissible rules in basic transitive logics, which additionally yields the following dichotomy: any canonical rule is either admissible in the logic, or it is equivalent to an assumption-free rule. Other applications of canonical rules include a generalization of the Blok–Esakia theorem and the theory of modal companions to systems of multiple-conclusion rules or (finitary structural global) consequence relations, and a characterization of splittings in the lattices of consequence relations over monomodal or superintuitionistic logics with the finite model property.

1 Introduction

Investigation of propositional logics usually revolves around provability of formulas. When we generalize the problem from formulas to inference rules, there arises an important distinction between *derivable* and *admissible* rules, introduced by Lorenzen [26]. A rule

$$\varphi_1, \dots, \varphi_n \,/\, \psi$$

is derivable if it can be derived from the postulated axioms and rules of the logic (such as modus ponens, or necessitation) by composition (see Section 2 for definition of rule derivation); and it is admissible if the set of theorems of the logic is closed under the rule. These two notions coincide for the standard consequence relation of classical logic, but nonclassical logics often admit rules which are not derivable. For example, all intermediate (superintuitionistic, si) logics admit the Kreisel–Putnam rule

$$\neg p \to q \vee r \,/\, (\neg p \to q) \vee (\neg p \to r),$$

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whereas many of these logics (such as *IPC* itself) do not derive this rule.

An obvious application of admissible rules is to shorten and simplify derivations in a given logic. An admissible rule applied to valid theorems of the logic again produces a valid theorem of the logic, even if the rule itself may be invalid. Admissible rules are also a fundamental structural property of the logic, as they form the largest consequence relation which yields the logic (as a set of formulas). The prominence of this consequence relation is stressed by the fact that it can be uniquely defined using only the set of theorems of the logic, unlike the usual consequence relation of derivable rules. There are also some practical applications of admissibility and its special case, unification, e.g., in description logics [1].

The research of admissible rules was stimulated by a question of H. Friedman [11], asking whether admissibility of rules in *IPC* is decidable. The problem was extensively investigated in a series of papers by Rybakov, who has shown that admissibility is decidable for a large class of modal and intermediate logics, found semantic criteria for admissibility, and obtained other results on various aspects of admissibility. His results on admissible rules in transitive modal and si logics are summarized in the monograph [29]. He also applied his method to tense logics [30, 31, 32]. Ghilardi [12, 13] discovered the connection of admissibility to projective formulas and unification, which provided another criteria for admissibility in certain modal and intermediate logics, and new decision procedures for admissibility in some modal and si systems. Ghilardi's results were utilized by Iemhoff [14, 15, 16] to construct an explicit basis of admissible rules for *IPC* and some other si logics, and to develop Kripke semantics for admissible rules. These results were extended to modal logics by Jeřábek [18]. We note that decidability of admissibility is by no means automatic. An artificial decidable modal logic with undecidable admissibility problem was constructed by Chagrov [5], and natural examples of bimodal logics with undecidable admissibility (or even unification) problem were found by Wolter and Zakharyaschev [35]. In terms of computational complexity, admissibility in basic transitive logics is *coNE*-complete by Jeřábek [20], in contrast to *PSPACE*-completeness of derivability in these logics.

In short, there are two basic approaches to the analysis of admissible rules that have been followed in the literature: one is the strategy of Rybakov, which relies on the combinatorics of universal frames of finite rank; the other is the strategy of Ghilardi and Iemhoff, using projectivity and extension properties of classes of finite frames. In this paper, we introduce a new approach based on *canonical rules*.

The idea of axiomatization of logics using frame-based (or algebra-based) formulas[1] first appeared in the frame formulas of Jankov [17] and Fine [9], and the subframe formulas of Fine [10]. A systematic investigation was undertaken by Zakharyaschev, who discovered *canonical formulas* which axiomatize all si logics, and all quasinormal extensions of $K4$ [37]. Canonical formulas for linear tense logics were found by Wolter [33]. Canonical formulas proved to be a powerful general tool for the study of various properties of modal and si logics, in particular the finite model property (cf. [38, 39, 6, 34]).

We will generalize this idea from formulas to rules. Following [18], we will actually work

[1] I.e., a formula whose definition involves a finite Kripke frame (or a modal or Heyting algebra) which also determines its semantical properties in a certain way.

with *multiple-conclusion rules*

$$\varphi_1, \ldots, \varphi_n \,/\, \psi_1, \ldots, \psi_m$$

which considerably simplifies the presentation. (Multiple-conclusion rules were also suggested by Kracht [24, 25].) We define certain frame-based multiple-conclusion rules, called the canonical rules, and we establish that every system of rules extending (the global consequence relation of) $K4$ or IPC can be axiomatized by canonical rules. We further define single-conclusion canonical rules, which are certain combinations of canonical rules expressible as one single-conclusion rule, and show that they axiomatize all single-conclusion rule systems (i.e., finitary structural global consequence relations) extending $K4$ or IPC.

We apply the general machinery to admissible rules of basic transitive or linearly ordered logics ($K4$, $S4$, GL, IPC, $K4.3$, $S4.3$, $GL.3$) as well as all logics inheriting their admissible rules in Section 4. We will show that for these logics, there exists a simple combinatorial criterion to recognize which canonical rules are admissible. In this way we obtain an alternative proof of decidability of admissibility in each of these systems, as well as a description of a basis of admissible rules. Furthermore, the proof gives a new result about admissible rules in these logics, a kind of dichotomy property: every canonical rule is either admissible, or equivalent to a rule without assumptions. Formulated without reference to canonical rules, this means that every rule is equivalent to a set of admissible rules and assumption-free rules. The single-conclusion version of the property also holds; note that assumption-free single-conclusion rules are just axioms.

As another application, we use canonical rules to develop the theory of modal companions of intuitionistic rule systems and consequence relations in Section 5. We show that the well-known basic properties of modal companions of si logics (cf. [27, 3, 8, 36]), including the Blok–Esakia theorem, generalize smoothly to rule systems.

In Section 6 we consider the generalization of canonical rules to nontransitive modal logics. While we cannot get any completeness theorem, we are at least able to define canonical rules so that they obey the expected refutation conditions (in contrast to canonical formulas). As an application of nontransitive canonical rules (actually, frame rules) we give a description of *splittings* in the lattices of rule systems or consequence relations extending a given rule system with the finite model property, which is an analogue of some classical results on the splittings in the lattice of normal modal logics (see e.g. [4, 28, 23]). Our final application concerns the famous open problem of describing the admissible rules of the basic modal logic K. This problem is important both theoretically, as K is the simplest and most fundamental modal logic, yet its admissible rules are much harder to understand than for transitive logics like $K4$, and practically, as K and its multimodal variants are among the most often used modal logics in computer science. We characterize canonical rules admissible in K, and give a simple set of axioms from which they are derivable. (We do not obtain any general result on admissibility of rules in K though, as canonical rules are incomplete for nontransitive rule systems.)

2 Preliminaries

Given a finite set C of propositional connectives, let Form_C be the set of propositional formulas built using connectives from C and countably many propositional variables p_n, $n < \omega$. A *substitution* is a mapping of propositional formulas to propositional formulas which commutes with connectives. A *single-conclusion rule* is an expression of the form

$$\Gamma \mathbin{/} \varphi,$$

where Γ is a finite subset of Form_C, and $\varphi \in \mathrm{Form}_C$. We will usually write Γ, Δ for $\Gamma \cup \Delta$, and omit curly brackets around sets of formulas. A set A of single-conclusion rules is a (structural finitary) *consequence relation* (see e.g. [25]), if it satisfies the following conditions for all formulas φ, ψ, finite sets of formulas Γ, Γ', and substitutions σ:

(i) $\varphi \mathbin{/} \varphi \in A$,

(ii) if $\Gamma \mathbin{/} \varphi \in A$ and $\Gamma, \varphi \mathbin{/} \psi \in A$, then $\Gamma \mathbin{/} \psi \in A$,

(iii) if $\Gamma \mathbin{/} \varphi \in A$, then $\Gamma, \Gamma' \mathbin{/} \varphi \in A$,

(iv) if $\Gamma \mathbin{/} \varphi \in A$, then $\sigma\Gamma \mathbin{/} \sigma\varphi \in A$.

If A is a consequence relation, we denote by $\mathrm{Ext}_1 A$ the complete lattice of all consequence relations extending A.

A (*multiple-conclusion*) *rule* is an expression of the form

$$\Gamma \mathbin{/} \Delta,$$

where Γ and Δ are finite sets of formulas. The intuitive meaning of $\Gamma \mathbin{/} \Delta$ is that if all formulas from Γ hold true, then some formula from Δ also holds true (for whatever meaning of "hold true" appropriate in a given context). If $\varrho = \Gamma \mathbin{/} \Delta$, we call $\varrho^a := \Gamma$ its set of *assumptions*, and $\varrho^c := \Delta$ its set of *conclusions*. Thus a rule ϱ is single-conclusion iff $|\varrho^c| = 1$. A rule ϱ is *assumption-free* if $\varrho^a = \varnothing$. Assumption-free single-conclusion rules are called *axioms* or *theorems*, and are identified with formulas. We will sometimes write

$$\frac{\Gamma}{\Delta}$$

instead of $\Gamma \mathbin{/} \Delta$. A set A of rules is a *rule system*, if it satisfies the following conditions for all formulas φ, finite sets of formulas Γ, Δ, Γ', Δ', and substitutions σ:

(i) $\varphi \mathbin{/} \varphi \in A$,

(ii) if $\Gamma \mathbin{/} \Delta, \varphi \in A$ and $\Gamma, \varphi \mathbin{/} \Delta \in A$, then $\Gamma \mathbin{/} \Delta \in A$,

(iii) if $\Gamma \mathbin{/} \Delta \in A$, then $\Gamma, \Gamma' \mathbin{/} \Delta, \Delta' \in A$,

(iv) if $\Gamma \mathbin{/} \Delta \in A$, then $\sigma\Gamma \mathbin{/} \sigma\Delta \in A$.

A set X of rules *entails* or *derives* a rule ϱ, written as $X \vdash \varrho$, if ϱ belongs to the smallest rule system A containing X. We also say that X is a *basis* of A, and that A is *axiomatized* by X.

Given a rule system A, a *rule system over* A is a rule system $A' \supseteq A$. The complete lattice of all rule systems over A is denoted by $\mathrm{Ext}_m A$. If X is a set of rules, then $A + X$ is the smallest rule system over A which contains X. The set X is called a basis of $A + X$ over A, and $A+X$ is axiomatized over A by X. If X consists of assumption-free (or single-conclusion) rules, we will call $A + X$ an assumption-free (single-conclusion, resp.) rule system over A.

If A is a consequence relation, then

$$A_m = \{\Gamma \;/\; \Delta, \varphi \mid \Delta \subseteq \mathrm{Form}_C, \Gamma \;/\; \varphi \in A\}$$

is a rule system. Conversely, for any rule system A, the set A_1 of single-conclusion rules in A is a consequence relation. Moreover, $(A_m)_1 = A$ for any consequence relation A, and $(A_1)_m = A$ for any rule system axiomatized by single-conclusion rules. We will therefore identify consequence relations with single-conclusion rule systems.

We assume that the reader is familiar with the theory of modal and superintuitionistic logics. However, we introduce some of the basic notions below to fix the notation, and to adapt it to the context of multiple-conclusion rule systems. We refer the reader to [6] for concepts unexplained here.

A *normal modal logic* is a rule system using the connectives $M = \{\to, \bot, \Box\}$, axiomatized by the *modus ponens* and *necessitation* rules

$$(MP) \qquad p, p \to q \;/\; q,$$

$$(Nec) \qquad p \;/\; \Box p,$$

and a set of axioms which includes classical propositional tautologies, and the axiom

$$\Box(p \to q) \to (\Box p \to \Box q).$$

(I.e., we identify a normal modal logic with what is usually called its *global* consequence relation, as opposed to the local consequence relation, which does not include the necessitation rule.) If L is a logic, we will also write $\Gamma \vdash_L \varphi$ instead of $\Gamma \;/\; \varphi \in L$. The smallest normal modal logic is denoted by K. Rule systems over K are called *normal modal rule systems.* We may sometimes omit the qualifier "normal". A *quasinormal modal logic* is a rule system axiomatized by MP, and a set of axioms which includes all theorems of K. If A is a rule system over K, let qA be the largest quasinormal logic contained in A.

Notice that as a consequence of these definitions, for any normal modal logic L and a set of formulas X, we denote by $L + X$ the *normal* closure of L and X (usually denoted by $L \oplus X$), whereas we denote the quasinormal closure of L and X (usually denoted by $L+X$) by $qL+X$. We decided for this incompatibility with common usage because the $\oplus/+$ distinction does not seem to have a natural generalization to the context of rule systems.

We define other Boolean connectives as abbreviations in the obvious way. The symbols $\Diamond\varphi$, $\boxdot\varphi$, $\diamondsuit\!\!\cdot\,\varphi$, $\Box^n\varphi$, and $\Box^{\leq n}\varphi$, are respectively abbreviations for $\neg\Box\neg\varphi$, $\varphi \wedge \Box\varphi$, $\varphi \vee \Diamond\varphi$, $\underbrace{\Box \cdots \Box}_{n}\varphi$, and $\bigwedge_{k \leq n} \Box^k\varphi$.

A *superintuitionistic* (*si*) or *intermediate* logic is a rule system using the connectives $I = \{\to, \wedge, \vee, \bot\}$, axiomatized by MP, and a set of axioms including all theorems of intuitionistic propositional logic (IPC). Rule systems over IPC are called *intuitionistic rule systems.* Rule systems over

$$K4 = K + \Box p \to \Box\Box p$$

or over IPC are called *transitive rule systems.*

Let L be a logic. A substitution σ is an *L-unifier* of a formula φ, if $\sigma\varphi \in L$. A rule $\Gamma \,/\, \Delta$ is *L-admissible*, if every common L-unifier of the formulas in Γ is also an L-unifier of some formula in Δ. The set of all L-admissible rules forms a rule system over L, which we denote by $\mathrel{|\!\sim}_L$. A *basis* of L-admissible rules is a basis of $\mathrel{|\!\sim}_L$ over L. The logic L is *structurally complete*, if $L = \mathrel{|\!\sim}_L$.

A *modal Kripke frame* is a pair $\langle W, R\rangle$, where R is a binary relation on a set W (which may be empty). We will most often work with transitive frames. In that case we will usually denote the accessibility relation R by the ordering symbol $<$, and its reflexive closure by $\leq$. (The notation $<$ does not imply that the relation is irreflexive. In particular, if the accessibility relation is already reflexive, then $< \, = \, \leq$.) For any $X \subseteq W$, we define

$$\begin{aligned}
X{\uparrow} &:= R[X] = \{y \mid \exists x \in X\,(x \mathrel{R} y)\},\\
X{\downarrow} &:= R^{-1}[X] = \{y \mid \exists x \in X\,(y \mathrel{R} x)\},\\
X{\Uparrow} &:= R^{*}[X] = \{y \mid \exists x \in X\,(x \mathrel{R^{*}} y)\},\\
X{\Downarrow} &:= (R^{*})^{-1}[X] = \{y \mid \exists x \in X\,(y \mathrel{R^{*}} x)\},
\end{aligned}$$

where R^* is the transitive reflexive closure of R. We will abbreviate $\{x\}{\uparrow}$ as $x{\uparrow}$. A *modal frame* is a triple $\langle W, R, P\rangle$, where $\langle W, R\rangle$ is a Kripke frame, and $P \subseteq \mathcal{P}(W)$ is closed under (finitary) Boolean operations, and the ${\downarrow}$ operation (or the operation $\Box X = W \smallsetminus (W \smallsetminus X){\downarrow}$). We will often denote the frame $\langle W, R, P\rangle$ by just W if there is no danger of confusion. A Kripke frame $\langle W, R\rangle$ may be identified with the frame $\langle W, R, \mathcal{P}(W)\rangle$. An *admissible valuation* is a relation $\Vdash \, \subseteq W \times \mathrm{Form}_M$ which satisfies

$$\begin{aligned}
&x \Vdash \varphi \to \psi \quad \text{iff} \quad x \nVdash \varphi \text{ or } x \Vdash \psi,\\
&x \nVdash \bot,\\
&x \Vdash \Box\varphi \quad \text{iff} \quad \forall y \in x{\uparrow}\; y \Vdash \varphi,\\
&\ \Vdash(p) := \{x \mid x \Vdash p\} \in P
\end{aligned}$$

for every $x \in W$, $\varphi, \psi \in \mathrm{Form}_M$, and every variable p. We will abbreviate the last condition as $\Vdash \, \in P$; it implies $\Vdash(\varphi) \in P$ for all formulas φ. A *Kripke model* is a pair $\langle W, \Vdash\rangle$, where W is a Kripke frame, and $\Vdash$ is a valuation in W.

