On the Discrete Maximum Principle for Higher Order Finite Elements

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Maximum Principle

Let $f \leq 0$ in $\Omega \subset \mathbb{R}^d$ and $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ be the solution of

$$-\sum_{i,j=1}^{d} a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} = f,$$

where $A(x) = \{a_{ij}\}_{i,j=1}^{d}$ is uniformly positive-definite in Ω.
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Discrete Maximum Principle (DMP):
Does it hold also for the finite element solution?

Answer: in general NO, but under suitable conditions YES.

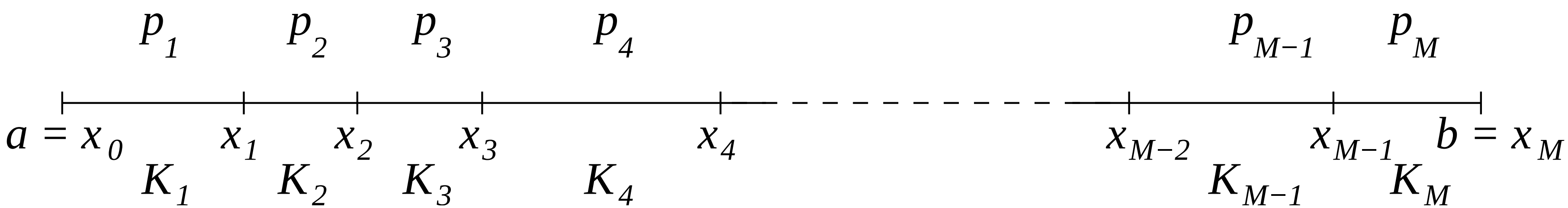
What is known about DMP?

Almost all results:

- n-D, $p = 1$, acute type condition $\Rightarrow$ DMP
  Proof – based on M-matrices.
- Various generalization:
  - weakened acute type condition
  - nonlinear problems
  - parabolic problems
  - ...
- W. Höhn, H. D. Mittelmann: *Some Remarks on the Discrete Maximum Principle for Finite Elements of Higher-Order*, Computing 27, pp. 145–154, 1981.

1D Model Problem

$$-u'' = f \quad \text{in } \Omega = (a,b); \quad u(a) = u(b) = 0$$



- $V_{hp} = \{v_{hp} \in H_0^1(\Omega) : v_{hp}|_{K_i} \in P^{p_i}(K_i)\}$
- $N = \dim(V_{hp}) = -1 + \sum_{i=1}^{M} p_i$

Find $u_{hp} \in V_{hp}$:

$$\int_a^b u'_{hp}(x) v'_{hp}(x)\,\mathrm{d}x = \int_a^b f(x) v_{hp}(x)\,\mathrm{d}x \qquad \text{for all } v_{hp} \in V_{hp}$$

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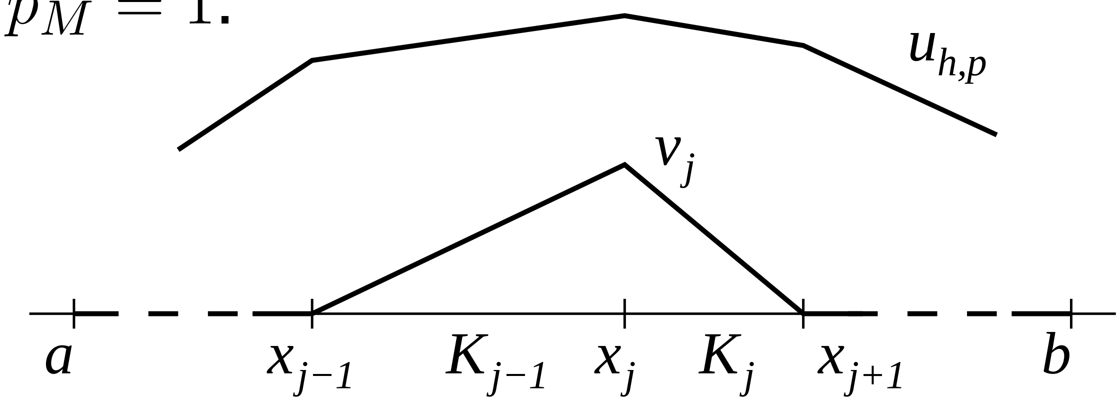
YES, if $p_1 = p_2 = \ldots = p_M = 1$.



