A posteriori error estimates in the finite element method

A survey of techniques

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Poisson problem

- Classical formulation: find $u \in C^2(\Omega) \cap C(\overline{\Omega})$:

$$-\Delta u = f \quad \text{in } \Omega \qquad u = 0 \quad \text{on } \partial\Omega$$

- Weak formulation: $V = H_0^1(\Omega)$

$$u \in V \ : \quad \underbrace{\mathcal{B}(u,v)}_{\int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x} = \underbrace{\mathcal{F}(v)}_{\int_\Omega f v \,\mathrm{d}x} \quad \forall v \in V$$

- Galerkin method $\quad V_h \subset V \quad \dim V_h < \infty$

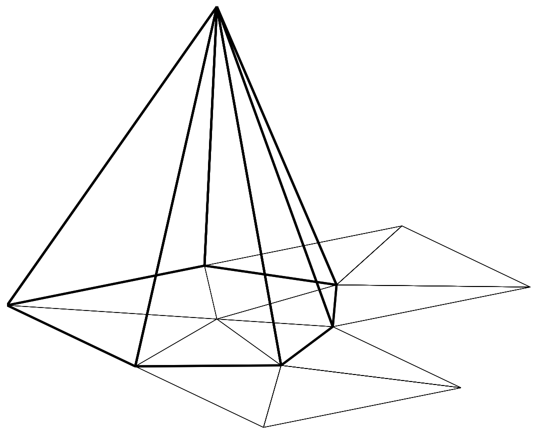
$$u_h \in V_h \ : \quad \mathcal{B}(u_h, v_h) = \mathcal{F}(v_h) \quad \forall v_h \in V_h \quad \Leftrightarrow \quad Ay = F$$

$$u_h(x) = \sum_{j=1}^N y_j \varphi_j(x) \qquad \sum_{j=1}^N y_j \underbrace{\mathcal{B}(\varphi_j, \varphi_i)}_{A_{ij}} = \underbrace{\mathcal{F}(\varphi_i)}_{F_i}$$

Finite element method (FEM)

- FEM $\quad V_h = \{v_h \in V : v_h|_K \in P^1(K) \ \forall K \in \mathcal{T}_h\}$

$\varphi_1, \dots, \varphi_N$ ... FEM basis functions $\quad \varphi_i(x_j) = \delta_{ij}$



Motivation – adaptive algorithm

1. Initialize: Construct the initial mesh $\mathcal{T}_h$.
2. Solve: Find u_h on $\mathcal{T}_h$.
3. Estimate error: Compute η_K for all $K \in \mathcal{T}_h$.
4. Stopping criterion: If $\sum_{K\in\mathcal{T}_h} \eta_K^2 \leq \mathrm{TOL}^2 \quad \Rightarrow \quad$ STOP.
5. Mark: If $\eta_K \geq \Theta \max_{K\in\mathcal{T}_h} \eta_K \quad \Rightarrow \quad$ mark K. $\qquad 0 < \Theta < 1$
6. Refine: Refine marked elements and build the new mesh $\mathcal{T}_h$.
7. GO TO 2.

A priori error estimates

- Discretization error: $e = u - u_h$
- Energy norm: $\|u\|^2 = \mathcal{B}(u,u) = |u|^2_{H^1(\Omega)}$
- Céa's lemma: $\|e\| = \inf_{v_h\in V_h} \|u - v_h\| = \operatorname{dist}(u, V_h)$
- Lagrange interpolation: $\pi_h^{\mathrm{Lag}} : C(\overline{\Omega}) \mapsto V_h$
  $v \in H^2(\Omega)\cap C(\overline{\Omega}) \quad \Rightarrow \quad \|v - \pi_h^{\mathrm{Lag}} v\|_{H^1(\Omega)} \leq C\, h\, |v|_{H^2(\Omega)}$
- $u \in H^2(\Omega)\cap C(\overline{\Omega}) \quad \Rightarrow \quad \|e\| \leq C\, h\, |u|_{H^2(\Omega)}$

A posteriori error estimates I

Definition

- $\|e\| \approx \eta \quad$ (or $\|e\| \leq \eta$, or $\eta \leq \|e\|$)
- $\eta = \eta(u_h, f, \Omega, \mathcal{T}_h, \dots)$

Properties

- Local: $\eta^2 = \sum_{K\in\mathcal{T}_h} \eta_K^2$
- Guaranteed upper (lower) bound: $\|e\| \leq \eta \quad (\eta \leq \|e\|)$
- Asymptotic exactness: $\lim_{h\to 0} I_{\mathrm{eff}} = 1, \quad I_{\mathrm{eff}} = \dfrac{\eta}{\|e\|}$
- Efficient and reliable: $C_1\eta \leq \|e\| \leq C_2\eta$
- Robust: C_1 and C_2 are independent from quantities like coefficients in the equation, mesh aspect ratio etc.