An *intuitionistic Kripke frame* is a partially ordered set $\langle W, \leq\rangle$. An *intuitionistic frame* is a triple $\langle W, \leq, P\rangle$, where $\langle W, \leq\rangle$ is a poset, and P is a set of upper subsets of W, closed under monotone Boolean operations, and the operation

$$A \to B := W \smallsetminus (A \smallsetminus B){\downarrow} = \{x \in W \mid \forall y \geq x\,(y \in A \Rightarrow y \in B)\}.$$

Admissible valuations are relations $\Vdash \subseteq W \times \mathrm{Form}_I$ such that $\Vdash \in P$, and

$$\begin{aligned}
x \Vdash \varphi \to \psi \quad &\text{iff} \quad \forall y \in x{\uparrow}\, (y \Vdash \varphi \Rightarrow y \Vdash \psi),\\
x \Vdash \varphi \wedge \psi \quad &\text{iff} \quad x \Vdash \varphi \text{ and } x \Vdash \psi,\\
x \Vdash \varphi \vee \psi \quad &\text{iff} \quad x \Vdash \varphi \text{ or } x \Vdash \psi,\\
x \nVdash \bot,&
\end{aligned}$$

for every $x \in W$, and $\varphi, \psi \in \mathrm{Form}_I$.

Let $\langle W, R, P\rangle$ be a modal or intuitionistic frame. A formula φ is *satisfied* by a valuation $\Vdash$, if $x \Vdash \varphi$ for every $x \in W$. A rule $\Gamma \mathbin{/} \Delta$ is *satisfied* by $\Vdash$, if some formula $\psi \in \Delta$ is satisfied by $\Vdash$, or some $\varphi \in \Gamma$ is not satisfied by $\Vdash$. A rule ϱ is *valid* in $\langle W, R, P\rangle$, written as $W \vDash \varrho$, if it is satisfied by all admissible valuations $\Vdash \in P$. A rule which is not valid in W is said to be *refuted* in W. If A is a rule system, then W is called an *A-frame* if A (i.e., every rule from A) is valid in W.

If $\mathcal{C}$ is a class of frames, then $\mathrm{Th}(\mathcal{C})$, $\mathrm{Th}_m(\mathcal{C})$, and $\mathrm{Th}_1(\mathcal{C})$ are the sets of formulas, multiple-conclusion rules, and single-conclusion rules (respectively) valid in all frames $W \in \mathcal{C}$. A rule system A is *Kripke complete*, if $A = \mathrm{Th}_m(\mathcal{C})$ for a class $\mathcal{C}$ of Kripke frames, and it has the *finite model property* (*fmp*), if $A = \mathrm{Th}_m(\mathcal{C})$ for a class of finite (Kripke, w.l.o.g.) frames. Notice that a nontransitive logic may fail to have fmp as a rule system even though it has fmp as a logic.

A *pointed frame* is a quadruple $\langle W, R, P, x\rangle$, where $\langle W, R, P\rangle$ is a modal frame, and $x \in W$. A formula is *valid* in $\langle W, R, P, x\rangle$, if $x \Vdash \varphi$ for every admissible valuation $\Vdash$.

A frame $\langle V, S, Q\rangle$ is a *generated subframe* of a frame $\langle W, R, P\rangle$, if $V \subseteq W$, $V{\uparrow} \subseteq V$, $S = R \restriction V$, and $Q = \{A \cap V \mid A \in P\}$. We will denote generated subframes by $V \subseteq\!\cdot\, W$. The frame W is *generated* by $X \subseteq W$, if $W = X{\Uparrow}$ (i.e., if W is the only generated subframe of itself which includes X.) If $X = \{x\}$, the frame W is *rooted*, and x is its root. If $* \in \{\bullet, \circ\}$, we define $\langle W^*, R^*, P^*\rangle$ as the frame obtained from W by attaching a new root r below W, such that r is reflexive if $* = \circ$ and irreflexive if $* = \bullet$ (i.e., $R^\bullet = R \cup (\{r\} \times W)$, $R^\circ = R \cup (\{r\} \times W^\circ)$), and $P^* = \{A \subseteq W^* \mid A \cap W \in P\}$.

Let $\langle W, <\rangle$ be a transitive Kripke frame. We put $x \sim y$ iff $x \le y \le x$. Equivalence classes of $\sim$ are called *clusters*, and we denote the cluster of a point x by $\mathrm{cl}(x)$. A cluster is *proper* if it has at least two points. Notice that every proper cluster is reflexive (i.e., all its points are). A cluster is *final* if it is a generated subframe of W. Dually, a cluster c is *initial*, if $c{\downarrow} \subseteq c$. An irreflexive final cluster (i.e., a point x such that $x{\uparrow} = \varnothing$) is called a *dead end*.

The *disjoint union* $\sum_i W_i$ of frames $\langle W_i, R_i, P_i\rangle$, $i \in I$, is the frame $\langle W, R, P\rangle$ defined as follows. We replace W_i with their isomorphic copies so that they are disjoint, and put $W = \bigcup_i W_i$, $R = \bigcup_i R_i$, and $P = \{A \subseteq W \mid \forall i \in I\, A \cap W_i \in P_i\}$. (On the other hand, we will denote disjoint union of *sets* by $X \mathbin{\dot\cup} Y$.)

An onto mapping $f \colon W \to V$ is a *p-morphism* of $\langle W, R, P\rangle$ to $\langle V, S, Q\rangle$, if

(i) $x \mathrel{R} y$ implies $f(x) \mathrel{S} f(y)$,

(ii) if $f(x) \mathrel{S} u$, there exists $y \in W$ such that $x \mathrel{R} y$ and $f(y) = u$,

(iii) $f^{-1}[A] \in P$,

for every $x, y \in W$, $u \in V$, and $A \in Q$.

A modal frame $\langle W, R, P \rangle$ is *refined* if

$$\forall X \in P\,(x \in X \Leftrightarrow y \in X) \Rightarrow x = y,$$
$$\forall X \in P\,(x \in \Box X \Rightarrow y \in X) \Rightarrow x\,R\,y,$$

for all $x, y \in F$. An intuitionistic frame $\langle W, \leq, P \rangle$ is refined if

$$\forall X \in P\,(x \in X \Rightarrow y \in X) \Rightarrow x \leq y.$$

A frame is *compact* if every subset of P (resp. $P \cup \{W \smallsetminus A \mid A \in P\}$ in the intuitionistic case) with the finite intersection property (fip) has a nonempty intersection. A compact refined frame is called *descriptive*.

A *modal algebra* $\langle A, \Rightarrow, 0, \Box \rangle$ is a Boolean algebra $\langle A, \Rightarrow, 0 \rangle$ with an additional unary operator $\Box$ which distributes over finite meets. A valuation v (i.e., a homomorphism of Form_M to A) *satisfies* a rule ϱ, if $v(\varphi) \neq 1$ for some $\varphi \in \varrho^a$, or $v(\psi) = 1$ for some $\psi \in \varrho^c$. A rule ϱ is *valid* in A, if it is satisfied by every valuation in A. If $\langle W, R, P \rangle$ is a modal frame, then its *dual* is the modal algebra $\langle P, \to, \varnothing, \Box \rangle$. A frame is *finitely generated*, or *$\varkappa$-generated* for a cardinal $\varkappa$, if its dual is. The intuitionistic case is analogous, using Heyting algebras in place of modal algebras.

We end this section with basic completeness and preservation results about rule systems.

Lemma 2.1 *If L is a normal modal or intermediate logic, and $\mathcal{C}$ a class of L-frames, then* $\mathrm{Th}_m(\mathcal{C})$ *is a rule system over L.*

Proof: Immediate from the definition. $\Box$

Theorem 2.2 *Let L be a normal modal or intermediate logic, A a rule system over L, and ϱ a rule. If $\varrho \notin A$, there is a descriptive L-frame W which validates A, and refutes ϱ.*

Proof: For any subsets $\Gamma \subseteq \varrho^a$ and $\Delta \subseteq \varrho^c$, the rule $\Gamma \mathbin{/} \Delta$ is not in A, by weakening. By Zorn's lemma, there exists a pair $\langle x, y \rangle$ of maximal sets of formulas such that $\varrho^a \subseteq x$, $\varrho^c \subseteq y$, and $\Gamma \mathbin{/} \Delta \in A$ for no finite $\Gamma \subseteq x$, $\Delta \subseteq y$. Clearly $x \cap y = \varnothing$. On the other hand, $x \cup y$ contains all formulas, as A is closed under cut. It follows that x is closed under all rules from A; i.e., if $\Gamma \mathbin{/} \Delta \in A$ and $\Gamma \subseteq x$, then $\Delta \cap x \neq \varnothing$.

Let $\langle C, R, P \rangle$ be the canonical L-frame, and $W := \{u \in C \mid u \supseteq x\}$. W is a generated subframe of C: in the intuitionistic case it is obvious from the definition, in the modal case it follows as x is closed under necessitation. As such, W is a refined L-frame, and it is easy to see that it is compact as well.

An admissible valuation $\Vdash$ in W, given by a substitution σ, satisfies a formula φ iff $\sigma\varphi \in x$: the right-to-left implication is obvious. If $\sigma\varphi \notin x$, the set $x \cup \{\neg\sigma\varphi\}$ is L-consistent, as A includes L. Therefore there exists an L-MCS $u \supseteq x$ such that $\sigma\varphi \notin u$, which means that $u \in W$, and $u \nVdash \varphi$. (The intuitionistic case is similar.)

Consequently, all rules from A are valid in W as A is closed under substitution, but ϱ is refuted by the valuation given by the identity substitution. $\Box$

Proposition 2.3 *Let A be a modal or intuitionistic rule system.*

(i) A is valid in a frame iff it is valid in its dual algebra.

(ii) Validity of A is preserved by p-morphisms.

Proof: (i) It is clear from the definition that validity of rules in a frame depends only on its algebra of admissible sets.

(ii) Let f be a p-morphism of $W \vDash A$ onto U, $\varrho \in A$, and let $\Vdash_U$ be an admissible valuation in U. As f is a p-morphism, there exists an admissible valuation $\Vdash_W$ in W such that $\Vdash_W(\varphi) = f^{-1}[\Vdash_U(\varphi)]$ for every formula φ. If $\Vdash_U$ satisfies ϱ^a in U, then $\Vdash_W$ satisfies ϱ^a in W, hence it also satisfies some $\psi \in \varrho^c$, thus ψ is satisfied by $\Vdash_U$. $\square$

Proposition 2.4 *Let A be a modal or intuitionistic rule system. The following are equivalent.*

(i) Validity of A is preserved under disjoint unions.

(ii) A has a single-conclusion basis.

(iii) For any rule $\varrho \in A$, there is $\psi \in \varrho^c$ such that $\varrho^a \mathbin{/} \psi \in A$.

Proof: (iii) $\to$ (ii) is trivial.

(ii) $\to$ (i): Let $W_i \vDash A$ for each $i \in I$, and put $W = \sum_{i\in I} W_i$. Let $\varrho \in A$ be a single-conclusion rule, and $\Vdash$ an admissible valuation in W which satisfies ϱ^a. For each $i \in I$, the restriction of $\Vdash$ to W_i is an admissible valuation which satisfies ϱ^a in W_i, hence it also satisfies ϱ^c. It follows that $\Vdash$ satisfies ϱ^c in W.

(i) $\to$ (iii): Assume that A is preserved under disjoint unions, and let ϱ be such that $\varrho_\psi := \varrho^a \mathbin{/} \psi \in A$ for no $\psi \in \varrho^c$. We may fix frames W_ψ validating A and refuting ϱ_ψ by a valuation $\Vdash_\psi$ for each $\psi \in \varrho^c$. Then the disjoint union $W = \sum_\psi W_\psi$ validates A and refutes ϱ by the valuation $\sum_\psi \Vdash_\psi$, thus $\varrho \notin A$. $\square$

Proposition 2.5 *Let A be a modal or intuitionistic rule system. The following are equivalent.*

(i) Validity of A is preserved by generated subframes.

(ii) A has an assumption-free basis over K or IPC.

(iii) For any rule ϱ, $\varrho \in A$ implies $\mathbin{/} \{\varphi \to \psi \mid \psi \in \varrho^c\} \in A$, where $\varphi = \bigwedge \varrho^a$ in the intuitionistic case, and $\varphi = \Box^{\le n} \bigwedge \varrho^a$ for some $n \in \omega$ in the modal case. (If A is transitive modal, we can take $n = 1$.)

Proof: (iii) $\to$ (ii) is trivial.

(ii) $\to$ (i): Let $W \vDash A$, $U \subseteq\cdot\, W$, $\Vdash$ an admissible valuation in U, and $\varrho \in A$ an assumption-free rule. We can extend $\Vdash$ to an admissible valuation in W. As $\Vdash$ satisfies some $\psi \in \varrho^c$ in W, it also satisfies ψ in U.

(i) $\to$ (iii): Assume that A is preserved by generated subframes, and $\mathbin{/} \{\varphi_n \to \psi \mid \psi \in \varrho^c\} \notin A$ for every n, where $\varphi_n = \Box^{\le n} \bigwedge \varrho^a$. As $\vdash_K \varphi_n \to \varphi_{n+1}$, we have $\mathbin{/} \Delta \notin A$ for every finite subset Δ of $S := \{\varphi_n \to \psi \mid n \in \omega, \psi \in \varrho^c\}$. By the proof of Theorem 2.2, there exists a descriptive frame $\langle W, R, P\rangle$ such that $W \vDash A$, and a valuation $\Vdash$ in W which refutes all formulas in S. For every $\psi \in \varrho^c$, the set

$$P_\psi := \{\Vdash(\neg\psi)\} \cup \{\Vdash(\varphi_n) \mid n \in \omega\} \subseteq P$$

has fip, hence there exists $x_\psi \in \bigcap P_\psi$. Then $x_\psi \nVdash \psi$, and $y \Vdash \varrho^a$ for every $y \in x\Uparrow$. Let $U = \{x_\psi \mid \psi \in \varrho^c\}\Uparrow$. Then U, as a generated subframe of W, validates A, but refutes ϱ by the valuation $\Vdash$.

The intuitionistic case is similar. $\square$

3 Canonical rules

In this section we will introduce our canonical rules (including their single-conclusion variants) as well as their refutation criteria, and prove their completeness for transitive rule systems, and some basic properties. We will discuss Zakharyaschev's canonical formulas along the way.

All frames in this section are assumed to be transitive.

Definition 3.1 Let $\langle W, <, P\rangle$ be a modal frame, and $\langle F, <\rangle$ a finite Kripke frame. A partial mapping f from W onto F is a *subreduction* of W to F, if for all $x, y \in W$ and $u \in F$,

(i) $x < y$ and $x, y \in \mathrm{dom}(f)$ implies $f(x) < f(y)$,

(ii) if $f(x) < u$, there exists $y \in \mathrm{dom}(f)$ such that $x < y$ and $f(y) = u$,

(iii) $f^{-1}[u] \in P$.

Subreductions of intuitionistic frames are defined in a similar way, except that the last condition is replaced with

(iii') $W \smallsetminus f^{-1}[u]\downarrow \in P$.