$$\int_{x_{j-1}}^{x_{j+1}} u_{hp}'(x) v_j'(x)\, \mathrm{d}x = \int_{x_{j-1}}^{x_{j+1}} \underbrace{f(x)}_{\geq 0} \underbrace{v_j(x)}_{\geq 0}\, \mathrm{d}x \geq 0,$$

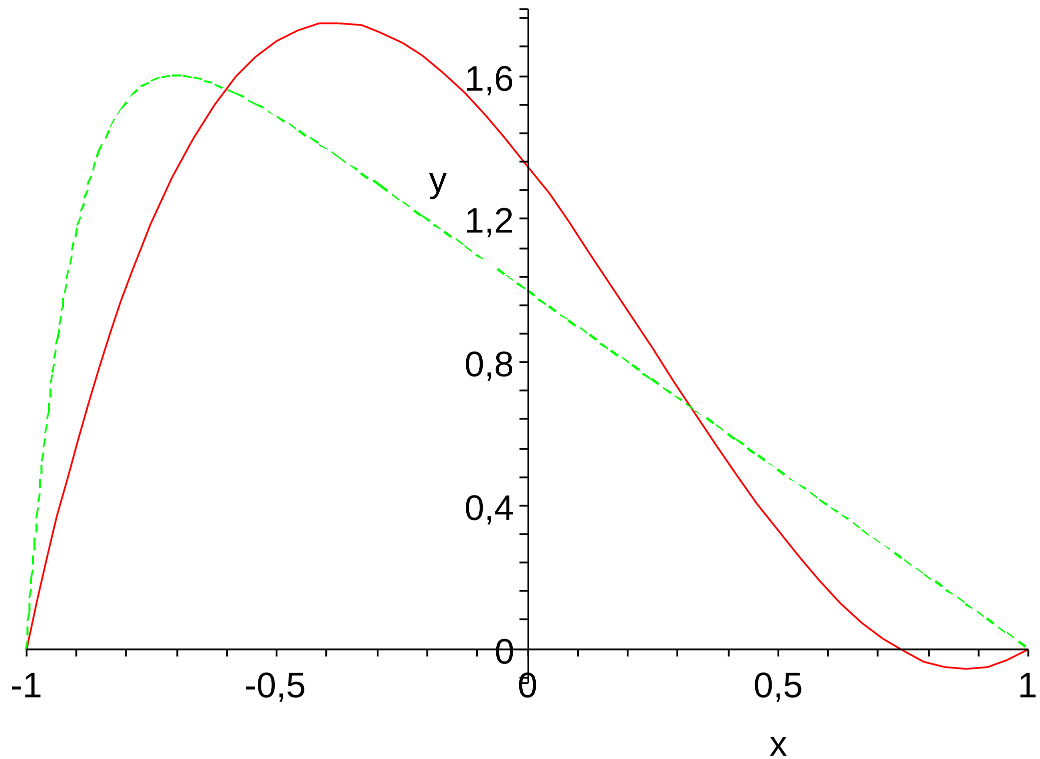
$$0 \leq Du_{hp}^{(j-1)} \frac{x_j - x_{j-1}}{x_j - x_{j-1}} - Du_{hp}^{(j)} \frac{x_{j+1} - x_j}{x_{j+1} - x_j} = Du_{hp}^{(j-1)} - Du_{hp}^{(j)}$$

Discrete Maximum Principle

Is $u_{hp}(x) \geq 0$ for any $f(x) \geq 0$? (for all $x \in \Omega$)

NO, for general $p_1, p_2, \ldots, p_M \geq 1$.

