Blending margins

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Abstract

We investigate properties of the formula $p \to \Box p$ in the basic modal logic K. We show that K satisfies an infinitary weak variant of the rule of margins, which leads to a description of a complete set of unifiers of $p \to \Box p$. Using this information, we establish that K has nullary unification type. Moreover, we show that the formula $p \to \Box p$ is admissibly saturated, but not exact.

1 Introduction

Equational unification studies the problem of making terms equivalent modulo an equational theory by means of a substitution. It has been thoroughly investigated for basic algebraic theories, such as the theory of commutative semigroups, see Baader and Snyder [2] for an overview. If L is a propositional logic algebraizable with respect to a class V of algebras, unification modulo the equational theory of V can be stated purely in terms of propositional logic: an L-unifier of a set Γ of formulas is a substitution which turns all formulas from Γ into L-tautologies.

In the realm of modal logics, the seminal results of Ghilardi [5] show that unification is at most finitary, decidable, and generally well-behaved for a representative class of transitive modal logics, including e.g. $K4$, $S4$, GL, Grz. Unification was also studied for fragments of description logics, which have industrial database applications; see Baader and Ghilardi [1].

Unification in propositional logics is closely connected to admissibility of inference rules: a multiple-conclusion rule $\Gamma \,/\, \Delta$ is L-admissible if every L-unifier of Γ also unifies some formula from Δ. Rybakov [9] proved that admissibility is decidable for a class of transitive modal logics (similar to the one mentioned above) and provided characterizations of their admissible rules. Some of these results can be alternatively obtained using Ghilardi's approach (cf. also [7]). It is also possible to treat intuitionistic and intermediate logics in parallel with the transitive modal case [9, 4, 6].

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In contrast to these results, not much is known about unification and admissibility in non-transitive modal logics. In particular, one of the main open problems in the area is decidability of unification or admissibility in the basic modal logic K. (Wolter and Zakharyaschev [13] have shown that unifiability is undecidable in the bimodal extension of K with the universal modality and in some description logics, but it is wide open whether one can extend these results to K itself.)

In this note we present some negative properties of unification and admissibility in K. The main result is that unification in K is nullary (i.e., of the worst possible type). We also show that there exists a formula (namely, $p \to \Box p$) which is admissibly saturated in the sense of [8], but it is not exact (or projective, for that matter). (The results of Ghilardi [5] imply that in well-behaved transitive modal logics such as $K4$, projective, exact, and admissibly saturated formulas coincide, and indeed this is an important precondition which makes possible the characterization of admissibility in terms of projective approximations.)

Our results are based on a classification of unifiers of the formula $p \to \Box p$. The main ingredient is establishing that K admits a weaker version of the so-called rule of margins:

$$\varphi \to \Box\varphi \;/\; \varphi, \neg\varphi.$$

The rule of margins was investigated by Williamson [10, 11, 12] in the context of epistemic logic. (The rule is supposed to express the ubiquity of vagueness. We read $\Box$ as "clearly". Since all our learning processes have a certain margin of error, the only way we can know for sure that φ is clearly true whenever it is true is that we know in fact whether φ is true or false.) The rule of margins is admissible e.g. in the logics KD, KT, KDB, and KTB, but not in K. However, we will show that K satisfies a variant of the rule whose conclusion is that either φ holds, or it is almost contradictory in the sense of implying $\Box^n\bot$ for some $n \in \omega$.

2 Preliminaries

We refer the reader to [3, 2] for background on modal logic and unification, respectively. We review below the needed definitions to fix the notation, and some relevant basic facts.