A posteriori error estimates II

Remarks:

- Locality
  $\Rightarrow$ fast evaluation of η
- Guaranteed upper bound
  $\Rightarrow$ adaptive algorithm guarantees $\|e\| \leq \mathrm{TOL}$
- Efficiency and reliability
  $\Rightarrow$ convergence of adaptive algorithm

Residual estimates

$$u \in V: \quad \mathcal{B}(u,v) = \mathcal{F}(v) \quad \forall v \in V$$
$$u_h \in V_h: \quad \mathcal{B}(u_h,v_h) = \mathcal{F}(v_h) \quad \forall v_h \in V_h$$

- Residual: $\mathcal{R}(v) = \mathcal{F}(v) - \mathcal{B}(u_h,v) \quad \forall v \in V$
- Residual equation: $e \in V: \quad \mathcal{B}(e,v) = \mathcal{R}(v) \quad \forall v \in V$

$$\left[\mathcal{B}(u,v) - \mathcal{B}(u_h,v) = \mathcal{F}(v) - \mathcal{B}(u_h,v) \quad \forall v \in V\right]$$

- Galerkin orthogonality: $\mathcal{B}(e,v_h) = 0 \quad \forall v_h \in V_h$
- $$\|e\| = \sup_{0\neq v\in V} \frac{|\mathcal{B}(e,v)|}{\|v\|} = \sup_{0\neq v\in V} \frac{|\mathcal{R}(v)|}{\|v\|} = \|\mathcal{R}\|_{V^*}$$

Explicit residual estimates I

- Residual splitting: $\mathcal{R}(v) = \sum_{K\in\mathcal{T}_h} \int_K rv \,\mathrm{d}x + \sum_\ell \int_\ell J_\ell v \,\mathrm{d}s$

  $r = f + \Delta u_h \quad J_\ell = (\nabla u_h^+ - \nabla u_h^-)\cdot \nu_\ell \quad \ell$ are edges in $\mathcal{T}_h$

$$\mathcal{R}(v) = \mathcal{F}(v) - \mathcal{B}(u_h,v) = \sum_{K\in\mathcal{T}_h}\left(\int_K fv\,\mathrm{d}x - \int_K \nabla u_h\cdot\nabla v\,\mathrm{d}x\right)$$
$$= \sum_{K\in\mathcal{T}_h}\left(\int_K fv\,\mathrm{d}x + \int_K \Delta u_h v\,\mathrm{d}x - \int_{\partial K} \nabla u_h\cdot\nu_K v\,\mathrm{d}s\right)$$
$$= \sum_{K\in\mathcal{T}_h}\int_K rv\,\mathrm{d}x + \sum_\ell \int_\ell (\nabla u_h^+ - \nabla u_h^-)\cdot\nu_\ell v\,\mathrm{d}s$$

Explicit residual estimates II

- Clément inter.: $\pi_h^{\mathrm{Cl}}: V \mapsto V_h \quad \|v - \pi_h^{\mathrm{Cl}} v\|_{0,K} \le C_1 |K|^{1/2} \|\nabla v\|_{0,\omega_K}$

  $\|v - \pi_h^{\mathrm{Cl}} v\|_{0,\ell} \le C_2 |\ell|^{1/2} \|\nabla v\|_{0,\omega_\ell}$

$$\|e\|^2 = \mathcal{B}(e,e) = \mathcal{R}(e) = \mathcal{R}(e - \pi_h^{\mathrm{Cl}} e)$$
$$= \sum_{K\in\mathcal{T}_h}\int_K r(e-\pi_h^{\mathrm{Cl}}e)\,\mathrm{d}x + \sum_\ell \int_\ell J_\ell(e-\pi_h^{\mathrm{Cl}}e)\,\mathrm{d}s$$
$$\le \sum_{K\in\mathcal{T}_h} C_1\|r\|_{0,K}|K|^{1/2}\|\nabla e\|_{0,\omega_K} + \sum_\ell C_2 \|J_\ell\|_{0,\ell}|\ell|^{1/2}\|\nabla e\|_{0,\omega_\ell}$$
$$\le C_3\left(\sum_{K\in\mathcal{T}_h}|K|\,\|r\|_{0,K}^2 + \sum_\ell |\ell|\,\|J_\ell\|_{0,\ell}^2\right) + \varepsilon\|e\|^2$$
$$\|e\|^2 \le C_4 \underbrace{\left(\sum_{K\in\mathcal{T}_h}|K|\,\|r\|_{0,K}^2 + \sum_\ell |\ell|\,\|J_\ell\|_{0,\ell}^2\right)}_{(\eta^{\mathrm{expl}})^2} \equiv C_4(\eta^{\mathrm{expl}})^2$$