A *domain* is an upper subset $d \subseteq F$. A subreduction f satisfies the *global closed domain condition* (*GCDC*) for a domain d, if there is no $x \in W \smallsetminus \mathrm{dom}(f)$ such that $f[x\uparrow] = d$. We say that f is *globally cofinal*, if it satisfies GCDC for the empty domain, i.e., $W = \mathrm{dom}(f)\Downarrow$. If D is a set of domains, f satisfies GCDC for D if it satisfies GCDC for every $d \in D$. If G is a generated subframe of F, we put $D \restriction G := \{d \in D \mid d \subseteq G\}$.

Definition 3.2 Let $\langle F, <\rangle$ be a finite Kripke frame, and D a set of domains in F. The *modal canonical rule* $\gamma(F, D)$ in variables $\{p_i \mid i \in F\}$ is defined as

$$\frac{\{p_i \vee p_j \mid i \neq j\}, \{\Box p_j \to p_i \mid i < j\}, \{p_i \vee \Box p_j \mid i \nless j\}, \Big\{\bigwedge_i p_i \wedge \bigwedge_{i \notin d} \Box p_i \to \bigvee_{i \in d} \Box p_i \;\Big|\; d \in D\Big\}}{\{p_i \mid i \in F\}}$$

where i, j range over elements of F. If F is an intuitionistic frame, the *intuitionistic canonical rule* $\delta(F, D)$ is

$$\frac{\Big\{\Big(\bigwedge_{k \ngeq j} p_k \to p_j\Big) \to p_i \;\Big|\; i \leq j\Big\}, \Big\{\bigwedge_{i \notin d} p_i \to \bigvee_{i \in d} p_i \;\Big|\; d \in D'\Big\}}{\{p_i \mid i \in F\}}$$

where $i, j \in F$, and D' is the set of domains $d \in D$ which are not rooted.

Lemma 3.3 *A modal (resp., intuitionistic) frame* $\langle W, <, P\rangle$ *refutes* $\gamma(F, D)$ *(resp.,* $\delta(F, D)$*) iff there is a subreduction of* W *to* F *with GCDC for* D*.*

Proof: Let $\langle W, <, P\rangle$ be a modal frame. Valuations $\Vdash$ to the variables $\{p_i \mid i \in F\}$ in the underlying Kripke frame $\langle W, <\rangle$ are in 1–1 correspondence to relations $f \subseteq W \times F$, using

$$x \Vdash p_i \quad \text{iff} \quad \langle x, i\rangle \notin f.$$

Then we observe that

- $\Vdash$ is admissible in W iff f satisfies condition (iii) of Definition 3.1,
- $p_i \vee p_j$ is satisfied by $\Vdash$ for all $i \neq j$ iff f is a partial function,
- $p_i \vee \Box p_j$ is satisfied for all $i \not< j$ iff f satisfies condition (i),
- $\Box p_j \to p_i$ is satisfied for all $i < j$ iff f satisfies condition (ii),
- $\bigwedge_i p_i \wedge \bigwedge_{i \notin d} \Box p_i \to \bigvee_{i \in d} \Box p_i$ is satisfied iff f satisfies GCDC for d,
- p_i is refuted for all i iff f is onto.

Hence $\Vdash$ refutes $\gamma(F, D)$ if and only if f is a subreduction of W to F with GCDC for D.

In the intuitionistic case, we define a subreduction from a valuation refuting $\delta(F, D)$ by

$$f(x) = i \quad \text{iff} \quad x \nVdash p_i \text{ and } x \Vdash \bigwedge_{j \not\geq i} p_j.$$

Conversely, given a subreduction, we can refute $\delta(F, D)$ by the valuation defined by

$$x \Vdash p_i \quad \text{iff} \quad x \notin f^{-1}[i]{\downarrow}.$$

Details are left to the reader. □

We define Zakharyaschev's canonical formulas and their refutation criteria below. We will only need canonical formulas for quasinormal extensions of $qK4$, but we also include canonical formulas axiomatizing normal extensions of $K4$ and si logics, for comparison with canonical rules.

Definition 3.4 Let $\langle W, <, P, r\rangle$ be a pointed frame, and F a finite rooted Kripke frame with root $0 \in F$.

If the root 0 of F is irreflexive, a *quasisubreduction* of W to F is a partial mapping from F onto W, which satisfies the conditions (ii) and (iii) from Definition 3.1, and for every $x, y \in W$,

(i') $x < y$ and $x, y \in \mathrm{dom}(f)$ implies $f(x) < f(y)$ or $f(x) = f(y) = 0$.

A subreduction or quasisubreduction f of W to F satisfies the *actual world condition* (*AWC*), if $f(r)$ is the root of F. For any domain d, f satisfies the *local closed domain condition* (*LCDC*) for d, if there is no $x \in \mathrm{dom}(f)\uparrow \smallsetminus \mathrm{dom}(f)$ such that $f[x\uparrow] = d$. f is *locally cofinal*, if it satisfies LCDC for $d = \varnothing$, i.e., if $\mathrm{dom}(f)\uparrow \subseteq \mathrm{dom}(f)\updownarrow$.

The (*normal*) *canonical formula* $\alpha(F, D)$ is defined as

$$\boxdot\Big(\bigwedge_{i\neq j}(p_i \vee p_j) \wedge \bigwedge_{i<j}(\Box p_j \to p_i) \wedge \bigwedge_{i\not<j}(p_i \vee \Box p_j) \wedge \bigwedge_{d\in D}\Big(\bigwedge_{i} p_i \wedge \bigwedge_{i\notin d}\Box p_i \to \bigvee_{i\in d}\Box p_i\Big)\Big) \to p_0,$$

where i, j range over elements of F. If 0 is irreflexive, the *quasinormal canonical formula* $\alpha^\bullet(F, D)$ is defined by omitting the conjunct $p_0 \vee \Box p_0$ from the antecedent of $\alpha(F, D)$. If F is an intuitionistic frame, the *intuitionistic canonical formula* $\beta(F, D)$ is defined as

$$\bigwedge_{i\leq j}\Big(\Big(\bigwedge_{k\not\geq j} p_k \to p_j\Big) \to p_i\Big) \wedge \bigwedge_{d\in D'}\Big(\bigwedge_{i\notin d} p_i \to \bigvee_{i\in d} p_i\Big) \to p_0,$$

where $i, j \in F$, and D' is the set of domains $d \in D$ which are not rooted. In other words, $\alpha(F, D) = \boxdot \bigwedge(\gamma(F, D))^a \to p_0$, and $\beta(F, D) = \bigwedge(\delta(F, D))^a \to p_0$.

Lemma 3.5 (Zakharyaschev [37]) *Let F be a finite rooted Kripke frame, and D a set of domains in F.*

(*i*) *A modal frame W refutes the canonical formula $\alpha(F, D)$ iff there is a subreduction of W to F with LCDC for D.*

(*ii*) *If F is intuitionistic, then an intuitionistic frame W refutes $\beta(F, D)$ iff there is a subreduction of W to F with LCDC for D.*

(*iii*) *A pointed modal frame $\langle W, r\rangle$ refutes $\alpha(F, D)$ iff there is a subreduction of W to F with AWC and LCDC for D.*

If the root of F is irreflexive, then $\langle W, r\rangle$ refutes $\alpha^\bullet(F, D)$ iff there is a quasisubreduction of W to F with AWC and LCDC for D.

□

Theorem 3.6 (Zakharyaschev [37])

(*i*) *For any $\varphi \in \mathrm{Form}_M$, there is a finite set Δ of normal canonical formulas such that $K4 + \varphi = K4 + \Delta$.*

(*ii*) *For any $\varphi \in \mathrm{Form}_I$, there is a finite set Δ of intuitionistic canonical formulas such that $IPC + \varphi = IPC + \Delta$.*

(*iii*) *For any $\varphi \in \mathrm{Form}_M$, there is a finite set Δ of normal and quasinormal canonical formulas such that $qK4 + \varphi = qK4 + \Delta$.*

Moreover, for each $\alpha(F, D)$ or $\alpha^\bullet(F, D)$ or $\beta(F, D)$ in Δ, we may assume that $\varnothing \in D$, $F \notin D$, and no $d \in D$ is generated by a reflexive point. There is an algorithm which, given φ, computes a suitable Δ. □

Remark 3.7 We have deviated from the original Zakharyaschev's presentation of canonical formulas in several details. We define domains as upper subsets of F, whereas Zakharyaschev uses antichains; we prefer the former as distinct antichains may generate the same subset of F (thus giving equivalent canonical formulas). We do not formally distinguish negation-free canonical formulas $\alpha(F, D)$ from the "regular" canonical formulas $\alpha(F, D, \bot)$, as the latter is syntactically and semantically equivalent to the special case $\alpha(F, D \cup \{\varnothing\})$ of the former, using the empty domain (or antichain). (Zakharyaschev apparently does not allow empty antichains in $\alpha(F, D)$, though we failed to find it stated explicitly in [37, 6].) We renamed Zakharyaschev's closed domain condition to local closed domain condition, as we also consider the global version thereof needed for canonical rules. (We actually find the global closed domain condition to be the more natural and more fundamental of the two, but we did not go so far as to call GCDC just "closed domain condition" to avoid confusion.)

The main result of this section is the completeness theorem below. We will split its proof into a few lemmas. The basic idea is to embed modal rule systems in quasinormal logics, use Zakharyaschev's canonical formulas for quasinormal extensions of $qK4$, and reduce intuitionistic rule systems to modal ones by Gödel's translation.

Theorem 3.8

(*i*) *If ϱ is a modal rule, there is a finite set Δ of modal canonical rules such that $K4 + \varrho = K4 + \Delta$.*

(*ii*) *If ϱ is an intuitionistic rule, there is a finite set Δ of intuitionistic canonical rules such that $IPC + \varrho = IPC + \Delta$.*

Moreover, for each $\gamma(F, D)$ or $\delta(F, D)$ in Δ, we may require that $\varnothing \in D$, and no $d \in D$ is generated by a reflexive point. There is an algorithm which, given ϱ, computes a suitable Δ.

Definition 3.9 The *characteristic formula* of a modal rule ϱ is

$$\chi(\varrho) = \bigwedge_{\varphi \in \varrho^a} \Box\varphi \to \bigvee_{\psi \in \varrho^c} \Box\psi.$$

Recall the definition of $W^\bullet$. Notice that the pointed frame $\langle W^\bullet, r\rangle$, where r is the root of $W^\bullet$, validates $\chi(\varrho)$ iff W validates ϱ.

Lemma 3.10 ([18]) *For any set R of modal rules,*

$$K4 + R = \chi^{-1}[qK4 + \chi[R]].$$

Proof: The inclusion $K4 + R \subseteq \chi^{-1}[qK4 + \chi[R]]$ is clear, as $\chi^{-1}[L]$ is a rule system for any quasinormal logic L. Assume that $\varrho \notin K4 + R$. By Theorem 2.2, there is a frame $\langle W, <, P\rangle$ which validates R and refutes ϱ. Then the pointed frame $\langle W^\bullet, r\rangle$ validates $qK4 + \chi[R]$, and refutes $\chi(\varrho)$. □

Lemma 3.11 *Let $\langle W, <, P\rangle$ be a frame, r the root of $W^\bullet$, F a finite Kripke frame, and D a set of domains in F.*

(i) If F has a reflexive root, then $\langle W^\bullet, r\rangle$ refutes $\alpha(F, D)$ iff W refutes $\gamma(F, D)$.

(ii) $\langle W^\bullet, r\rangle$ refutes $\alpha(F^\bullet, D)$ iff W refutes $\gamma(F, D)$.

(iii) If $F \notin D$, then $\langle W^\bullet, r\rangle$ refutes $\alpha^\bullet(F^\bullet, D)$ iff W refutes $\gamma(F, D)$.

(iv) If $F \in D$ does not have a reflexive root, then $\langle W^\bullet, r\rangle$ refutes $\alpha^\bullet(F^\bullet, D)$ iff W refutes $\gamma(F, D \smallsetminus \{F\})$.

Proof: We only indicate how to construct the (quasi)subreductions, leaving the details to the reader.

Right-to-left: given a subreduction f of W to F, we extend it to $W^\bullet$ by mapping r to the root of F or $F^\bullet$ (as appropriate) in cases (i)–(iii). In case (iv), we map all points x such that $f[x\!\uparrow] = F$ to the root of $F^\bullet$.

Left-to-right: given a (quasi)subreduction f of $W^\bullet$ to F or $F^\bullet$, we construct a subreduction of W to F by taking $f \restriction W$ in cases (i) and (ii), and $f \restriction (W \smallsetminus f^{-1}[0])$ in cases (iii) and (iv), where 0 is the root of $F^\bullet$. ☐

Definition 3.12 If φ is an intuitionistic formula, its *Gödel–McKinsey–Tarski translation* $\mathsf{T}\varphi$ is obtained by prefixing $\Box$ before each subformula of φ which is an implication or variable. If ϱ is an intuitionistic rule, we define $\mathsf{T}\varrho$ as $\{\mathsf{T}\varphi \mid \varphi \in \varrho^a\} \,/\, \{\mathsf{T}\psi \mid \psi \in \varrho^c\}$.

If $\langle W, \leq, P\rangle$ is an intuitionistic frame, let σW be the modal frame $\langle W, \leq, \sigma P\rangle$, where σP is the Boolean closure of P. If $\langle W, \leq, P\rangle$ is a reflexive transitive (i.e., preordered) modal frame, its *skeleton* ϱW is the intuitionistic frame $\langle W/{\sim}, \leq/{\sim}, \varrho P\rangle$ (i.e., the points of ϱW are the clusters of W), where

$$\varrho P = \{A/{\sim} \mid A \in P, A = \Box A\}.$$

Recall that for any intuitionistic frame W, we have $W \simeq \varrho\sigma W$, and $\sigma W \vDash Grz = K4 + \alpha(\circ\!\!\to\!\!\circ) + \alpha(\bullet)$ [6] (hence $\sigma W \vDash \gamma(F, D)$ whenever F is not an intuitionistic frame).

Lemma 3.13 *Let W be a preordered frame, π an intuitionistic rule, F a finite intuitionistic Kripke frame, and D a set of domains in F.*

(i) $\varrho W \vDash \pi$ iff $W \vDash \mathsf{T}\pi$.

(ii) $\varrho W \vDash \delta(F, D)$ iff $W \vDash \gamma(F, D)$.

Proof: (i) On the one hand, assume $W \vDash \mathsf{T}\pi$, and let $\Vdash_{\varrho W} \in \varrho P$. There exists a valuation $\Vdash_W \in P$ such that

$$x \Vdash_W \Box p \Leftrightarrow x \Vdash_W p \Leftrightarrow \mathrm{cl}(x) \Vdash_{\varrho W} p$$

for every variable p, and $x \in W$. Then

$$(*) \qquad x \Vdash_W \mathsf{T}\varphi \Leftrightarrow \mathrm{cl}(x) \Vdash_{\varrho W} \varphi$$

for every $\varphi \in \mathrm{Form}_I$ by induction on its complexity (see [6, §8.3]), hence the validity of $\mathsf{T}\pi$ in W implies that π is satisfied by $\Vdash_{\varrho W}$.

On the other hand, let $\varrho W \vDash \pi$, and $\Vdash_W \in P$. There exits a valuation $\Vdash_{\varrho W} \in \varrho P$ such that $\mathrm{cl}(x) \Vdash_{\varrho W} p \Leftrightarrow x \Vdash_W \Box p$. Then again $(*)$ holds, hence $\mathsf{T}\pi$ is satisfied by $\Vdash_W$.

(ii) On the one hand, let f be a subreduction of ϱW to F with GCDC for D, and define a partial function $g\colon W \to F$ by

$$g(x) = u \quad \text{iff} \quad \mathrm{cl}(x) \in f^{-1}[u]{\downarrow} \smallsetminus \bigcup_{v \ngeq u} f^{-1}[v]{\downarrow}.$$

We have $g^{-1}[u] \in P$, $g(x) = f(\mathrm{cl}(x))$ whenever $\mathrm{cl}(x) \in \mathrm{dom}(f)$, and $g[x{\uparrow}] = f[\mathrm{cl}(x){\uparrow}]$, whence it is easy to see that g is a subreduction of W to F with GCDC for D.