We work with formulas in the propositional modal language using propositional variables p_n, $n < \omega$ (we will often write just p for p_0), classical propositional connectives (including the nullary connectives $\bot, \top$), and the unary modal connective $\Box$. We will use lower-case Greek letters to denote formulas, and upper-case Greek letters for finite sets of formulas. We define $\Diamond\varphi$, $\Box^n\varphi$, $\Box^{<n}\varphi$, $\Diamond^n\varphi$ as shorthands for $\neg\Box\neg\varphi$, $\underbrace{\Box\dots\Box}_{n\text{ times}}\varphi$, $\bigwedge_{i=0}^{n-1}\Box^i\varphi$, and $\neg\Box^n\neg\varphi$, respectively. (As a special case, $\Box^0\varphi = \varphi$ and $\Box^{<0}\varphi = \top$.) The *modal degree* $\mathrm{md}(\varphi)$ of a formula φ is defined so that $\mathrm{md}(p_i) = 0$, $\mathrm{md}(\circ(\varphi_0, \dots, \varphi_{k-1})) = \max_{i<k}\mathrm{md}(\varphi_i)$ for a k-ary propositional connective $\circ$, and $\mathrm{md}(\Box\varphi) = 1 + \mathrm{md}(\varphi)$.

We use $\vdash$ to denote the *global consequence relation* of K. That is, $\Gamma \vdash \varphi$ iff there exists a sequence of formulas $\varphi_0, \dots, \varphi_n$ such that $\varphi_n = \varphi$, and each φ_i is an element of Γ, a classical propositional tautology, an instance of the axiom

$$\Box(\alpha \to \beta) \to (\Box\alpha \to \Box\beta),$$

or it is derived from some of the formulas φ_j, $j < i$, by an instance of necessitation $\alpha \;/\; \Box\alpha$ or modus ponens $\alpha, (\alpha \to \beta) \;/\; \beta$.

A *Kripke model* is a triple $\langle F, R, \vDash\rangle$, where the *accessibility relation* R is a binary relation on F, and the *valuation* $\vDash$ is a relation between elements of F and formulas, written as $F, x \vDash \varphi$, which commutes with propositional connectives and satisfies

$$F, x \vDash \Box\varphi \quad \text{iff} \quad \forall y \in F\,(x\; R\; y \Rightarrow F, y \vDash \varphi).$$

If there is no danger of confusion, we will denote the model $\langle F, R, \vDash\rangle$ just by F. We write $F \vDash \varphi$ if $F, x \vDash \varphi$ for every $x \in F$, and $F \vDash \Gamma$ if $F \vDash \varphi$ for every $\varphi \in \Gamma$. The *completeness theorem* for K states

Fact 2.1 *$\Gamma \vdash \varphi$ iff $F \vDash \Gamma$ implies $F \vDash \varphi$ for every model $\langle F, R, \vDash\rangle$.*

Let $R(x) = \{y : x\; R\; y\}$, let

$$R^n = \{\langle x_0, x_n\rangle \in F^2 : \exists x_1, \dots, x_{n-1} \in F\, \forall i < n\, x_i\; R\; x_{i+1}\}$$

be the n-fold composition of R (where the case $n = 0$ is understood to mean $R^0 = \{\langle x, x\rangle : x \in F\}$), and $R^{\leq n} = \bigcup_{i\leq n} R^i$. We say that x is a *root* of F if $F = \bigcup_{n\in\omega} R^n(x)$.

Fact 2.2 *If $\nvdash \varphi$, then there exists a model $\langle F, R, \vDash\rangle$ based on a finite irreflexive intransitive tree with root x such that $F, x \nvDash \varphi$.*

(That is, R is the edge relation of a directed tree in the sense of graph theory, with edges oriented away from x and no self-loops.)

A model $\langle F', R', \vDash'\rangle$ is the restriction of $\langle F, R, \vDash\rangle$ to F', denoted as $\langle F, R, \vDash\rangle \restriction F'$, if $F' \subseteq F$, $R' = R \cap F'^2$, and $F, x \vDash p_i$ iff $F', x \vDash p_i$ for every $x \in F'$ and i.

Fact 2.3 *If $n \geq \mathrm{md}(\varphi)$, $x \in F \cap G$, and $\langle F, R, \vDash\rangle \restriction R^{\leq n}(x) = \langle G, S, \vDash\rangle \restriction S^{\leq n}(x)$, then $F, x \vDash \varphi$ iff $G, x \vDash \varphi$.*

A *p-morphism* between models $\langle F, R, \vDash\rangle$ and $\langle G, S, \vDash\rangle$ is a function $f\colon F \to G$ such that

(i) $x\; R\; y$ implies $f(x)\; S\; f(y)$,

(ii) if $f(x)\; S\; z$, there exists $y \in F$ such that $x\; R\; y$ and $f(y) = z$,

(iii) $F, x \vDash p_i$ iff $G, f(x) \vDash p_i$.