Implicit residual estimates – Dirichlet I

Construction:

- Local Dirichlet problems:

  $e_K^{\mathrm{Dir}} \in H_0^1(K): \quad \mathcal{B}_K(e_K^{\mathrm{Dir}}, v) = \mathcal{R}_K(v) \quad \forall v \in H_0^1(K)$
- Approximate local problems: $V_{0,h}(K) \subset H_0^1(K)$

  $e_{K,h}^{\mathrm{Dir}} \in V_{0,h}(K): \quad \mathcal{B}_K(e_{K,h}^{\mathrm{Dir}}, v_h) = \mathcal{R}_K(v_h) \quad \forall v_h \in V_{0,h}(K)$
- $\eta_K^{\mathrm{Dir}} = \|e_{K,h}^{\mathrm{Dir}}\|_K \quad (\eta^{\mathrm{Dir}})^2 = \sum_{K\in\mathcal{T}_h}(\eta_K^{\mathrm{Dir}})^2$
- Notation: $\mathcal{B}_K(u,v) = \int_K \nabla u\cdot\nabla v\,\mathrm{d}x$

  $\mathcal{R}_K(v) = \int_K fv\,\mathrm{d}x - \int_K \nabla u_h\cdot\nabla v\,\mathrm{d}x$

  $\|v\|_K^2 = \mathcal{B}_K(v,v)$

Implicit residual estimates – Dirichlet II

Guaranteed lower bound:

- $e^{\mathrm{Dir}}|_K = e_K^{\mathrm{Dir}} \quad \forall K \in \mathcal{T}_h; \quad V_0 = \{v \in V : v|_K \in H_0^1(K)\} \subset V$
- $e_h^{\mathrm{Dir}}|_K = e_{K,h}^{\mathrm{Dir}} \quad \forall K \in \mathcal{T}_h; \quad V_{0,h} = \{v \in V : v|_K \in V_{0,h}(K)\} \subset V_0$
- Theorem: $\|e_h^{\mathrm{Dir}}\| \le \|e^{\mathrm{Dir}}\| \le \|e\|$
  Proof:
  - $\|e^{\mathrm{Dir}}\| \le \|e\|$:
    $e^{\mathrm{Dir}} \in V_0 \quad v \in V_0$
    $\mathcal{B}(e - e^{\mathrm{Dir}}, v) = \mathcal{R}(v) - \mathcal{R}(v) = 0$
    $\Rightarrow \quad \mathcal{B}(e, e^{\mathrm{Dir}}) = \|e^{\mathrm{Dir}}\|^2$
    $\Rightarrow \quad \|e - e^{\mathrm{Dir}}\|^2 = \|e\|^2 - 2\mathcal{B}(e, e^{\mathrm{Dir}}) + \|e^{\mathrm{Dir}}\|^2$
    $= \|e\|^2 - \|e^{\mathrm{Dir}}\|^2 \ge 0$
  - $\|e_h^{\mathrm{Dir}}\| \le \|e^{\mathrm{Dir}}\|$:
    $\mathcal{B}(e^{\mathrm{Dir}} - e_h^{\mathrm{Dir}}, v_h) = \mathcal{R}(v_h) - \mathcal{R}(v_h) = 0 \quad \forall v_h \in V_{0,h}$
    Similarly.