On the other hand, let g be a subreduction of W to F with GCDC for D. Notice that $g(x) = g(y)$ for every $x, y \in \mathrm{dom}(g)$ such that $x \sim y$. We define a partial function $f\colon \varrho W \to F$ by

$$f(c) = u \quad \text{iff} \quad \exists x \in c\, g(x) = u.$$

We have $\mathrm{cl}(x) \notin f^{-1}[u]{\downarrow}$ iff $\forall y \geq x\, \neg(g(y) = u)$, hence $\varrho W \smallsetminus f^{-1}[u]{\downarrow} = \Box(W \smallsetminus g^{-1}[u]) \in \varrho P$. Also trivially $f(\mathrm{cl}(x)) = g(x)$ whenever $x \in \mathrm{dom}(g)$, and we have $f[\mathrm{cl}(x){\uparrow}] = g[x{\uparrow}]$, thus it is easy to see that f is a subreduction of ϱW to F with GCDC for D. $\Box$

Proof (*of Theorem* 3.8)*:* Consider the modal case first. Let

$$A = K4 + \{\gamma(F, D) \mid \gamma(F, D) \in K4 + \varrho\},$$

we need to show $A = K4 + \varrho$. Clearly, $A \subseteq K4 + \varrho$. Assume for contradiction that $\varrho \notin A$, and let W be a frame validating A but refuting ϱ, which exists by Theorem 2.2. Then the pointed frame $\langle W^\bullet, r\rangle$ validates $\chi[A]$ and refutes $\chi(\varrho)$. By Theorem 3.6, there exists a normal or quasinormal canonical formula $\alpha \in qK4 + \chi(\varrho)$ which is refuted in $\langle W^\bullet, r\rangle$. Let γ be the corresponding canonical rule as in Lemma 3.11. Then W refutes γ, thus $\gamma \notin A$, and by the definition of A, $\gamma \notin K4 + \varrho$. By Theorem 2.2, there is a frame U validating ϱ, and refuting γ. Then $\langle U^\bullet, r\rangle$ validates $\chi(\varrho)$ and refutes α, which contradicts $\alpha \in qK4 + \chi(\varrho)$.

The intuitionistic case can be reduced to the modal case as follows. Let

$$A = \{\delta(F, D) \mid \gamma(F, D) \in K4 + \mathsf{T}\varrho\}.$$

For any intuitionistic frame W, we have

$$\begin{aligned} W \vDash \varrho \quad &\text{iff} \quad \sigma W \vDash \mathsf{T}\varrho \\ &\text{iff} \quad \sigma W \vDash \{\gamma(F, D) \mid \delta(F, D) \in A\} \\ &\text{iff} \quad W \vDash A, \end{aligned}$$

where the first and last equivalence come from Lemma 3.13, and the second one from completeness of modal canonical rules which we have just established, using the fact that canonical rules $\gamma(F, D)$ which are not based on an intuitionistic frame (i.e., F contains an irreflexive point or a proper cluster) are valid in σW. Thus, by Theorem 2.2, $IPC + \varrho = IPC + A$.

We have shown that any rule ϱ is equivalent to a set R of canonical rules. By definition, this means that ϱ is derivable from R and finitely many initial rules using substitution, cut, and weakening. As such a derivation is finitary, we may assume R to be finite. Moreover, the derivation is checkable by an algorithm, thus the set of pairs $\langle \varrho, R \rangle$ such that R is a finite sequence of canonical rules equivalent to ϱ is recursively enumerable. As every total r.e. relation contains a graph of a recursive function, we can compute one such R from ϱ. (Alternatively, we may construct an algorithm using the effectiveness of Theorem 3.6 and Lemma 3.11.) □

Most results below have a modal version and an intuitionistic version. Often the proofs of both are essentially identical, except that we have to replace $\gamma(F, D)$, $\alpha(F, D)$ with $\delta(F, D)$, $\beta(F, D)$, and omit $\boxdot$'s. In such cases we will generally skip details of the proof of the intuitionistic part to avoid unnecessary cluttering of the text.

Theorem 3.8 implies that domains generated by a reflexive cluster are redundant. In more detail, we have the following.

Proposition 3.14 *If d is generated by a reflexive point, then $K4 + \gamma(F, D \cup \{d\}) = K4 + \gamma(F, D)$, and $IPC + \delta(F, D \cup \{d\}) = IPC + \delta(F, D)$.*

Proof: Let f be a subreduction of W to F with GCDC for D, and r a generator of d. Put

$$g(x) = \begin{cases} f(x), & \text{if } x \in \mathrm{dom}(f), \\ r, & \text{if } x \notin \mathrm{dom}(f),\ f[x\!\uparrow] = d, \\ \text{undefined} & \text{otherwise.} \end{cases}$$

Then $f[x\!\uparrow] = g[x\!\uparrow]$ for any $x \in W$, and it is easy to see that g is a subreduction of W to F with GCDC for $D \cup \{d\}$. □

Definition 3.15 We introduce shorthand notation for some common types of canonical rules by analogy with Zakharyaschev's notation for canonical formulas. We put $\gamma(F, D, \bot) := \gamma(F, D \cup \{\varnothing\})$. We define the *subframe rules* $\gamma(F) := \gamma(F, \varnothing)$, the *cofinal-subframe rules* $\gamma(F, \bot) := \gamma(F, \{\varnothing\})$, the *frame rules* $\gamma^\sharp(F, \bot) := \gamma(F, F^\sharp)$, and the *dense-subframe rules* $\gamma^\sharp(F) := \gamma(F, F^\sharp \smallsetminus \{\varnothing\})$, where $F^\sharp$ is the set of all domains in F. The intuitionistic variants $\delta(F)$, $\delta(F, \bot)$, $\delta^\sharp(F)$, $\delta^\sharp(F, \bot)$ are defined similarly.

We will depict frames using $\bullet$ for irreflexive points, and $\circ$ for reflexive points. We will write $*$ as a shorthand when we intend to include both the reflexive and irreflexive version, for example

$$\gamma^\sharp(*\,*, \bot) = \gamma^\sharp(\bullet\,\bullet, \bot) + \gamma^\sharp(\bullet\,\circ, \bot) + \gamma^\sharp(\circ\,\circ, \bot).$$

Definition 3.16 Let $\langle F, < \rangle$ be a finite Kripke frame, D a set of domains in F, and $X \subseteq F$. We define the *restricted modal canonical rule*

$$\gamma(F, D, X) := (\gamma(F, D))^a \;/\; \{p_i \mid i \in X\},$$

and if F is an intuitionistic frame, the *restricted intuitionistic canonical rule*

$$\delta(F, D, X) := (\delta(F, D))^a \;/\; \{p_i \mid i \in X\}.$$

Lemma 3.17 *Let $\gamma(F, D, X)$ be a restricted modal canonical rule.*

(i) *A frame W refutes $\gamma(F, D, X)$ iff there is a subreduction of W to a generated subframe G of F with GCDC for D, such that $X \subseteq G$.*

(ii) $K4 + \gamma(F, D, X) = K4 + \{\gamma(G, D \restriction G) \mid X \subseteq G \subseteq\cdot F\}$.

(iii) *If $Y \subseteq F$ is such that $X{\updownarrow} = Y{\updownarrow}$, then $K4 + \gamma(F, D, X) = K4 + \gamma(F, D, Y)$.*

The same holds for intuitionistic rules with δ in place of γ.

Proof: (i) The proof of Lemma 3.3 shows that $W \nvDash \gamma(F, D, X)$ iff there exists a subreduction f into (not necessarily onto) F with GCDC for D, such that $\mathrm{rng}(f) \supseteq X$. By condition (ii) of Definition 3.1, $G = \mathrm{rng}(f)$ is a generated subframe of F. Notice that it makes no difference whether we consider GCDC for D relative to F or G, as GCDC trivially holds for all domains d such that $d \nsubseteq G$. Thus, f is a subreduction of W to G with GCDC for $D \restriction G$.

(ii) and (iii) follow from (i). The intuitionistic case is analogous. □

Theorem 3.18 *Let A be a rule system over $K4$. The following are equivalent.*

(i) *A is equivalent to a set of single-conclusion rules.*

(ii) *For any $\gamma(F, D) \in A$, F is nonempty, and whenever $F_1, F_2 \subseteq\cdot F$ are such that $F = F_1 \cup F_2$, then $\gamma(F_\alpha, D \restriction F_\alpha) \in A$ for some $\alpha = 1, 2$.*

(iii) *A is equivalent to a set of restricted canonical rules of the form $\gamma(F, D, \{i\})$, $i \in F$.*

Moreover, we may require in (iii) that D satisfies the conditions from Theorem 3.8, and i belongs to an initial cluster of F.

The same holds for intuitionistic rules with δ in place of γ.

Proof: (iii) → (i) is trivial.

(i) → (ii): The empty frame validates all single-conclusion rules, and refutes $\gamma(\varnothing, D)$.

Assume that $A \nvdash \gamma(F_\alpha, D)$ for each α. Let f_α be a subreduction of a frame $W_\alpha \vDash A$ to F_α with GCDC for D. Put $W = W_1 + W_2$, and $f = f_1 + f_2$. Then $W \vDash A$ by Proposition 2.4, and f is a subreduction of W to F with GCDC for D, hence $A \nvdash \gamma(F, D)$.

(ii) → (iii): By Theorem 3.8, A is axiomatized by canonical rules $\gamma(F, D)$. Let $\gamma(F, D) \in A$, it suffices to show that $\gamma(F, D, \{i\}) \in A$ for some $i \in F$. Assume for contradiction that there is no such i. By Lemma 3.17, for each i there exists a generated subframe $G_i \subseteq\cdot F$ such that $i \in G_i$, and $A \nvdash \gamma(G_i, D \restriction G_i)$. As $\bigcup_i G_i = F$, we obtain $A \nvdash \gamma(F, D)$ by repeated application of (ii), which is a contradiction.

The intuitionistic case is analogous. □

Example 3.19 The introduction of restricted canonical rules is necessary, if we want to axiomatize consequence relations (single-conclusion rule systems) by single-conclusion rules: the set of single-conclusion canonical rules is not complete for such systems. First, notice that a canonical rule $\gamma(F, D)$ is single-conclusion iff F is rooted (hint: if $G \subsetneq F$, the frame G separates $\gamma(G, D \upharpoonright G)$ from $\gamma(F, D)$). Consider the logic

$$D4 = K4 + \Diamond\top,$$

viewed as a rule system. We claim that

$$D4 \vdash \gamma(F, D) \quad \text{iff} \quad \varnothing \in F \text{ and } F \text{ contains a dead end.}$$

On the one hand, if W cofinally subreduces to an F which contains a dead end, then W itself contains a dead end, thus $W \nvDash D4$. On the other hand, if F contains no dead end, then F is a $D4$-frame refuting $\gamma(F, D)$; if F contains a dead end and $\varnothing \notin D$, we may put an extra reflexive point above F.

If F is a rooted frame with a dead end, and $\varnothing \in D$, then $\gamma(F, D)$ is valid in the two-element frame $\bullet\,\circ$, which is not a $D4$-frame. Thus, $D4$ cannot be axiomatized by single-conclusion canonical rules.

Incidentally, we note that the above considerations show

$$D4 = K4 + \gamma(\bullet\,\circ, \bot, \{\bullet\}) = K4 + \gamma(\bullet, \bot) + \gamma(\bullet\,\circ, \bot).$$

Example 3.20 Canonical rules rely on certain phenomena, which a reader familiar with Zakharyaschev's canonical formulas may find surprising. Apart from the fact (discussed above) that frames are no longer rooted, we have rules $\gamma(F, D)$ where F is empty, or where $F \in D$. Both are essential, as we explain below.

There are two canonical rules based on the empty frame, viz.

$$\overline{} = \gamma(\varnothing),$$

$$Con := \frac{\bot}{} = \gamma(\varnothing, \bot).$$

The first one generates the inconsistent rule system, and by Theorem 3.8, it is redundant: we can express it e.g. as

$$\gamma(\varnothing) = \gamma(\bullet\,\circ, \bot, \varnothing) = \gamma(\varnothing, \bot) + \gamma(\bullet) + \gamma(\circ).$$

On the other hand, the *consistency rule*[2] *Con*, which is valid in a frame W iff W is nonempty, cannot be written as a combination of other canonical rules, as it is the only consistent canonical rule with empty conclusion. Such a rule cannot be derived from any set of rules with nonempty conclusions. However, if we intend to deal only with rules with nonempty conclusions, we can dispense with $\gamma(\varnothing, \bot)$.

Rules $\gamma(F, D)$ where $F \in D$ can express "downwards unboundedness" conditions. As an example, the *unboxing rule*

$$Unb := \frac{\Box p}{p} = \gamma^\sharp(\bullet)$$

[2]Named as such because a logic L admits *Con* iff L is consistent.

rule	canonical form
$/$	$\gamma(\varnothing)$
$\bot\,/$	$\gamma(\varnothing, \bot)$
$\Box p \to p\,/\,p$	$\gamma(\circ)$
$\Diamond\top\,/\,\bot$	$\gamma(\circ, \bot)$
$/\,\Box p \to p$	$\gamma(\bullet)$
$\Diamonddot\Box\bot\,/\,\bot$	$\gamma(\bullet, \bot)$
$\Box p\,/\,p$	$\gamma^\sharp(\bullet)$
$\Box\bot\,/\,\bot$	$\gamma^\sharp(\bullet, \bot)$
$/\,\Diamond\top, \Diamonddot\Box\bot$	$\gamma(\bullet\;\circ, \bot)$
$/\,\bot$	$\delta(\circ)$
$/\,p \to q, q \to p$	$\delta(\circ\;\circ)$
$/\,\neg p, \neg\neg p$	$\delta(\circ\;\circ, \bot)$
$p \vee q\,/\,p, q$	$\delta^\sharp(\circ\;\circ)$
$p \vee \neg p\,/\,p, \neg p$	$\delta^\sharp(\circ\;\circ, \bot)$

Table 1: simple canonical rules

is valid in a Kripke frame W iff every irreflexive point $x \in W$ has a predecessor. We cannot express *Unb* using only canonical rules $\gamma(F, D)$ such that $F \notin D$. In fact, *no* such rule is derivable from *Unb*: the frame F° validates *Unb*, and refutes $\gamma(F, D)$.

Another interesting example is the *weak disjunction property rule*

$$WDP := \frac{\boxdot p \vee \boxdot q}{p, q} = \gamma(*\,*, \{\{*\,*\}\}),$$

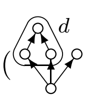
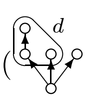
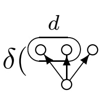
which is valid in a Kripke frame $\langle W, <\rangle$ iff $\leq$ is downwards directed (i.e., every pair of points has a lower bound). The full *disjunction property* rule system axiomatized by

$$(DP) \qquad \frac{\Box p_1 \vee \cdots \vee \Box p_k}{p_1, \ldots, p_k}, \quad k \in \omega,$$

can be written as

$$DP = Con + Unb + WDP = \gamma(\varnothing, \bot) + \gamma^\sharp(\bullet) + \gamma(*\,*, \{\{*\,*\}\}).$$

Example 3.21 Some of the simplest canonical rules are listed in Table 1. For a more complicated example, the Kreisel–Putnam rule can be expressed as

$\delta($$, \{\varnothing, d\}) + \delta($$, \{\varnothing, d\}) + \delta($$, \{\varnothing, d\}).$

Every intermediate logic or normal extension of $K4$ is a rule system, and thus can be axiomatized by canonical rules. In general, we know of no simple method to convert a canonical formula $\alpha(F, D)$ to a set of canonical rules. Roughly speaking, one has to consider variants of $\gamma(F, D)$ where extra points are inserted in "harmless" places in F. However, cofinal-subframe logics can be axiomatized by cofinal-subframe canonical rules in an easy way. We first check that Zakharyaschev's proof of the finite model property for cofinal-subframe logics [38] carries on to rule systems.