Fact 2.4 *If $f\colon F \to G$ is a p-morphism, then $F, x \vDash \varphi$ iff $G, f(x) \vDash \varphi$ for every formula φ.*

A *substitution* is a mapping from formulas to formulas which commutes with all connectives. A *unifier* of Γ is a substitution σ such that $\vdash \sigma(\varphi)$ for all $\varphi \in \Gamma$. Let $U(\Gamma)$ be the set of all unifiers of Γ. The *composition* of substitutions σ, τ is the substitution $\sigma \circ \tau$ such that $(\sigma \circ \tau)(\varphi) = \sigma(\tau(\varphi))$. Let $\sigma \equiv \tau$ if $\vdash \sigma(p_i) \leftrightarrow \tau(p_i)$ for every i. A substitution τ is *more general* than σ, written as $\sigma \preceq \tau$, if there exists a substitution υ such that $\sigma \equiv \upsilon \circ \tau$. We write $\sigma \approx \tau$ if $\sigma \preceq \tau$ and $\tau \preceq \sigma$, and $\sigma \prec \tau$ if $\sigma \preceq \tau$ but $\tau \npreceq \sigma$. Note that $\preceq$ is a preorder, and $\approx$ is its kernel equivalence relation. A *complete set of unifiers of* Γ is a cofinal subset C of $\langle U(\Gamma), \preceq \rangle$ (i.e., a set of unifiers of Γ such that every unifier of Γ is less general than some element of C). If $\{\sigma\}$ is a complete set of unifiers of Γ, then σ is a *most general unifier* (*mgu*) of Γ.

If $\langle P, \leq \rangle$ is a nonempty poset, let M be the set of its maximal elements (i.e., $x \in P$ such that $x < y$ for no $y \in P$). If every element of P is below an element of M, we say that $\langle P, \leq \rangle$ is of

- type 1 (unitary), if $|M| = 1$,
- type ω (finitary), if M is finite and $|M| > 1$,
- type ∞ (infinitary), if M is infinite.

Otherwise, it is of type 0 (nullary).

The *unification type* of Γ is the type of the quotient poset $\langle U(\Gamma), \preceq \rangle / \approx$. Note that Γ is of unitary type iff it has an mgu. The unification type of a logic (that is, for us, of K) is the maximal type of a unifiable finite set of formulas Γ, where we order the unification types as $1 < \omega < \infty < 0$.

A *multiple-conclusion rule* is an expression $\Gamma \;/\; \Delta$, where Γ, Δ are finite sets of formulas. A rule $\Gamma \;/\; \Delta$ is *derivable* if $\Gamma \vdash \psi$ for some $\psi \in \Delta$. A rule $\Gamma \;/\; \Delta$ is *admissible*, written as $\Gamma \mathrel{|\!\!\sim} \Delta$, if every unifier of Γ also unifies some $\psi \in \Delta$. Note that all derivable rules are admissible, but not vice versa. A set of formulas Γ is *admissibly saturated* [8], if every admissible rule of the form $\Gamma \;/\; \Delta$ is derivable. Γ is *exact* if there exists a substitution σ such that

$$\Gamma \vdash \psi \quad \text{iff} \quad \vdash \sigma(\psi)$$

for every formula ψ. Γ is *projective* if it has a unifier σ such that

$$\Gamma \vdash p_i \leftrightarrow \sigma(p_i)$$

for every i. Note that σ is then an mgu of Γ: if $\tau \in U(\Gamma)$, then $\tau \equiv \tau \circ \sigma$.