Implicit residual estimates – Neumann I

Construction:

- Weak f.: $e_K^{\mathrm{Neu}} \in H_E^1(K) = \{v \in H^1(K) : v = 0 \text{ on } \partial K \cap \partial\Omega\}$:
$$\mathcal{B}_K(e_K^{\mathrm{Neu}}, v) = \int_K f v \,\mathrm{d}x - \mathcal{B}_K(u_h, v) + \int_{\partial K} g_K v \,\mathrm{d}s \quad \forall v \in H_E^1(K)$$
- Classical f.:
$$-\Delta(e_K^{\mathrm{Neu}} + u_h) = f \quad \text{in } K$$
$$\nabla(e_K^{\mathrm{Neu}} + u_h) \cdot \nu_K = g_K \quad \text{on } \partial K \setminus \partial\Omega$$
$$e_K^{\mathrm{Neu}} + u_h = 0 \quad \text{on } \partial K \cap \partial\Omega$$
- $g_K|_\ell \in P^1(\ell)$, $\ell \subset \partial K$, $K \in \mathcal{T}_h$, $\quad g_K \approx \nabla u|_K \cdot \nu_K$ on ∂K
- Compatibility condition: $g_K|_\ell + g_{K^*}|_\ell = 0$ for $\ell = \partial K \cap \partial K^*$
- p-order equilibration condition ($p = 0, 1$):
$$\int_K f\varphi \,\mathrm{d}x - \mathcal{B}_K(u_h, \varphi) + \int_{\partial K} g_K \varphi \,\mathrm{d}s = 0 \quad \forall \varphi \in P^p(K)$$
- $\eta_K^{\mathrm{Neu}} = \|e_K^{\mathrm{Neu}}\|_K \quad (\eta^{\mathrm{Neu}})^2 = \sum_{K\in\mathcal{T}_h} (\eta_K^{\mathrm{Neu}})^2$

Implicit residual estimates – Neumann II

Guaranteed upper bound:

- Theorem: Compatibility cond. $\Rightarrow \quad \|e\| \le \eta^{\mathrm{Neu}}$
  Proof: $\quad v \in V$
$$\mathcal{B}(e, v) = \mathcal{R}(v) = \sum_{K\in\mathcal{T}_h} \left(\int_K f v \,\mathrm{d}x - \mathcal{B}_K(u_h, v) + \int_{\partial K} g_K v \,\mathrm{d}s \right)$$
$$= \sum_{K\in\mathcal{T}_h} \mathcal{B}_K(e_K^{\mathrm{Neu}}, v) \le \sum_{K\in\mathcal{T}_h} \|e_K^{\mathrm{Neu}}\|_K \|v\|_K \le \left(\sum_{K\in\mathcal{T}_h} \|e_K^{\mathrm{Neu}}\|_K^2 \right)^{\frac{1}{2}} \|v\|$$
- Remark: Approximate Neumann problems:
  $e_{K,h}^{\mathrm{Neu}} \in V_h^{\mathrm{Neu}} \subset H_E^1(K) \quad \Rightarrow \quad \sum_{K\in\mathcal{T}_h} \|e_{K,h}^{\mathrm{Neu}}\|_K^2 \le (\eta^{\mathrm{Neu}})^2$
- In general: $\|e\| \not\le \eta^{\mathrm{Neu}}$

Hierarchic (residual) estimates

- $\widehat{V}_h = V_h \oplus Y_h \quad Y_h \subset V \quad V_h \cap Y_h = \{0\}$
- $\widehat{u}_h \in \widehat{V}_h : \quad \mathcal{B}(\widehat{u}_h, \widehat{v}_h) = \mathcal{F}(\widehat{v}_h) \quad \forall \widehat{v}_h \in \widehat{V}_h$
- $\|e\| \approx \|\widehat{u}_h - u_h\| \equiv \|\widehat{e}_h\|$
- $\bar{e}_h \in Y_h : \quad \mathcal{B}(\bar{e}_h, y_h) = \mathcal{R}(y_h) \quad \forall y_h \in Y_h$
- $\|e\| \approx \|\bar{e}_h\| \equiv \eta^{\mathrm{Hie}}$
- Saturation assumption:
  $\exists \beta < 1: \quad \|u - \widehat{u}_h\| \le \beta \|u - u_h\|$
- Strengthenned Cauchy-Schwarz inequality:
  $\exists \gamma < 1: \quad |\mathcal{B}(v_h, y_h)| \le \gamma \|v_h\| \|y_h\| \quad \forall v_h \in V_h, \ y_h \in Y_h$
- $\|\bar{e}_h\| \le \|\widehat{e}_h\| \le \|e\| \le \frac{1}{(1-\beta^2)^{\frac{1}{2}}} \|\widehat{e}_h\| \le \frac{1}{(1-\beta^2)^{\frac{1}{2}}(1-\gamma^2)^{\frac{1}{2}}} \|\bar{e}_h\|$