Lemma 3.22 *A transitive rule system axiomatized by subframe and cofinal-subframe canonical rules has the finite model property.*

Proof: Let A be the rule system, and assume that $\varrho \notin A$. By Theorem 3.8, there exists a canonical rule $\pi = \gamma(F, D)$ (or $\delta(F, D)$, in the intuitionistic case) such that $\varnothing \in D$, $\varrho \vdash \pi$, and $\pi \notin A$. The latter implies that there exists a frame $W \vDash A$, and a globally cofinal (i.e., with GCDC for $\varnothing \in D$) subreduction f of W to F. As $F \nvDash \varrho$, it suffices to show that $F \vDash A$. Let $\sigma = \gamma(G, E) \in A$, where $E = \varnothing$ or $E = \{\varnothing\}$. If we assume for contradiction that $F \nvDash \sigma$, there exists a (globally cofinal if $E \neq \varnothing$) subreduction g of F to G. Then $g \circ f$ is a (globally cofinal if appropriate) subreduction of W to G, hence $W \nvDash \sigma$, a contradiction. $\square$

Lemma 3.23 *Let L be a transitive logic. If L has the finite model property as a logic, then it also has the finite model property as a rule system.*

Proof: Let $L \nvDash \varrho$. For each $\psi \in \varrho^c$, $L \nvDash \boxdot \bigwedge \varrho^a \to \psi$, hence there exists a finite rooted L-model W_ψ which refutes $\boxdot \bigwedge \varrho^a \to \psi$ in the root. Then $W := \sum_\psi W_\psi \vDash L$, and $W \nvDash \varrho$. The intuitionistic case is analogous. $\square$

Proposition 3.24 *Let F be a finite rooted transitive frame.*

(i) $K4 + \alpha(F) = K4 + \gamma(F)$, $IPC + \beta(F) = IPC + \delta(F)$,

(ii) $IPC + \beta(F, \bot) = IPC + \delta(F, \bot)$,

(iii) $K4 + \alpha(F, \bot) = K4 + \gamma(F \bullet \circ, \bot, F)$.

Moreover, we may omit the extra $\bullet$ or $\circ$ in (iii) if F has an irreflexive or reflexive final singleton cluster.

Proof: (i) is obvious from the refutation criteria.

(ii), (iii): By Lemmas 3.22 and 3.23, it suffices to show that both sides have the same finite models. Given a locally cofinal subreduction of a finite frame to F, we extend it to a globally cofinal subreduction to $F \bullet \circ$ by mapping all final clusters to a final singleton cluster, reflexive or irreflexive as needed. On the other hand, if f a globally cofinal subreduction to $F \bullet \circ$, we pick a point x such that $f(x)$ is the root of F, and we restrict f to $x{\uparrow}$ to obtain a locally cofinal subreduction to F. $\square$

For example, apart from the above mentioned $D4 = K4 + \gamma(\bullet \circ, \bot, \{\bullet\})$, we have

$$K4.1 := K4 + \boxdot\Diamond p \to \dot{\Diamond}\Box p = K4 + \gamma(\circ\!\leftrightarrow\!\circ_0 \circ \bullet, \bot, \{0\}),$$

$$K4.2 := K4 + \Diamond\boxdot p \to \Box\dot{\Diamond} p = K4 + \gamma(* \, {}^{*}\!\nwarrow\!*_a\!\nearrow\!{}^{*}, \bot, \{a\}).$$

4 Admissible rules

We are going to apply canonical rules to the analysis of admissible rules in some well-known logics. The basic idea is that subreductions to frames with sufficiently many tight predecessors (see below) can be lifted up from a generated subframe, which implies that the corresponding canonical rule is assumption-free using Proposition 2.5. However, we begin with a few general observations on admissibility and derivability of canonical rules.

Lemma 4.1 *A normal extension L of $K4$ derives $\gamma(F,D)$ if and only if $\alpha(G, D \restriction G) \in L$ for some rooted generated subframe G of F.*

A si logic L derives $\delta(F,D)$ if and only if $\beta(G, D \restriction G) \in L$ for some rooted generated subframe G of F.

Proof: Comparison of the refutation conditions shows immediately that $K4 + \alpha(G, D \restriction G)$ derives $\gamma(F,D)$. Conversely, if L derives $\gamma(F,D)$, then it also derives $\gamma(F,D,\{i\})$ for some i by Theorem 3.18. In particular, $L \vdash \gamma(G, D \restriction G)$ for $G = i\uparrow$, hence $L \vdash \alpha(G, D \restriction G) = \boxdot(\gamma(G, D \restriction G))^a \to (\gamma(G, D \restriction G))^c$ by Proposition 2.5.

The intuitionistic case is analogous. □

The following holds despite Example 3.19.

Proposition 4.2 *A normal extension L of $K4$ is structurally complete if and only if for every rooted F, if $\gamma(F,D) \in \mathrel{|\!\!\sim}_L$, then $\alpha(F,D) \in L$.*

A si logic L is structurally complete if and only if for every rooted F, if $\delta(F,D) \in \mathrel{|\!\!\sim}_L$, then $\beta(F,D) \in L$.

Proof: The left-to-right implication follows from Lemma 4.1. Assume that the RHS holds, and L admits a single-conclusion rule ϱ. By Theorem 3.18, we may assume $\varrho = \gamma(F,D,\{i\})$. Let G be the subframe of F generated by i, and $D' = D \restriction G$. As L admits $\gamma(G,D')$, our assumption gives $\alpha(G,D') \in L$, hence L derives ϱ by Lemma 4.1.

The intuitionistic case is analogous. □

Definition 4.3 A transitive logic L has the *rule dichotomy property*, if every rule is over L equivalent to a set of rules which are either L-admissible or assumption-free.

L has the *single-conclusion rule dichotomy property*, if every single-conclusion rule is over L equivalent to a set of single-conclusion rules which are either L-admissible or assumption-free (i.e., formulas).

L has the *strong rule dichotomy property*, if every canonical rule is L-admissible or equivalent over L to an assumption-free rule.

L has the *strong single-conclusion rule dichotomy property*, if every restricted canonical rule $\gamma(F,D,\{i\})$ (or $\delta(F,D,\{i\})$ in the intuitionistic case) is L-admissible or equivalent over L to a formula.

Observation 4.4 *If L has the strong (single-conclusion) rule dichotomy property, then it has the (single-conclusion) rule dichotomy property.*

If L has the (strong) rule dichotomy property, and M inherits L-admissible rules, then M has the (strong) rule dichotomy property.

If L has the (strong) single-conclusion rule dichotomy, and M inherits L-admissible single-conclusion rules, then M has the (strong) single-conclusion rule dichotomy. □

Corollary 4.5 *If L has the (single-conclusion) rule dichotomy property, and $M \supseteq L$, then (single-conclusion) M-admissible rules have a basis consisting of rules admissible in both L and M.*

Proof: M-admissible assumption-free rules are M-derivable, thus redundant in a basis. □

Proposition 4.6 *If a logic L has the strong rule dichotomy property, then it has the single-conclusion rule dichotomy property. In more detail, if ϱ is a single-conclusion rule, there exists a formula φ, and restricted canonical rules $\varrho_i = \gamma(F_i, D_i, \{\alpha_i\})$, $i < k$ (resp. $\delta(F_i, D_i, \{\alpha_i\})$ in the intuitionistic case), such that*

$$L + \varrho = L + \varphi + \{\varrho_i \mid i < k\},$$

and $\varrho_i \in\, \mathrel{|\!\!\sim}_L$ for every $i < k$.

Proof: Let $\gamma(F, D) \in L + \varrho$. By Theorem 3.18, $\gamma(F, D, \{\alpha\}) \in L + \varrho$ for some $\alpha \in F$. If $\gamma(F, D, \{\alpha\}) \in\, \mathrel{|\!\!\sim}_L$, we are done. Otherwise Lemma 3.17 implies that there exists a $G \subseteq\!\cdot\, F$ such that $\alpha \in G$, and $\pi := \gamma(G, D \restriction G)$ is assumption-free over L. We have

$$\frac{}{\{\boxdot\pi^a \to \psi \mid \psi \in \pi^c\}} \in L + \pi$$

by Proposition 2.5. As $\pi \in L + \varrho$ by Lemma 3.17, we obtain $L + \varrho \vdash \varphi := (\boxdot\pi^a \to \psi)$ for some $\psi \in \pi^c$ by Proposition 2.4. We have $L + \varphi \vdash \gamma(F, D)$ by Lemma 4.1.

The intuitionistic case is analogous. □

Remark 4.7 It is not clear whether we can weaken the assumption of Proposition 4.6 to the rule dichotomy property, or whether we can strengthen the conclusion to the strong single-conclusion dichotomy property.

The rule dichotomy property, specifically Corollary 4.5, generalizes a property noticed by Iemhoff [15], which states that Visser's rules (i.e., a basis of admissible rules of *IPC*) are a basis of admissible rules for any si logic where they are admissible. This follows easily from the characterization of admissibility in terms of projective formulas, using the fact that projectivity is preserved in extensions (thus it also holds for modal logics where the projectivity-based characterization works, see [18]). Corollary 4.5 shows that a similar property holds even for extensions which do not inherit the admissible rules of the base logic.

We are going to establish rule dichotomy for several basic modal logics as well as intuitionistic logic. We obtain for free an alternative proof of decidability of admissibility in these logics (originally due to Rybakov, see [29]), and an explicit description of bases (after Iemhoff [14] and Jeřábek [18, 21]). We first introduce some concepts from [14, 18].

logic	IPC	$K4$	GL	$S4$	$K4.3$	$GL.3$	$S4.3$
basis	V	$A^\bullet + A^\circ$	$A^\bullet$	A°	$A^{\bullet,1} + A^{\circ,1}$	$A^{\bullet,1}$	$A^{\circ,1}$

Table 2: bases of admissible rules

Definition 4.8 We define the sets of rules

$$(A^\bullet)\qquad \Box q \to \bigvee_{i<n} \Box p_i \Big/ \{\boxdot q \to p_i \mid i < n\},$$

$$(A^\circ)\qquad \boxdot(q \leftrightarrow \Box q) \to \bigvee_{i<n} \Box p_i \Big/ \{\boxdot q \to p_i \mid i < n\},$$

$$(V)\qquad \bigwedge_{j<n} (p_j \to q_j) \to \bigvee_{i<n+m} p_i \Big/ \Big\{ \bigwedge_{j<n} (p_j \to q_j) \to p_i \Bigm| i < n+m \Big\},$$

where $n, m \in \omega$. Let $A^{\circ,1}$ and $A^{\bullet,1}$ be the restrictions of A° and $A^\bullet$ to $n \leq 1$.

Let $\langle W, < \rangle$ be a Kripke frame, and $X \subseteq W$. A point $t \in W$ is an *irreflexive tight predecessor* ($\bullet$*-tp*) *of* X, if $t{\uparrow} = X{\Uparrow}$, and it is a *reflexive tight predecessor* ($\circ$*-tp*) *of* X, if $t{\uparrow} = \{t\} \cup X{\Uparrow}$. (Notice that if X is generated by a reflexive point r, then r is both a $\circ$-tp and an $\bullet$-tp of X; in particular, irreflexive tight predecessors do not have to be irreflexive points.)

Notice that a subframe logic L derives the rule $\gamma(F, D)$ iff $F \nvDash L$.

Theorem 4.9 *If L is one of the logics IPC, $K4$, GL, $S4$, $K4.3$, $S4.3$, or $GL.3$, then L has the strong rule dichotomy property. In more detail, the following are equivalent for modal L.*

(i) *$\gamma(F, D)$ is derivable in L, or it is not assumption-free over L.*

(ii) *$\gamma(F, D)$ is admissible in L.*

(iii) *$\gamma(F, D)$ is derivable in the rule system axiomatized over L by the rules given in Table 2.*

(iv) *$F \nvDash L$, or there exists $d \in D$ and $* \in \{\bullet, \circ\}$ such that $d^* \vDash L$, and d has no $*$-tp in F.*

The same holds for IPC with $\delta(F, D)$ in place of $\gamma(F, D)$.

Proof: (ii) $\to$ (i): An admissible assumption-free rule is derivable by the definition of admissibility.

(iii) $\to$ (ii): Let $* \in \{\bullet, \circ\}$ be an L-frame, and $n \in \omega$ be such that $n \leq 1$ if $L \supseteq K4.3$. Consider a substitution σ such that $\nvdash_L \sigma(\boxdot q \to p_i)$ for each $i < n$. Fix finite rooted Kripke L-models $W_i, x_i \Vdash \sigma(\boxdot q \wedge \neg p_i)$, and put $W = (\sum_i W_i)^*$, with r being its root. Then $W \vDash L$, $W, r \Vdash \sigma\big(\Box q \wedge \bigwedge_i \neg \Box p_i\big)$ if $* = \bullet$, and $W, r \Vdash \sigma\big(\boxdot(q \leftrightarrow \Box q) \wedge \bigwedge_i \neg \Box p_i\big)$ if $* = \circ$. The intuitionistic case is similar.

(iv) $\to$ (iii): Let A be the rule system from Table 2, and assume that there exists a frame $W \vDash L + A$, and a subreduction f of W to F with GCDC for D. Clearly $F \vDash L$, as L is a subframe logic. Let $d \in D$, and $* \in \{\bullet, \circ\}$ be such that $d^* \vDash L$, we will show that d has a $*$-tp in F. Fix $d' \subseteq d$ such that $d'\Uparrow = d$, and $|d'| \le 1$ if $L \supseteq K4.3$. Let $\Vdash$ be the valuation such that $\Vdash(p_i) = W \smallsetminus f^{-1}[i]$. For each $i \in d'$, there exists $x_i \Vdash \neg p_i \wedge \boxdot \bigwedge_{j \notin d} p_j$ as f is onto.

If $* = \bullet$, we can use the validity of $A^\bullet$ (or $A^{\bullet,1}$, as appropriate) in W to find $x \in W$ such that

$$x \Vdash \Box \bigwedge_{j \notin d} p_j \wedge \bigwedge_{i \in d'} \neg \Box p_i.$$

By the definition of $\Vdash$, we obtain $f[x\uparrow] = d$. We have $x \in \mathrm{dom}(f)$ by GCDC for d. If $f(x) = i$, then $i\uparrow = d$, hence i is an $\bullet$-tp of d.

If $* = \circ$, let S be a selector on the clusters of F, and define

$$x \Vdash q \quad \text{iff} \quad x \notin \mathrm{dom}(f) \vee f(x) \in d \vee \exists i \gneq f(x)\,(i \notin d \wedge i \in S).$$

If $f(x) = i \notin d$, let $j \ge i$ be in a maximal cluster such that $j \notin d$, and w.l.o.g. $j \in S$. Then $y \nVdash q$ for any $y \ge x$ such that $f(y) = j$. It follows that

$$x \Vdash \boxdot q \quad \text{iff} \quad f[x\Uparrow] \subseteq d.$$

As $x_i \Vdash \neg p_i \wedge \boxdot q$ for each $i \in d'$, there exists an $x \in W$ such that

$$x \Vdash \boxdot(q \leftrightarrow \Box q) \wedge \bigwedge_{i \in d'} \neg \Box p_i$$

by validity of A° (or $A^{\circ,1}$) in W. Clearly $f[x\uparrow] \supseteq d$. If $f[x\Uparrow] \subseteq d$, then $x \in \mathrm{dom}(f)$ by GCDC, hence $f(x) \in d$ is a $\circ$-tp of d. Otherwise $x \nVdash \boxdot q$, which implies $x \nVdash \Box q$ and $x \nVdash q$. The latter implies $x \in \mathrm{dom}(f)$, thus let $i = f(x)$. We have $i \notin d$, thus let $j \ge i$ be in a maximal cluster such that $j \notin d$, and $j \in S$. If $j \ne i$, we have $x \Vdash q$, a contradiction. Thus $j = i$, which means that $i\Uparrow = d \cup \mathrm{cl}(i)$. If i is irreflexive, then $x \Vdash \Box q$, a contradiction. If $\mathrm{cl}(i)$ is proper, then $y \Vdash q \wedge \neg \Box q$ for any $y > x$ such that $f(y) \in \mathrm{cl}(i) \smallsetminus \{i\}$. Hence $\mathrm{cl}(i)$ is a reflexive singleton, which means that i is a $\circ$-tp of d.