Fact 2.5

(*i*) *If* Γ *is projective, it is exact.*

(*ii*) *If* Γ *is exact, it is admissibly saturated.*

A *projective approximation* of Γ is a finite set Π_Γ of projective formulas such that $\Gamma \mathrel{|\!\!\sim} \Pi_\Gamma$, and $\pi \vdash \varphi$ for every $\pi \in \Pi_\Gamma$ and $\varphi \in \Gamma$. More generally, an *admissibly saturated approximation* is a set with properties as above, except that its elements are only required to be admissibly saturated instead of projective. If Π_Γ is an admissibly saturated approximation, it is easy to see that

$$\Gamma \mathrel{|\!\!\sim} \Delta \quad \text{iff} \quad \forall \pi \in \Pi_\Gamma \, \exists \psi \in \Delta \, \pi \vdash \psi.$$

Since any admissibly saturated approximation of an admissibly saturated Γ has to include a formula deductively equivalent to $\bigwedge \Gamma$, we have:

Fact 2.6 *The following are equivalent for any modal logic:*

(*i*) *Every* Γ *has a projective approximation.*

(*ii*) *Every* Γ *has an admissibly saturated approximation, and every admissibly saturated formula is projective.*

In $K4$, $S4$, GL, and other transitive logics satisfying the assumptions of Ghilardi [5], every formula has a projective approximation, hence admissibly saturated, exact, and projective formulas coincide.

3 Results

As all of our results concern properties of the formula $p \to \Box p$, our first task is to describe a complete set of unifiers of this formula. Without further ado, this set will consist of the following substitutions.

Definition 3.1 For any $n \in \omega$, we introduce the substitutions

$$\sigma_n(p) = \Box^{<n} p \wedge \Box^n \bot,$$
$$\sigma_\top(p) = \top,$$

where $\sigma_\alpha(q) = q$ for every variable $q \neq p$ and $\alpha \in \omega_+ := \omega \cup \{\top\}$.

Lemma 3.2 σ_α *is a unifier of* $p \to \Box p$ *for every* $\alpha \in \omega_+$*.*

Proof: We have

$$\vdash \Box^{<n} p \wedge \Box^n \bot \to \Box^{\leq n} p \to \Box\Box^{<n} p$$

and

$$\vdash \Box^n \bot \to \Box^{n+1} \bot,$$

the rest is clear. $\Box$

We start with simple criteria for recognizing that a given unifier of $p \to \Box p$ is below σ_α.

Lemma 3.3 *If* σ *is a unifier of* $p \to \Box p$ *and* $n \in \omega$*, the following are equivalent:*

(*i*) $\sigma \preceq \sigma_n$,

(*ii*) $\sigma \equiv \sigma \circ \sigma_n$,

(*iii*) $\vdash \sigma(p) \to \Box^n \bot$.

Proof: (ii) $\to$ (i) follows from the definition of $\preceq$.

(i) $\to$ (iii): if $\sigma \equiv \tau \circ \sigma_n$, then $\vdash \sigma_n(p) \to \Box^n \bot$ implies $\vdash \tau(\sigma_n(p)) \to \tau(\Box^n \bot)$, i.e., $\vdash \sigma(p) \to \Box^n \bot$.

(iii) $\to$ (ii): put $\varphi = \sigma(p)$. Since σ is a unifier of $p \to \Box p$, we have $\vdash \varphi \to \Box\varphi$, hence $\vdash \varphi \to \Box^{<n} \varphi$ by induction. Since we also assume $\vdash \varphi \to \Box^n \bot$, we have $\vdash \sigma(p) \to \sigma(\sigma_n(p))$. The other implication is trivial. $\Box$

Definition 3.4 For any substitution σ, let $\sigma \upharpoonright p$ be the substitution τ such that $\tau(p) = \sigma(p)$, and $\tau(q) = q$ for every variable $q \neq p$.

Lemma 3.5 *If σ is a substitution, the following are equivalent:*

(i) $\sigma \preceq \sigma_\top$,

(ii) $\sigma \equiv \sigma \circ \sigma_\top$,

(iii) $\sigma \upharpoonright p \equiv \sigma_\top$,

(iv) $\vdash \sigma(p)$.