Error majorants

- Friedrichs inequality: $\|v\|_{0,\Omega} \le C_\Omega \|\nabla v\|_{0,\Omega} \quad \forall v \in V = H_0^1(\Omega)$
- $\|e\| \le C_\Omega \|f + \operatorname{div} \mathbf{y}\|_{0,\Omega} + \|\mathbf{y} - \nabla u_h\|_{0,\Omega} \quad \forall \mathbf{y} \in \mathbf{H}(\operatorname{div}, \Omega)$

Proof:
$v \in V \quad \mathbf{y} \in \mathbf{H}(\operatorname{div}, \Omega)$

$$\begin{aligned}\mathcal{B}(e, v) &= \mathcal{R}(v) + \int_\Omega \mathbf{y} \cdot \nabla v \,\mathrm{d}x + \int_\Omega \operatorname{div} \mathbf{y} v \,\mathrm{d}x + \int_{\partial\Omega} \mathbf{y} \cdot \nu v \,\mathrm{d}s \\ &= \int_\Omega (f + \operatorname{div} \mathbf{y}) v \,\mathrm{d}x + \int_\Omega (\mathbf{y} - \nabla u_h) \cdot \nabla v \,\mathrm{d}x \\ &\le \|f + \operatorname{div} \mathbf{y}\|_{0,\Omega} \|v\|_{0,\Omega} + \|\mathbf{y} - \nabla u_h\|_{0,\Omega} \|\nabla v\|_{0,\Omega} \\ &\le \left(C_\Omega \|f + \operatorname{div} \mathbf{y}\|_{0,\Omega} + \|\mathbf{y} - \nabla u_h\|_{0,\Omega} \right) \|\nabla v\|_{0,\Omega}\end{aligned}$$

Put $v = e$.

Postprocessing

- Recovered gradient: $\nabla u_h \mapsto \mathcal{G}(u_h)$
- $\|e\| \approx \eta^{\text{post}} = \|\mathcal{G}(u_h) - \nabla u_h\|_{0,\Omega}$
- Superconvergence: $\|\nabla u - \mathcal{G}(u_h)\|_{0,\Omega} \le C_1 h^{1+\epsilon}$
- Assumption: $\|e\| \ge C_2 h$
- Theorem (asymptotic exactness): $\lim_{h \to 0} \dfrac{\eta^{\text{post}}}{\|e\|} = 1$

Proof:

$$\frac{\eta^{\text{post}}}{\|e\|} \le \frac{\|\nabla u - \nabla u_h\|_0}{\|e\|} + \frac{\|\nabla u - \mathcal{G}(u_h)\|_0}{\|e\|} \le 1 + \frac{C_1 h^{1+\epsilon}}{C_2 h} \to 1$$

$$\frac{\eta^{\text{post}}}{\|e\|} \ge \frac{\|\nabla u - \nabla u_h\|_0}{\|e\|} - \frac{\|\nabla u - \mathcal{G}(u_h)\|_0}{\|e\|} \ge 1 - \frac{C_1 h^{1+\epsilon}}{C_2 h} \to 1$$

Quantity of interest

- Quantity of interest: $\Phi \in V^*$
- Adjoint problem: $z \in V: \quad \mathcal{B}(v, z) = \Phi(v) \quad \forall v \in V$
- Approx. adjoint prob.: $z_h \in V_h: \quad \mathcal{B}(v_h, z_h) = \Phi(v_h) \quad \forall v_h \in V_h$
- Error representation formula:

$$\Phi(e) = \mathcal{B}(e, z) = \mathcal{R}(z) = \mathcal{R}(z - z_h) = \mathcal{B}(u - u_h, z - z_h)$$

$$\begin{aligned}|\Phi(e)| &\le \|u - u_h\| \, \|z - z_h\| \\ &\le \eta^{\text{pri}} \eta^{\text{adj}}\end{aligned}$$

Summary

A posteriori error estimators

- Explicit residual – fast, simple, reliable, (efficient)
- Implicit residual
  - Dirichlet type – guaranteed lower bound, (reliable)
  - Neumann type – upper bound (not guaranteed), (efficient)
- Hierarchic (residual) – efficient and reliable
- Error majorants – guaranteed upper bound, demanding
- Postprocessing – fast, simple, superconvergence ⇒ asympt. exact
- Quantity of interest – if energy norm is not the goal

Recommended books

- M. Ainsworth, J. T. Oden, A posteriori error estimation in finite element analysis, Wiley, New York, 2000.
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- S. Repin, A posteriori estimates for partial differential equations, de Gruyter, Berlin, 2008.
- R. Verfürth, A review of a posteriori error estimation and adaptive mesh-refinement techniques., Wiley-Teubner, Chichester/Stuttgart, 1996.