In the intuitionistic case, we consider $W \vDash V$, a subreduction f of W to F with GCDC for D, $d \in D$, and the valuation $\Vdash$ such that $\Vdash(p_i) = W \smallsetminus f^{-1}[i]\downarrow$. For each $i \in d$, there exists $x_i \nVdash \bigwedge_{j \notin d} p_j \to p_i$, hence also $x_i \nVdash \bigwedge_{j \notin d, k \in d}(p_k \to p_j) \to p_i$. Using V, we obtain an $x \in W$ such that

$$(*) \qquad x \Vdash \bigwedge_{\substack{j \notin d \\ k \in d}} (p_k \to p_j), \qquad x \nVdash \bigvee_{i \in d} p_i.$$

The latter implies $f[x\uparrow] \supseteq d$. If $f[x\uparrow] = d$, then there exists an i such that $f(x) = i$ by GCDC, hence i is a $\circ$-tp of $d = i\uparrow$. Otherwise we can pick a maximal $j \in f[x\uparrow] \smallsetminus d$, and $y \ge x$ such that $f(y) = j$. We have $y \nVdash p_k$ for any $k \in D$ by $(*)$, hence $j\uparrow = f[y\uparrow] \supseteq d$. On the other hand, $j\uparrow \subseteq d \cup \{j\}$ by the choice of j, hence j is a $\circ$-tp of d.

(i) → (iv): assume that $F \vDash L$, and all $d \in D$ such that d^* have a $*$-tp in F. The former implies $L \nvdash \gamma(F, D)$. We will show that $\gamma(F, D)$ is assumption-free over L using the criterion of Proposition 2.5. Assume that $W \vDash L$, $U \subseteq\cdot\, W$, and f is a subreduction of U to F with GCDC for D, we want to extend f to W. We may assume w.l.o.g. that no $d \in D$ is generated by a reflexive point. Let $\{d_i \mid i < n\}$ be an enumeration of D such that $d_i \subseteq d_j$ only if $i \leq j$. By induction on $k \leq n$, we will construct a subreduction f_k of W to F with GCDC for $D_k := \{d_i \mid i < k\}$ such that $f_k \restriction U = f$, and the corresponding valuation $\Vdash_k$ on W refuting $\gamma(F, D_k)$.

$k = 0$: Let $\Vdash^0$ be a valuation in W such that

$$x \nVdash^0 p_i \Leftrightarrow f(x) = i$$

for every $x \in U$. Define

$$x \Vdash_0 p_i \Leftrightarrow x \Vdash^0 \boxdot(\gamma(F))^a \to p_i,$$
$$f_0(x) = i \Leftrightarrow x \nVdash_0 p_i.$$

Then f_0 is a subreduction, and $f = f_0 \restriction U$ as $(\gamma(F))^a$ is valid in U under $\Vdash$.

$k \mapsto k+1$: Let t be an $\bullet$-tp of d_k in F if one exists, otherwise we adjoin a new point t to F which makes an $\bullet$-tp of d_k, and extend $\Vdash_k$ to make p_t true everywhere in W. Let s be a $\circ$-tp of d_k constructed in a similar way. As d_k does not have a reflexive root, we have $t, s \notin d_k$, and $t \nleq s \nleq t$. Put $F^+ = F \cup \{t, s\}$,

$$\varphi_k = \bigwedge_i p_i \wedge \bigwedge_{i \notin d_k} \Box p_i \to \bigvee_{i \in d_k} \Box p_i,$$

$$x \Vdash^{k+1} p_i \Leftrightarrow \begin{cases} x \Vdash_k p_i, & i \neq t, s, \\ x \Vdash_k p_i \wedge (\Box\varphi_k \to \varphi_k), & i = t, \\ x \Vdash_k p_i \wedge (\boxdot(\Box\varphi_k \to \varphi_k) \to \varphi_k), & i = s, \end{cases}$$

$$x \Vdash_{k+1} p_i \Leftrightarrow x \Vdash^{k+1} \boxdot(\gamma(F^+))^a \to p_i,$$

$$f_{k+1}(x) = i \Leftrightarrow x \nVdash_{k+1} p_i.$$

Clearly, f_{k+1} is a subreduction of W to $\mathrm{rng}(f_{k+1})$. L is a subframe logic, hence $\mathrm{rng}(f_{k+1}) \vDash L$. As d_k has a $*$-tp in F whenever $d_k^* \vDash L$, we must have $\mathrm{rng}(f_{k+1}) \subseteq F$, and we may forget about the extra points. For any $x \in W$, f_{k+1} coincides with f_k on $x{\Uparrow}$ unless there exists $y \geq x$ such that $y \Vdash_k p_i$ and $y \nVdash^{k+1} p_i$, where $i \in \{t, s\}$. In particular, $y \nVdash_k \varphi_k$, whence it is easy to see that $y \Vdash^{k+1} \boxdot(\gamma(F^+))^a$, and

$$f_k[y{\uparrow}] = d_k, \quad y \notin \mathrm{dom}(f_k), \quad f_{k+1}(y) \in \{t, s\}.$$

In particular, $f_{k+1} \restriction U = f_k \restriction U = f$, as f satisfies GCDC for D. As $\mathrm{rng}(f_k) = F$, f_{k+1} is a subreduction to F.

We need to show that f_{k+1} satisfies GCDC for D_{k+1}. Assume for contradiction that $x \notin \mathrm{dom}(f_{k+1})$, and $f_{k+1}[x{\uparrow}] = d_i$ for some $x \in W$, and $i \leq k$. As d_i is not a proper superset of d_k, we have $t, s \notin f_{k+1}[x{\uparrow}]$, thus $f_{k+1} \restriction x{\Uparrow} = f_k \restriction x{\Uparrow}$. Therefore $f_k[x{\uparrow}] = d_i$, and $x \notin \mathrm{dom}(f_k)$. If $i < k$, this contradicts GCDC of f_k for d_i. If $i = k$, then $x \nVdash_k \varphi_k$. Either $y \nVdash_k \Box\varphi_k \to \varphi_k$ for some $y \geq x$, in which case $f_{k+1}(y) = t$, or $x \Vdash_k \boxdot(\Box\varphi_k \to \varphi_k)$, in which case $f_{k+1}(x) = s$. Both possibilities contradict either $f_{k+1}[x{\uparrow}] = d_k$ or $x \notin \mathrm{dom}(f_{k+1})$.

In the intuitionistic case, we take $\Vdash^0$ such that $\Vdash^0(p_i) \cap U = U \smallsetminus f^{-1}[i]{\downarrow}$, and define

$$f_0(x) = i \Leftrightarrow x \nVdash^0 p_i, x \Vdash^0 (\delta(F))^a \wedge \bigwedge_{j \ngeq i} p_j,$$

$$x \Vdash_0 p_i \Leftrightarrow x \Vdash^0 (\delta(F))^a \to p_i.$$

In the step from k to $k+1$, we take s to be a $\circ$-tp of d_k, and define

$$\varphi_k = \bigwedge_{i \notin d_k} p_i \to \bigvee_{i \in d_k} p_i,$$

$$x \Vdash^{k+1} p_i \Leftrightarrow \begin{cases} x \Vdash_k p_i, & i \neq s, \\ x \Vdash_k p_i \wedge \varphi_k, & i = s, \end{cases}$$

$$f_{k+1}(x) = i \Leftrightarrow x \nVdash^{k+1} p_i, x \Vdash^{k+1} (\delta(F))^a \wedge \bigwedge_{j \ngeq i} p_j,$$

$$x \Vdash_{k+1} p_i \Leftrightarrow x \Vdash^{k+1} (\delta(F))^a \to p_i.$$

Verification of the properties of f_k is then analogous to the modal case. □

We formulated Theorem 4.9 for seven basic logics for convenience, nevertheless Observation 4.4 immediately gives a generalization to all so-called extensible and linear extensible logics [18, 19] (including, e.g., *Grz*, *S*4.1, and all extensions of *S*4.3):

Corollary 4.10 *If a logic L inherits multiple-conclusion rules admissible in $L_0 = IPC$, $K4$, GL, $S4$, $K4.3$, $S4.3$, or $GL.3$, then L has the strong rule dichotomy property, and a recursive basis of admissible multiple-conclusion rules consisting of the basis for L_0 from Table 2.* □

It is easy to see from the definition that admissibility in any decidable logic L is Π^0_1. If furthermore L has a Σ^0_1 basis of admissible rules, then admissibility in L is Σ^0_1, hence it is decidable. Thus admissibility of multiple-conclusion rules in decidable logics L meeting the assumption of Corollary 4.10 is decidable. We can give a more concrete algorithm as follows. Given a rule ϱ, we first compute a set of canonical rules $\{\gamma(F_i, D_i) \mid i < n\}$ equivalent to ϱ over L_0 by Theorem 3.8. Then L admits ϱ iff it admits all $\gamma(F_i, D_i)$. Using (iv) of Theorem 4.9, we see that $\gamma(F_i, D_i)$ is L-admissible iff there is $d \in D_i$ and $* \in \{\bullet, \circ\}$ such that $d^* \vDash L_0$ and d has no $*$-tp in F_i, or $\gamma(F_i, D_i)$ is derivable in L. By Lemma 4.1, the latter is equivalent to $\vdash_L \alpha(G, D_i \restriction G)$ for some rooted generated subframe G of F_i, which we can check using a decision algorithm for L. However, this algorithm is only of theoretical interest because of its prohibitive complexity: already the first step (expressing ϱ in terms of canonical rules) is nonelementary.

The above mentioned logics have the single-conclusion rule dichotomy property by Proposition 4.6. We can do better by a direct proof:

Theorem 4.11 *If a logic L inherits single-conclusion admissible rules of IPC, $K4$, GL, $S4$, $K4.3$, $S4.3$, or $GL.3$, then L has the strong single-conclusion rule dichotomy property.*

Proof: By Observation 4.4, we may assume L to be one of the seven basic systems named in the theorem. We will concentrate on the modal cases, the intuitionistic case is analogous.

Consider a restricted canonical rule $\gamma(F, D, \{i\})$. If all the rules $\gamma(G, D \upharpoonright G)$ such that $i \in G \subseteq\cdot F$ are L-admissible, then $\gamma(F, D, \{i\})$ is also L-admissible. Otherwise there exists a G such that $i \in G \subseteq\cdot F$, and $\gamma(G, D \upharpoonright G)$ is assumption-free over L, and nonderivable in L. Fix an L-frame V, and a subreduction g of V to G with GCDC for D. We will show that $\gamma(F, D, \{i\})$ is assumption-free over L, using Proposition 2.5.

Let W be an L-frame, and $U \subseteq\cdot W$ such that $U \nvDash \gamma(F, D, \{i\})$. Fix a subreduction f of U to $H \subseteq\cdot F$, $i \in H$, with GCDC for D. The restriction of f to

$$U' = \{x \in U \mid f[x\updownarrow] \subseteq G\}$$

is a subreduction of $U' \subseteq\cdot W$ to $G \cap H$ with GCDC for D, thus we may assume $H \subseteq G$ w.l.o.g. Then $f + g$ is a subreduction of $U + V$ to G with GCDC for D. By the proof of Theorem 4.9, there exists a subreduction h of $W + V$ to G with GCDC for D such that $h \supseteq f + g$. The restriction $h \upharpoonright W$ is a subreduction of W to $K \subseteq\cdot F$ with GCDC for D. As $h \supseteq f$, we have $i \in K$, thus W refutes $\gamma(F, D, \{i\})$. $\Box$

Similar to Corollary 4.10, we can extend the proof of Theorem 4.11 to provide explicit bases, which also implies decidability of admissibility of single-conclusion rules in decidable logics meeting the assumption of the theorem.

Notice that Corollary 4.10 and Theorem 4.11 cover the known analysis (constructions of explicit bases of admissible rules and proofs of decidability) of admissible rules following the strategy of projective formulas, as given in [14, 15, 16, 18]. The methods of Rybakov [29] show decidability of admissibility for a larger class of logics, however no bases of admissible rules are known for these logics. Just like with the projectivity-based approach, it is not clear whether we can generalize the analysis of admissibility based on canonical rules to a wider class of logics. Presumably more refined techniques might be needed, as the rule dichotomy is a very strong property which is unlikely to hold for a substantial class of logics.

5 Blok–Esakia isomorphism

In this section we investigate modal companions of intuitionistic rule systems. We will show that every rule system has the smallest and the largest modal companion (the latter being the former plus *Grz*), we establish a Blok–Esakia isomorphism of $\mathrm{Ext}_m\, IPC$ and $\mathrm{Ext}_m\, Grz$, and we describe the companion maps in terms of their frames, and their canonical form. Recall that the *Grzegorczyk logic* is

$$\begin{aligned} Grz &= K + \Box(\Box(p \to \Box p) \to p) \to p = S4 + \Box(\Box(p \to \Box p) \to p) \to p \\ &= K4 + \alpha(\circ\!\!\leftrightarrow\!\!\circ) + \alpha(\bullet) = K4 + \gamma(\circ\!\!\leftrightarrow\!\!\circ) + \gamma(\bullet). \end{aligned}$$

Definition 5.1 If A is a rule system over $S4$, we define

$$\varrho A = \{\pi \mid \mathsf{T}\pi \in A\}.$$

We say that A is a *modal companion* of ϱA. If B is an intuitionistic rule system, we put

$$\tau B = S4 + \{\mathsf{T}\pi \mid \pi \in B\},$$
$$\sigma B = Grz + \tau B.$$

Lemma 5.2 *$\varrho\colon \mathrm{Ext}_m\, S4 \to \mathrm{Ext}_m\, IPC$, $\tau\colon \mathrm{Ext}_m\, IPC \to \mathrm{Ext}_m\, S4$, and $\sigma\colon \mathrm{Ext}_m\, IPC \to \mathrm{Ext}_m\, Grz$ are monotone maps.*

Proof: We observe that ϱA is a rule system. The rest follows trivially from the definition. □

Lemma 5.3 *If F is a finite intuitionistic frame, and D a set of domains in F, then $S4 + \mathsf{T}\delta(F, D) = S4 + \gamma(F, D)$.*

Proof: Follows from Lemma 3.13, as $S4$ is complete wrt preordered frames. □

Lemma 5.4

(*i*) *If B is a rule system over IPC, then $\varrho\tau B = \varrho\sigma B = B$.*

(*ii*) *If A is a rule system over $S4$, then $\tau\varrho A \subseteq A \subseteq \sigma\varrho A$.*

Proof: (i) $B \subseteq \varrho\tau B \subseteq \varrho\sigma B$ is obvious. If $\pi \notin B$, there exists a frame $W \vDash B$ such that $W \nvDash \pi$, thus $\sigma W \vDash \sigma B$ and $\sigma W \nvDash \mathsf{T}\pi$ by Lemma 3.13. Consequently $\mathsf{T}\pi \notin \sigma B$, thus $\pi \notin \varrho\sigma B$.

(ii) $\tau\varrho A \subseteq A$ is trivial from the definition. Assume that $\gamma(F, D) \in A$. If F is not an intuitionistic frame (i.e., it contains an irreflexive point, or a proper cluster), then $\gamma(F, D) \in Grz \subseteq \sigma\varrho A$. Otherwise $\mathsf{T}\delta(F, D) \in A$ by Lemma 5.3, hence $\delta(F, D) \in \varrho A$ and $\mathsf{T}\delta(F, D) \in \tau\varrho A$, which implies $\gamma(F, D) \in \tau\varrho A \subseteq \sigma\varrho A$. It follows that $A \subseteq \sigma\varrho A$ using Theorem 3.8. □

Theorem 5.5

(*i*) *$A \in \mathrm{Ext}_m\, S4$ is a modal companion of $B \in \mathrm{Ext}_m\, IPC$ if and only if $\tau B \subseteq A \subseteq \sigma B$.*

(*ii*) *The mappings σ and $\varrho \restriction \mathrm{Ext}_m\, Grz$ are mutually inverse isomorphisms of the lattices $\mathrm{Ext}_m\, IPC$ and $\mathrm{Ext}_m\, Grz$.*

Proof: (i) On the one hand, any rule system between τB and σB is a companion of B by Lemma 5.4 (i). On the other hand, if $\varrho A = B$, then $\tau B \subseteq A \subseteq \sigma B$ by Lemma 5.4 (ii).