Proof: Similar to Lemma 3.3. $\square$

The crucial element in the description of $U(p \to \Box p)$ is to show that one of the conditions in Lemma 3.3 or 3.5 applies to every unifier. This amounts to a variant of the rule of margins, as alluded to in the introduction.

Proposition 3.6 *If $\vdash \varphi \to \Box\varphi$, then $\vdash \varphi$ or $\vdash \varphi \to \Box^n\bot$, where $n = \mathrm{md}(\varphi)$.*

Proof: Assume $\nvdash \varphi$ and $\nvdash \varphi \to \Box^n\bot$. By Fact 2.2, the latter implies that there exists a finite irreflexive intransitive tree $\langle F, R, \vDash\rangle$ with root x_0 such that $F, x_0 \vDash \varphi \wedge \Diamond^n\top$. This means that there exists a sequence $x_0\ R\ x_1\ R \cdots R\ x_n$ of elements of F, and as R is an intransitive tree, $x_n \notin R^{<n}(x_0)$. Since $\nvdash \varphi$, there exists a model $\langle G, S, \vDash\rangle$ and a point $x_{n+1} \in G$ such that $G, x_{n+1} \nvDash \varphi$. Let $\langle H, T, \vDash\rangle$ be the disjoint union of F and G, where we additionally put $x_n\ T\ x_{n+1}$. Since $F \upharpoonright R^{\leq n}(x_0) = H \upharpoonright T^{\leq n}(x_0)$, we have $H, x_0 \vDash \varphi$ by Fact 2.3. On the other hand, $H, x_{n+1} \nvDash \varphi$, hence there exists $i \leq n$ such that $H, x_i \vDash \varphi$ and $H, x_{i+1} \nvDash \varphi$. Then $H, x_i \nvDash \varphi \to \Box\varphi$. $\square$

Ignoring the explicit dependence of n on φ, we can rephrase Proposition 3.6 by saying that the infinitary multiple-conclusion rule

$$p \to \Box p \ / \ \{p \to \Box^n\bot : n \in \omega\} \cup \{p\}$$

is admissible in K.

Corollary 3.7 *The substitutions $\{\sigma_\alpha : \alpha \in \omega_+\}$ form a complete set of unifiers for the formula $p \to \Box p$.*

Proof: By Lemmas 3.2, 3.3, and 3.5, and Proposition 3.6. $\square$

Corollary 3.8 *Unification in K is nullary.*

Proof: Since $\vdash \sigma_n(p) \to \Box^{n+1}\bot$ and $\nvdash \sigma_{n+1}(p) \to \Box^n\bot$, Lemma 3.3 shows that $\sigma_n \prec \sigma_{n+1}$. By a similar argument, σ_n and $\sigma_\top$ are incomparable. Thus, none of the σ_n is majorized by a maximal element in $U(p \to \Box p)/{\approx}$. $\square$

Now we turn to the (non)equivalence of exact and admissibly saturated formulas. That $p \to \Box p$ is inexact follows easily from Corollary 3.7:

Proposition 3.9 *$p \to \Box p$ is not exact, and a fortiori not projective.*

Proof: Assume for contradiction that σ is a substitution such that

$$p \to \Box p \vdash \psi \quad \text{iff} \quad \vdash \sigma(\psi)$$

for every ψ. In particular, σ is a unifier of $p \to \Box p$, hence $\sigma \preceq \sigma_\alpha$ for some $\alpha \in \omega_+$ by Corollary 3.7. If $\alpha \in \omega$, we have $\vdash \sigma_\alpha(p \to \Box^\alpha \bot)$, hence $\vdash \sigma(p \to \Box^\alpha \bot)$. However, $p \to \Box p \nvdash p \to \Box^\alpha \bot$, a contradiction. If $\alpha = \top$, we obtain a contradiction similarly using $\vdash \sigma(p)$. $\Box$

We remark that σ_n and $\sigma_\top$ are projective unifiers of the formulas $p \to \Box p \wedge \Box^n \bot$ and p, respectively.