(ii) We have $\varrho\sigma B = B$, and $\sigma\varrho A = Grz + \tau\varrho A \subseteq A \subseteq \sigma\varrho A$ by Lemma 5.4, whenever $A \supseteq Grz$. Thus σ and $\varrho \restriction \mathrm{Ext}_m\, Grz$ are mutually inverse bijections. Being monotone, they are also order isomorphisms, hence lattice isomorphisms. □

Theorem 5.6 *Let $\{F_i \mid i \in I\}$ be a sequence of finite intuitionistic frames, $\{F_i \mid i \in J\}$ a sequence of finite transitive frames which are not intuitionistic frames, and $\{D_i \mid i \in I \cup J\}$ an appropriate sequence of sets of domains. Then*

$$(i) \qquad \varrho\Big(S4 + \sum_{i \in I \cup J} \gamma(F_i, D_i)\Big) = IPC + \sum_{i \in I} \delta(F_i, D_i),$$

$$(ii) \qquad \tau\Big(IPC + \sum_{i \in I} \delta(F_i, D_i)\Big) = S4 + \sum_{i \in I} \gamma(F_i, D_i),$$

$$(iii) \qquad \sigma\Big(IPC + \sum_{i \in I} \delta(F_i, D_i)\Big) = Grz + \sum_{i \in I} \gamma(F_i, D_i).$$

Proof: (i): On the one hand, $\delta(F_i, D_i) \in \varrho(S4 + \gamma(F_i, D_i))$ by Lemma 5.3. On the other hand, the same Lemma implies

$$S4 + \sum_{i \in I \cup J} \gamma(F_i, D_i) \subseteq Grz + \sum_{i \in I} \gamma(F_i, D_i) \subseteq \sigma\Big(IPC + \sum_{i \in I} \delta(F_i, D_i)\Big),$$

hence using Lemma 5.4,

$$\varrho\Big(S4 + \sum_{i \in I \cup J} \gamma(F_i, D_i)\Big) \subseteq \varrho\sigma\Big(IPC + \sum_{i \in I} \delta(F_i, D_i)\Big) = IPC + \sum_{i \in I} \delta(F_i, D_i).$$

(ii): We have $\gamma(F_i, D_i) \in \tau(IPC + \delta(F_i, D_i))$ by Lemma 5.3. On the other hand,

$$\tau\Big(IPC + \sum_{i \in I} \delta(F_i, D_i)\Big) = \tau\varrho\Big(S4 + \sum_{i \in I} \gamma(F_i, D_i)\Big) \subseteq S4 + \sum_{i \in I} \gamma(F_i, D_i)$$

by (i), and Lemma 5.4.

(iii) follows from (ii). □

Definition 5.7 Let $\langle W, \leq, P\rangle$ be an intuitionistic frame, and $0 < k < \omega$. We define the modal frame $\tau_k W = \langle kW, k{\leq}, kP\rangle$, where $kW = W \times k$, $\langle x, i\rangle \; k{\leq} \; \langle y, j\rangle$ iff $x \leq y$, and

$$kP = \Big\{ \bigcup_{i<k} (X_i \times \{i\}) \;\Big|\; \forall i < k \; X_i \in \sigma P \Big\}.$$

Notice that $\tau_1 W \simeq \sigma W$, and $W \simeq \varrho\tau_k W$.

Lemma 5.8 *If F is a finite preordered frame with clusters of size at most $k < \omega$, and $\sigma W \nvDash \gamma(\varrho F, \varrho D)$, then $\tau_k W \nvDash \gamma(F, D)$.*

Proof: Let f be a subreduction of σW to F with GCDC for D. Enumerate (not necessarily injectively) each cluster $c \in \varrho F$ as $c = \{u_{c,i} \mid i < k\}$. Define a partial function $g \colon \tau_k W \to F$ by

$$g(x, i) = u_{c,i} \quad \text{iff} \quad f(x) = c.$$

Then it is easy to see that g is a subreduction of $\tau_k W$ to F with GCDC for D. □

Theorem 5.9 *Let $\mathcal{C}$ be a class of preordered frames, $\mathcal{D}$ a class of intuitionistic frames, $A \in \mathrm{Ext}_m\, S4$, and $B \in \mathrm{Ext}_m\, IPC$.*

(i) If $A = \mathrm{Th}_m(\mathcal{C})$, then $\varrho A = \mathrm{Th}_m(\{\varrho W \mid W \in \mathcal{C}\})$.

(ii) If $B = \mathrm{Th}_m(\mathcal{D})$, then $\sigma B = \mathrm{Th}_m(\{\sigma W \mid W \in \mathcal{D}\})$.

(iii) If $B = \mathrm{Th}_m(\mathcal{D})$, then $\tau B = \mathrm{Th}_m(\{\tau_k W \mid W \in \mathcal{D}, 0 < k < \omega\})$.

Proof: The inclusion $\subseteq$ in all three cases follows from Lemma 3.13, using $\varrho\sigma W \simeq \varrho\tau_k W \simeq W$. We will to show the other inclusion.

(i) If $\pi \notin \varrho A$, then $\mathsf{T}\pi \notin A$, hence there exists $W \in \mathcal{C}$ such that $W \nvDash \mathsf{T}\pi$, thus $\varrho W \nvDash \pi$ by Lemma 3.13.

(ii) If $\gamma(F, D) \notin \sigma B$, then F is an intuitionistic frame, as $\sigma B \supseteq Grz$. Thus $\delta(F, D) \notin B$ by Lemma 5.3, therefore there exists $W \in \mathcal{D}$ such that $W \nvDash \delta(F, D)$. We obtain $\sigma W \nvDash \gamma(F, D)$.

(iii) Assume that $\gamma(F, D) \notin \tau B$. Any subreduction to F with GCDC for D can be composed with the cluster quotient map to give a subreduction to ϱF, hence $\gamma(\varrho F, \varrho D) \notin \tau B$. We have $\delta(\varrho F, \varrho D) \notin B$ by Lemma 5.3, thus there exists $W \in \mathcal{D}$ such that $W \nvDash \delta(\varrho F, \varrho D)$. We have $\sigma W \nvDash \gamma(\varrho F, \varrho D)$, thus $\tau_k W \nvDash \gamma(F, D)$ for sufficiently large k by Lemma 5.8. $\square$

Theorem 5.10 *The mappings $\varrho\colon \mathrm{Ext}_m\, S4 \to \mathrm{Ext}_m\, IPC$ and $\tau\colon \mathrm{Ext}_m\, IPC \to \mathrm{Ext}_m\, S4$ are complete lattice homomorphisms such that $\varrho \circ \tau = \mathrm{id}$.*

Proof: We have $\varrho \circ \tau = \mathrm{id}$ from Lemma 5.4. It is obvious from the definition that ϱ preserves (finite or infinite) meets, and τ preserves joins. Theorem 5.6 implies that ϱ preserves joins as well. If $\{B_i \mid i \in I\} \subseteq \mathrm{Ext}_m\, IPC$, we have $\bigcap_i \tau B_i \supseteq \tau \bigcap_i B_i$ by monotony, we will show the other inclusion. If $\gamma(F, D) \notin \tau \bigcap_i B_i$, then also $\gamma(\varrho F, \varrho D) \notin \tau \bigcap_i B_i$ by the proof of Theorem 5.9, thus $\delta(\varrho F, \varrho D) \notin \bigcap_i B_i$, and we may fix $i \in I$ such that $\delta(\varrho F, \varrho D) \notin B_i$. There exists a frame $W \vDash B_i$ such that $W \nvDash \delta(\varrho F, \varrho D)$. Then $\sigma W \nvDash \gamma(\varrho F, \varrho D)$, hence $\tau_k W \nvDash \gamma(F, D)$ for some k by Lemma 5.8. However, $\tau_k W \vDash \tau B_i \supseteq \bigcap_i \tau B_i$. $\square$

Remark 5.11 ϱ, τ, and σ map single-conclusion rule systems to single-conclusion rule systems, hence the results above also hold for Ext_1 in place of Ext_m.

6 The nontransitive case

In this section, we take a look on the possibility of using canonical rules also in the context of nontransitive modal logics. We first observe that the definition of canonical rules and their basic properties readily generalize to K.

Definition 6.1 Let $\langle F, R\rangle$ be a finite Kripke frame, D a set of arbitrary subsets of F, and X a subset of F. We define the canonical rules $\gamma(F, D)$ and $\gamma(F, D, X)$ as in Definitions 3.2 and 3.16.

Lemma 6.2 *Let W be a frame.*

(*i*) *W refutes* $\gamma(F, D)$ *iff there exists a subreduction of* W *to* F *with GCDC for* D.

(*ii*) *W refutes* $\gamma(F, D, X)$ *iff there exists a subreduction of* W *to a generated subframe* G *of* F *with GCDC for* D, *such that* $X \subseteq G$.

In particular, W *refutes* $\gamma^\sharp(F, \bot)$ *iff* F *is a p-morphic image of* W.

Proof: As in Lemmas 3.3 and 3.17. □

Thus unlike canonical formulas, we can define canonical rules in a straightforward way even for nontransitive frames while preserving the refutation conditions. Nevertheless, their usefulness is considerably limited by their incompleteness:

Example 6.3 The modal logic $D = K + \Diamond\top$ is not equivalent to any set of canonical rules over K.

Proof: We start with a characterization of canonical rules valid in D:

Claim 1 $D \vdash \gamma(F, D)$ *if and only if* F *contains a dead end (i.e.,* $F \nvDash \Diamond\top$*), and* $D = F^\sharp$.

Proof: "←": Clearly, any logic derives the frame rules of all finite frames which do not validate it.

"→": If F contains no dead end, then $F \vDash D$, and $F \nvDash \gamma(F, D)$. If $d \notin D$, we define a frame $G = F \,\dot\cup\, \{a\}$ by making $x\ R\ a\ R\ u$ for every $x \in G$, and $u \in d$. We have $G \vDash D$, and the identity mapping from $F \subseteq G$ to F is a subreduction with GCDC for D. □ (Claim 1)

To finish the proof, consider the Kripke frame $W = \langle \omega, > \rangle$. On the one hand, $W \nvDash D$. On the other hand, every point of W is definable by a variable-free formula, hence W has no finite p-morphic image. In particular, all frame rules $\gamma^\sharp(F, \bot)$ are valid in W. □

The main application we have for nontransitive canonical rules is a description of splittings in the lattice of rule systems.

Definition 6.4 A *splitting pair* in a lattice L is a pair $\langle a, b \rangle$ of elements of L such that $L = a{\downarrow} \,\dot\cup\, b{\uparrow}$. If this holds, then b is uniquely determined by a and vice versa. We write $b = L/a$, and we say that a *splits* L, and b is a *splitting companion* of a. We will usually write A/B instead of $\mathrm{Ext}_m A/B$ or $\mathrm{Ext}_1 A/B$. See [6, 25] for more details on splittings. In particular, observe that if a splits L, then it is *strongly meet-prime*: if $X \subseteq L$ is such that $\bigwedge X \le a$, then $x \le a$ for some $x \in X$.

Theorem 6.5 *Let* A *be a rule system over* K.

(*i*) *If* $F \vDash A$ *is a finite Kripke frame, then* $\langle \mathrm{Th}_m(F), A + \gamma^\sharp(F, \bot) \rangle$ *is a splitting pair in* $\mathrm{Ext}_m A$.

(*ii*) *If* A *has the finite model property, then all splittings of* $\mathrm{Ext}_m A$ *are of the form given in* (*i*).

The same holds for intuitionistic rule systems, with δ *in place of* γ.

Proof: (i) Clearly $F \nvDash \gamma^\sharp(F, \bot)$, hence $A \subseteq \mathrm{Th}_m(F)$ and $A + \gamma^\sharp(F, \bot) \nsubseteq \mathrm{Th}_m(F)$. Let $B \supseteq A$ be a rule system such that $B \not\ni \gamma^\sharp(F, \bot)$. By Theorem 2.2, there exists a frame $W \vDash B$ such that $W \nvDash \gamma^\sharp(F, \bot)$. By Lemma 6.2 F is a p-morphic image of W, hence $F \vDash B$, i.e., $B \subseteq \mathrm{Th}_m(F)$.

(ii) Let $\langle B, A/B \rangle$ be a splitting pair. By assumption

$$B \supseteq A = \bigcap \{\mathrm{Th}_m(F) \mid F \vDash A, F \text{ finite}\},$$

hence $B \supseteq \mathrm{Th}_m(F)$ for some finite F as B is strongly meet-prime. If $n = |F|$, then F validates the rule

$$\mathit{Size}_n := \frac{}{p_0, p_0 \to p_1, p_0 \wedge p_1 \to p_2, \ldots, \bigwedge_{i<n} p_i \to p_n}.$$

On the other hand, any refined frame validating Size_n is easily seen to have at most n points, hence B is complete with respect to a set of Kripke frames of size at most n. This means

$$B = \bigcap \{\mathrm{Th}_m(F) \mid F \vDash B, F \text{ finite}\},$$

thus using once again its meet-primality we obtain $B = \mathrm{Th}_m(F)$ for some finite frame F. Clearly $F \vDash A$, as $A \subseteq B$. By (i) and the uniqueness of splitting companions, we must have $A/B = A + \gamma^\sharp(F, \bot)$.

The intuitionistic case is analogous (and indeed follows from the modal case by the results of Section 5). □

Remark 6.6 Theorem 6.5, stating that every finite frame splits $\mathrm{Ext}_m K$, should be contrasted with the result of Blok [4] that only *acyclic* rooted frames split the lattice $\mathrm{NExt}\, K$ of normal modal logics. (As we will see below, not all finite frames split the lattice $\mathrm{Ext}_1 K$ of consequence relations, nevertheless every rooted finite frame still does, with no acyclicity condition.)

Lemma 6.7 *Let F be a finite Kripke frame, A a consequence relation extending K, and $r \in F$. Then the following conditions are equivalent:*

- *(i) For every $s \in F$ and every $W \subseteq\!\cdot\, F$ such that $r \in W$ and $W \vDash A$, there exists a $U \subseteq\!\cdot\, F$ such that $s \in U$ and U is a p-morphic image of W.*
- *(ii) For every $W \subseteq\!\cdot\, F$ such that $r \in W$ and $W \vDash A$, we have $F \vDash \mathrm{Th}_1(W)$.*
- *(iii) For every $s \in F$, $A + \gamma^\sharp(F, \bot, \{s\}) \vdash \gamma^\sharp(F, \bot, \{r\})$.*

The same holds for intuitionistic rule systems, with δ in place of γ.

Proof: (i) → (ii): If $F \nvDash \varrho$, and ϱ is single-conclusion, there exists a valuation $\Vdash$ in F such that $F \Vdash \varrho^a$ and $F, s \nVdash \varrho^c$ for some $s \in F$. By (i), there exists $U \subseteq\!\cdot\, F$ such that $s \in U$ and U is a p-morphic image of W. We have $U \nvDash \varrho$, hence $W \nvDash \varrho$.

(ii) → (iii): If V is a frame such that $V \vDash A + \gamma^\sharp(F, \bot, \{s\})$ and $V \nvDash \gamma^\sharp(F, \bot, \{r\})$, there exists $W \subseteq\!\cdot\, F$ such that $r \in W$ and W is a p-morphic image of V by Lemma 6.2. In particular,

$W \vDash A$, thus $F \vDash \mathrm{Th}_1(W)$ by (ii). However, $F \nvDash \gamma^\sharp(F, \bot, \{s\})$ and $W \vDash \gamma^\sharp(F, \bot, \{s\})$, a contradiction.

(iii) $\to$ (i): We have $W \vDash A$ and $W \nvDash \gamma^\sharp(F, \bot, \{r\})$, thus $W \nvDash \gamma^\sharp(F, \bot, \{s\})$ by (iii), which implies the existence of U by Lemma 6.2.

The intuitionistic case is analogous. □

Remark 6.8 Let us say that a *coroot*[3] of a finite Kripke frame $\langle F, R\rangle$ is a point $x \in F$ belonging to an initial cluster of the transitive closure of R; in other words, if $y \in x\Uparrow$ for every y such that $x \in y\Uparrow$. In conditions (i) and (iii), it obviously suffices to consider only points s which are coroots (and we only need to check one coroot in every initial cluster).

Theorem 6.9 *Let A be a consequence relation extending K.*

- *(i) If $F \vDash A$ is a finite Kripke frame, $r \in F$, and the condition of Lemma* 6.7 *holds, then* $\langle \mathrm{Th}_1(F), A + \gamma^\sharp(F, \bot, \{r\})\rangle$ *is a splitting pair in* $\mathrm{Ext}_1\, A$.
- *(ii) If A has the finite model property, then all splittings of the lattice* $\mathrm{Ext}_1\, A$ *are of the form given in (i).*

Moreover, we may assume that r is a coroot of F. The same holds for intuitionistic rule systems, with δ in place of γ.

Proof: (i) Clearly $A \subseteq \mathrm{Th}_1(F)$ and $\gamma^\sharp(F, \bot, \{r\}) \notin \mathrm{Th}_1(F)$. Let $B \supseteq A$ be a consequence relation such that $\gamma^\sharp(F, \bot, \{r\}) \notin B$. By Lemma 6.2, there exists $W \subseteq\!\cdot\, F$ such that $r \in W$ and $W \vDash B$. In particular, $W \vDash A$, hence $F \vDash B$ by (ii) of Lemma 6.7.

(ii) Let $\langle B, A/B\rangle$ be a splitting pair. As B is strongly meet-prime and

$$B \supseteq A = \bigcap\{\mathrm{Th}_1(F) \mid F \vDash A, F \text{ finite}\},$$

we have $B \supseteq \mathrm{Th}_1(F)$ for some finite frame F. In particular, $\mathrm{Th}(B)$ is a locally tabular logic; as B is complete wrt a class of finitely generated descriptive $\mathrm{Th}(B)$-frames, it has the finite model property. Using its meet-primality again, we obtain $B = \mathrm{Th}_1(F)$ for some finite F.

For every $s \in F$, we have $F \nvDash \gamma^\sharp(F, \bot, \{s\})$, hence $A + \gamma^\sharp(F, \bot, \{s\}) \supseteq A/B$. Pick a single-conclusion rule $\varrho \in A/B$ such that $F \nvDash \varrho$. Clearly $A/B = A + \varrho$. There exists a valuation $\Vdash$ in F and $r' \in F$ such that $F \Vdash \varrho^a$, and $F, r' \nVdash \varrho^c$. Let r be any coroot of F such that $r' \in r\Uparrow$. Then $W \nvDash \varrho$ for every $W \subseteq\!\cdot\, F$ such that $r \in F$, hence $A/B = A + \varrho \vdash \gamma^\sharp(F, \bot, \{r\})$. Consequently, condition (iii) of Lemma 6.7 holds, and $A/B = A + \gamma^\sharp(F, \bot, \{r\})$.

The intuitionistic case is analogous. □

Remark 6.10 We may have $\mathrm{Th}_1(F) = \mathrm{Th}_1(G)$ even if F and G are nonisomorphic frames, and in principle it could happen that only one of them satisfies the conditions of Lemma 6.7. However, the proof of Theorem 6.9 actually shows that $B = \mathrm{Th}_1(F)$ splits $\mathrm{Ext}_1\, A$ iff F satisfies the conditions of Lemma 6.7, hence we do not have to worry about alternative representations of B.

[3]Think cochairman, not coproduct.

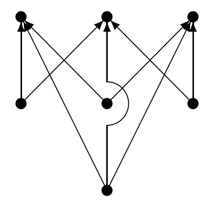


Figure 1: see Remark 6.11

Remark 6.11 Condition (i) of Lemma 6.7 is implied by the simpler condition

(i') For every $s \in F$ there exists a p-morphism f of F onto its generated subframe such that $s \in f(r)\Uparrow$.

Furthermore, it suffices to check only coroots s. However, condition (i') is strictly stronger than (i), as exemplified by the frame in Figure 1, which satisfies (i) but not (i').

Our final task is to characterize the canonical rules admissible in K, in the spirit of Theorem 4.9.

Definition 6.12 Let LR denote Löb's rule

$$\Box p \to p \;/\; p.$$

We introduce the sequences of rules

$$(E^\circ) \qquad \bigwedge_{j<m} (q_j \leftrightarrow \Box q_j) \to \bigvee_{i<n} \Box p_i \Big/ \Big\{ \bigwedge_{j<m} q_j \to p_i \Bigm| i < n \Big\},$$

$$(E^\bullet) \qquad \Box q \to \bigvee_{i<n} \Box p_i \;/\; \{q \to p_i \mid i < n\}.$$

In nontransitive context, we redefine a reflexive (irreflexive) tight predecessor of a subset X of a frame W to be a point t such that $t\uparrow = \{t\} \cup X$ ($t\uparrow = X$, respectively).

We will need the following combinatorial principle used in the theory of the Rudin–Keisler ordering [7], known as *Katětov's lemma on three sets*:

Theorem 6.13 (Katětov [22]) *If $f\colon X \to X$ has no fixpoints, we can partition X into three disjoint sets X_i, $i = 0, 1, 2$, such that $f[X_i] \cap X_i = \varnothing$ for every i.* $\Box$

The following lemma is somewhat stronger than what we really need, but we find it instructive to know the frame conditions corresponding to the rules we deal with. Recall that a frame $\langle W, R\rangle$ is *converse well-founded*, if every nonempty subset $X \subseteq W$ has an R-maximal element (i.e., an $x \in X$ such that $x \; R \; y$ for no $y \in X$). In particular, every converse well-founded frame is irreflexive. A finite frame is converse well-founded iff it contains no *cycle*: a sequence of points $\{x_i \mid i \le n\} \subseteq W$ such that $x_0 \; R \; x_1 \; R \cdots R \; x_n = x_0$ for some $n \ge 1$.

Lemma 6.14

(*i*) *LR is admissible in K, and corresponds to converse well-founded Kripke frames.*

(*ii*) *E° is admissible in K. It is d-persistent, and corresponds to frames where every finite subset has a reflexive tight predecessor.*

(*iii*) *$E^\bullet$ is admissible in K. It is d-persistent, and corresponds to frames where every finite subset has an irreflexive tight predecessor.*

Proof: (i) If LR fails in W under a valuation $\Vdash$, then $\{x \mid x \nVdash p\}$ is a nonempty set without an R-maximal element, and vice versa. The rule is K-admissible, as K is complete wrt finite irreflexive intransitive trees, which are converse well-founded.

(ii) Assume that $\nvdash_K \bigwedge_j \psi_j \to \varphi_i$ for every $i < n$, and pick Kripke models $\langle F_i, \Vdash\rangle$ such that $F_i, x_i \Vdash \bigwedge_j \psi_j \wedge \neg\varphi_i$ for some x_i. Define a new model F by taking the disjoint union of F_i together with a new reflexive point a, which sees exactly the points x_i. Then $F, a \Vdash \bigwedge_j(\psi_j \leftrightarrow \Box\psi_j) \wedge \bigwedge_i \neg\Box\varphi_i$.

Let W be a frame which has $\circ$-tp of all finite subsets, and $\Vdash$ a valuation which refutes $(E^\circ)^c$. Pick points $x_i \Vdash \bigwedge_j q_j \wedge \neg p_i$, and let t be a $\circ$-tp of $\{x_i \mid i < n\}$. Then $t \Vdash \bigwedge_j(q_j \leftrightarrow \Box q_j) \wedge \bigwedge_i \neg\Box p_i$, thus $W \vDash E^\circ$.

Conversely, let W be a Kripke frame of size at least 2 (the other cases are easy) which validates E°, and let $X = \{x_i \mid i < n\}$ be its finite subset. Choose a function f on W such that

$$\begin{cases} f(x) \neq x, f(x) \notin X, x\,R\,f(x) & \text{if possible,} \\ f(x) \neq x & \text{otherwise.} \end{cases}$$

By Katětov's lemma, there exists a partitioning $W = W_0 \dot\cup W_1 \dot\cup W_2$ such that $f[W_j] \cap W_j = \varnothing$. We define a valuation $\Vdash$ by

$$u \Vdash p_i \Leftrightarrow u \neq x_i,$$
$$u \Vdash q_j \Leftrightarrow u \in W_j \cup X.$$

As $x_i \Vdash \bigwedge_j q_j \wedge \neg p_i$ for every i, and W validates E°, there exists a $t \in W$ such that $t \Vdash \bigwedge_j(q_j \leftrightarrow \Box q_j) \wedge \bigwedge_i \neg\Box p_i$. Clearly $t\,R\,x_i$ for every i. We claim that $t{\uparrow} \subseteq X \cup \{t\}$: if not, then $f(t) \in t{\uparrow} \smallsetminus (X \cup \{t\})$. However, let j be such that $t \in W_j$. As $t \Vdash q_j$, we have $t \Vdash \Box q_j$, thus $f(t) \Vdash q_j$, and $f(t) \in X \cup W_j$, a contradiction. Finally, if t is not reflexive, then $t{\uparrow} \subseteq X$, hence $t \Vdash \bigwedge_j \Box q_j$, thus $t \Vdash \bigwedge_j q_j$. This is only possible for $t \in X$, but then $t\,R\,t$, a contradiction. Thus t is a $\circ$-tp of X.

Let $\langle W, R, P\rangle$ be a descriptive frame validating E°, let $X = \{x_i \mid i < n\}$ be its finite subset, and consider the set

$$S = \{\neg\Box A \mid A \in P, X \nsubseteq A\} \cup \{A \leftrightarrow \Box A \mid A \in P, X \subseteq A\}.$$

The set $\{A \in P \mid x_i \in A\}$ is a filter for any i. If S' is a finite subset of S such that $\bigcap S' = \varnothing$, we thus have

$$\bigcap_j (B_j \leftrightarrow \Box B_j) \subseteq \bigcup_i \Box A_i$$

for some $A_i \in P$ such that $x_i \notin A_i$, and $B_j \in P$ such that $X \subseteq B_j$. By $W \vDash E^\circ$, we obtain $\bigcap_j B_j \subseteq A_i$ for some i. However, $x_i \in \bigcap_j B_j \smallsetminus A_i$, a contradiction. We have thus verified that S has fip.

By compactness of W, there exists a point $t \in \bigcap S$. If $t \in \Box A$, we have $t \in A$ and $X \subseteq A$, hence $t\ R\ X, t$ by refinement of W. On the other hand, if $z \neq t$ and $z \notin X$, we can find $A \in P$ such that $z \notin A$ and $\{t\} \cup X \subseteq A$ as W is refined. Then $t \in (A \leftrightarrow \Box A) \in S$, hence $t \in \Box A$, which implies $\neg(t\ R\ z)$. Consequently t is a $\circ$-tp of X.

(iii) is similar to (ii), but easier. $\Box$

Incidentally, we observe that $\mid\!\sim_K$ is "Kripke inconsistent":

Corollary 6.15 *No Kripke frame validates simultaneously the special case*

$$\neg(p \leftrightarrow \Box p)\ /$$

of E°*, and Löb's rule.*

Proof: If $W \vDash LR$, then W is converse well-founded by Lemma 6.14. We can define a set $X \subseteq W$ satisfying

$$x \in X \quad \text{iff} \quad \exists y\,(x\ R\ y \wedge y \notin X)$$

by well-founded recursion, hence $\neg(p \leftrightarrow \Box p)\ /$ fails in W. $\Box$

Theorem 6.16 *The following are equivalent for any canonical rule* $\varrho = \gamma(F, D)$*:*

(*i*) K *admits* ϱ.

(*ii*) ϱ *is derivable in* $K + LR$ *or* $K + E^\circ$.

(*iii*) F *contains a cycle, or* D *is nonempty.*

Proof: (ii) $\to$ (i) follows from Lemma 6.14.

(iii) $\to$ (ii): If F contains a cycle $x_0\ R\ x_1\ R\ \cdots\ R\ x_n = x_0$, then the assumptions of $\gamma(F, D)$ contain the formulas $\Box p_{x_{i+1}} \to p_{x_i}$ for each $i < n$. Putting $\alpha = \bigwedge_i p_{x_i}$, we obtain

$$(\gamma(F, D))^a \vdash_K \Box\alpha \to \alpha \vdash_{LR} \alpha \vdash_K p_{x_0} \in (\gamma(F, D))^c.$$

Assume that F is acyclic, and $d \in D$, we will show that $\gamma(F, D) \in K + E^\circ$. Assume for contradiction that $W \vDash E^\circ$ is a descriptive frame which refutes $\gamma(F, D)$, and let f be the induced subreduction of W to F with GCDC for D. We can pick $x_a \in f^{-1}[a]$ for each $a \in d$. By Lemma 6.14, there exists a $\circ$-tp t of the set $\{x_a \mid a \in d\}$. As F is irreflexive, we have $t \notin \mathrm{dom}(f)$. Then $f[t\!\uparrow] = d$ contradicts GCDC.

(i) $\to$ (iii): Let F be a finite acyclic (i.e., converse well-founded) Kripke frame, and $D = \varnothing$. As F is descriptive and finitely generated, it is (isomorphic to) a generated subframe of a canonical frame $C = C_n(K)$ for some $n < \omega$ (cf. [6, Thm. 8.60]). We have $F \vDash \Box^k\bot$ for some $k < \omega$. The subset of C defined by $\Box^k\bot$ is finite, and C is refined, hence F as well as all its elements are definable in C. It follows that the identity mapping on $F \subseteq C$ is a subreduction of C to F, thus $C \nvDash \gamma(F) = \gamma(F, D)$, and $\gamma(F, D) \notin\ \mid\!\sim_K$. $\Box$

The easiness of the proof of Theorem 6.16 suggests that rather than being a progress towards the solution of the long-standing problem of admissibility in K, it is just a further testimony that nontransitive canonical rules are plagued with incompleteness. We can indeed supply a specific counterexample:

Example 6.17 $K + LR + E^\circ$ does not derive $E^\bullet$, hence no rule system between $K + E^\bullet$ and $\mathrel{|\!\sim}_K$ is axiomatizable by canonical rules.

Proof (sketch): Let F be any finite acyclic frame. We construct a new frame by repeating the following procedure: we take its finite subset X, and expand the frame by new points $\{z_{X,n} \mid n \in \omega\}$, where $z_{X,0}$ is a dead end, and $z_{X,n+1}\uparrow = X \cup \{z_{X,n}\}$. We can arrange the order of the steps so that the frame F^∞ we obtain as the union after ω steps has all its finite subsets taken care of. It is easy to see that F^∞ is converse well-founded, hence $F^\infty \vDash LR$. Let P be the set of subsets $A \subseteq F^\infty$ such that for each X, $\{n \mid z_{X,n} \in A\}$ is finite or cofinite. One can check that the chain $\{z_{X,n} \mid n < \omega\}$ behaves essentially as a $\circ$-tp of X wrt valuations from P, hence the frame $\langle F^\infty, P\rangle$ validates E°.

Now, take $F = \{a\}$ to be the irreflexive singleton, and let $\Vdash$ be the valuation on F^∞ such that $u \Vdash p$ iff $u = a$. Then $\Vdash$ makes $\Box p \to \Box\bot$ true in F^∞, but $p \to \bot$ fails in a, hence $\langle F^\infty, P\rangle \nvDash E^\bullet$. $\Box$

As we have seen, canonical rules, and specifically frame rules, can be applied in nontransitive context to a certain extent mimicking the behaviour of frame formulas in the transitive case. While this gives us more power than what we can achieve in similar situations with canonical formulas, we cannot expect serious progress with admissible rules and other areas comparable to the transitive case without a completeness theorem for canonical rules. It remains open whether we can generalize canonical rules to obtain complete axiomatization of rule systems over K. In principle, it could be easier to devise such complete canonical rules than canonical formulas, nevertheless both seem to meet similar serious combinatorial obstacles, and the goal remains elusive.

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