We complement Proposition 3.9 by showing that $p \to \Box p$ is admissibly saturated. We mention another pathological property of $p \to \Box p$ which will arise from the proof. Intuitively, it is not so surprising that a formula φ with an infinite cofinal chain of unifiers like σ_n can be admissibly saturated, as the unifiers high enough in the chain eventually become "indistinguishable" when applied to any particular formula ψ. However, if a formula has two incomparable maximal unifiers, say σ, σ', we would expect it *not* to be admissibly saturated: presumably, we can find formulas ψ, ψ' unified by σ and σ', respectively, but not vice versa. Then $\varphi \mathrel{|\!\sim} \psi, \psi'$, but not $\varphi \mathrel{|\!\sim} \psi$ or $\varphi \mathrel{|\!\sim} \psi'$. By the same token, we would expect that a formula like $p \to \Box p$, whose set of unifiers consists of two incomparable parts (a chain and a maximal unifier, in our case), is not admissibly saturated either.

What happens here is that when we apply the unifiers σ_n to a particular formula, they not only become "indistinguishable" from each other for n large enough, but they also "cover" the unifier $\sigma_\top$, despite that it is not comparable to any element of the chain. Returning to our weak rule of margins, one can imagine that the margins of error about the approximate falsities $\Box^n \bot$ gradually blend into the margin about the truth $\top$ as n goes to infinity.

Proposition 3.10 *$p \to \Box p$ is admissibly saturated.*

Proof: Let $p \to \Box p \mathrel{|\!\sim} \Delta$, and pick $n > \max\{\mathrm{md}(\psi) : \psi \in \Delta\}$. Since σ_n unifies $p \to \Box p$, there exists $\psi \in \Delta$ such that $\vdash \sigma_n(\psi)$. We claim

$$p \to \Box p \vdash \psi.$$

If not, there exists a Kripke model $\langle F, R, \vDash\rangle$ such that $F \vDash p \to \Box p$ and $F, x_0 \nvDash \psi$ for some $x_0 \in F$. First, we unravel F to a tree: let $\langle G, S, \vDash\rangle$ be the model where G consists of sequences $\langle x_0, \dots, x_m\rangle$ such that $m \in \omega$, $x_i \in F$, $x_i \; R \; x_{i+1}$; we put $\langle x_0, \dots, x_m\rangle \; S \; \langle x_0, \dots, x_m, x_{m+1}\rangle$; and $G, \langle x_0, \dots, x_m\rangle \vDash p_j$ iff $F, x_m \vDash p_j$ for each variable p_j. The mapping $f \colon G \to F$ given by $f(\langle x_0, \dots, x_m\rangle) = x_m$ is a p-morphism, hence it preserves the valuation of formulas by Fact 2.4. In particular, $G \vDash p \to \Box p$ and $G, \langle x_0\rangle \nvDash \psi$.

Let H be the submodel of G consisting of sequences $\langle x_0, \dots, x_m \rangle$ where $m < n$. We still have $H \vDash p \to \Box p$. Moreover, $H \vDash \Box^n \bot$, hence $H \vDash p \leftrightarrow \sigma_n(p)$. On the other hand, $G \supseteq H \supseteq G \upharpoonright S^{\leq \mathrm{md}(\psi)}(\langle x_0 \rangle)$, hence $H, \langle x_0 \rangle \nvDash \psi$ by Fact 2.3. Together, these properties imply $H, \langle x_0 \rangle \nvDash \sigma_n(\psi)$, contradicting $\vdash \sigma_n(\psi)$. □

We remark that unlike Proposition 3.6, we could not directly take a finite irreflexive intransitive tree for F in the proof above, because K is not finitely strongly complete with respect to such frames. (Every finite irreflexive tree is converse well-founded, and therefore validates Löb's rule $\Box p \to p \,/\, p$, which is admissible but not derivable in K.)

To conclude the paper, we have provided examples confirming that unification and admissibility in the basic modal logic K involves peculiar phenomena not encountered in the familiar case of transitive modal logics with frame extension properties. This might be seen as enhancing credence in the possibility that admissibility (or even unifiability) in K is undecidable. However, it might also mean that while admissibility in K could be decidable, showing this will require methods more powerful than what we are used to from the transitive case.

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