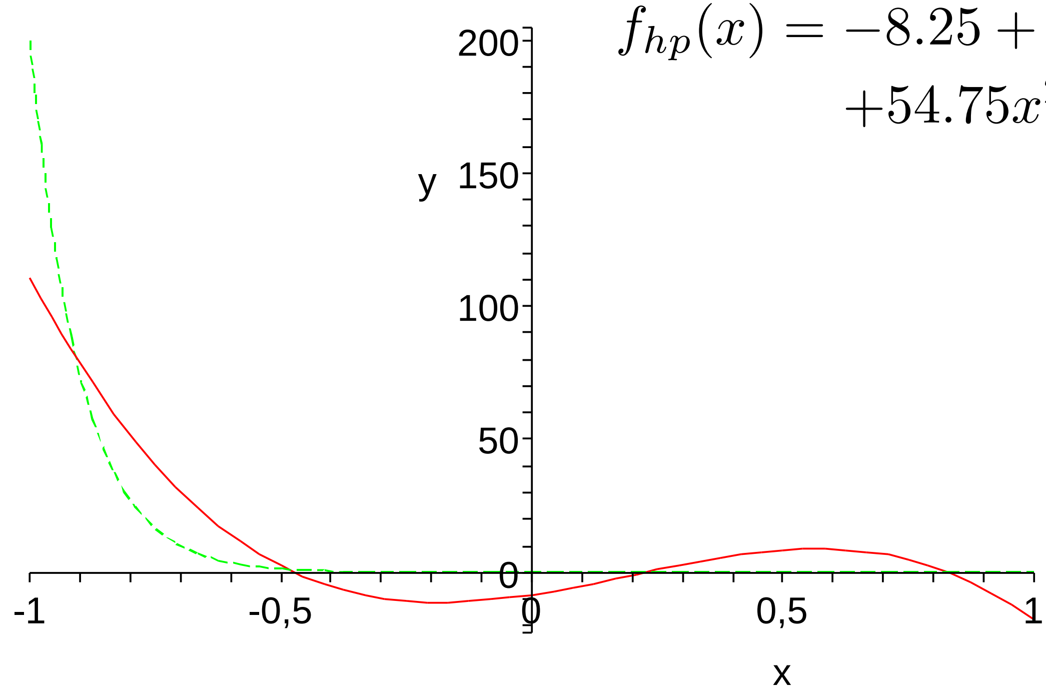
$\Omega = (-1, 1)$, $\mathcal{T}_{hp} = \{K_1\}$, $p_1 = 3$, $f(x) = 200e^{-10(x+1)} \geq 0$



Why?

L^2 projection of $f(x) = 200e^{-10(x+1)}$ to $P^3(\Omega) \supset V_{hp}$

$$\int_a^b f_{hp}(x) v_{hp}(x)\,\mathrm{d}x = \int_a^b f(x) v_{hp}(x)\,\mathrm{d}x \quad \text{for all } v_{hp} \in P^3(\Omega),$$



$$f_{hp}(x) = -8.25 + 29.175x + 54.75x^2 - 93.625x^3$$

Weak DMP

Let $f_{hp} \geq 0$, where f_{hp} is the L^2-projection of f to

$$W = \{v \in C(\overline{\Omega}); v|_{K_i} \in P^{p_i}(K_i),\ 1 \leq i \leq M\}.$$

Then (for the model problem $-u'' = f$, $u(a) = u(b) = 0$), $u_{hp} \geq 0$.

Proof:

It is enough to consider:
$(a,b) = (-1,1)$ and $\mathcal{T}_{hp} = \{K_1\}$, $p \geq 2$.

$$V_{hp} = V_{hp}^{(v)} \oplus V_{hp}^{(b)}$$

$$u_{hp} = u_{hp}^{(v)} + u_{hp}^{(b)}$$

$$\int_a^b \left(u_{hp}^{(v)}\right)' v_{hp}' \,\mathrm{d}x = \int_a^b f\, v_{hp} \,\mathrm{d}x \quad \forall v_{hp} \in V_{hp}^{(v)}$$

$$\int_a^b \left(u_{hp}^{(b)}\right)' v_{hp}' \,\mathrm{d}x = \int_a^b f\, v_{hp} \,\mathrm{d}x \quad \forall v_{hp} \in V_{hp}^{(b)}$$

Proof: reference element

Reference element: $(-1,1)$

Lobatto shape functions $l_2, l_3, \dots, l_{10}$:

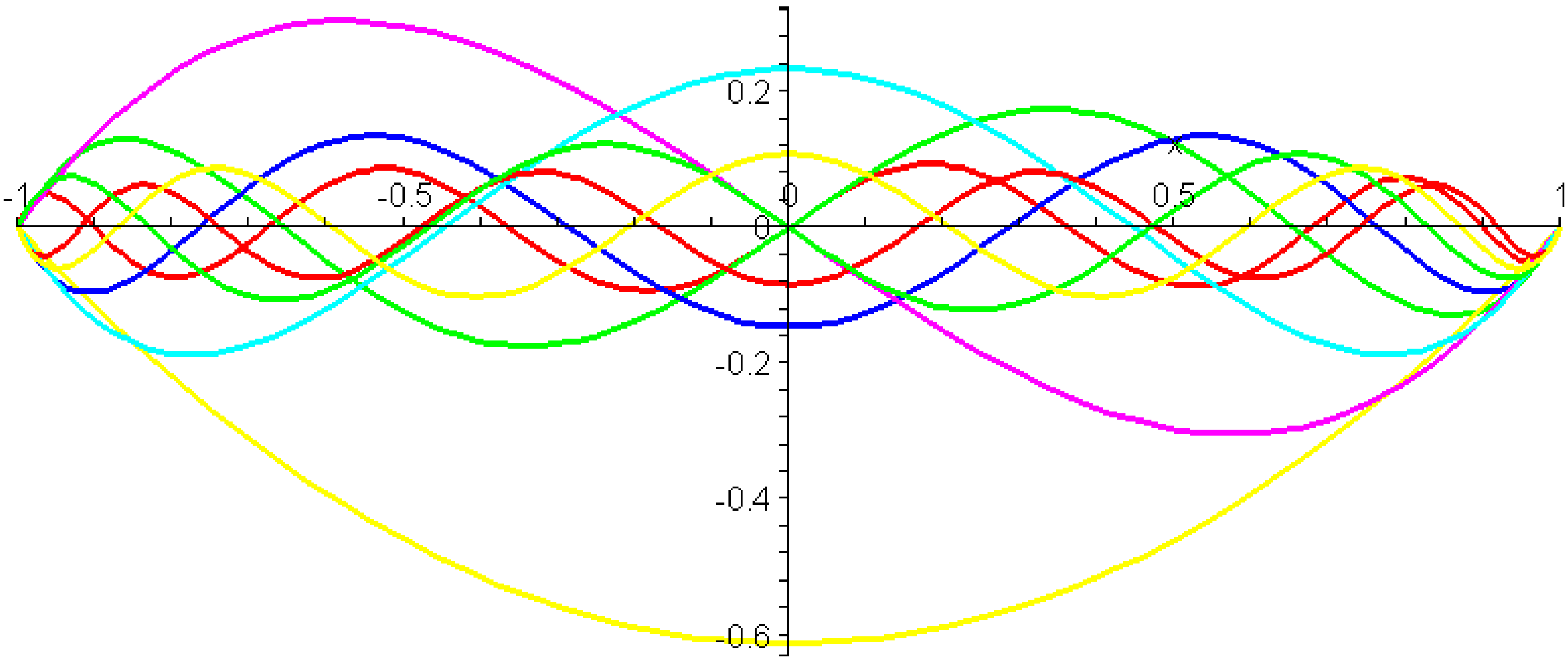
$$l_k(x) \;=\; \frac{1}{\|L_{k-1}\|_{L^2}} \int_{-1}^{x} L_{k-1}(\xi)\,\mathrm{d}\xi, \quad 2 \le k,$$

Important property: $\int_{-1}^{1} l_i'(x) l_j'(x)\,\mathrm{d}x = \delta_{ij}.$

Proof: reference element

$$
\begin{aligned}
l_2(x) &= \frac{1}{2}\sqrt{\frac{3}{2}}(x^2-1),\\
l_3(x) &= \frac{1}{2}\sqrt{\frac{5}{2}}(x^2-1)x,\\
l_4(x) &= \frac{1}{8}\sqrt{\frac{7}{2}}(x^2-1)(5x^2-1),\\
l_5(x) &= \frac{1}{8}\sqrt{\frac{9}{2}}(x^2-1)(7x^2-3)x,\\
l_6(x) &= \frac{1}{16}\sqrt{\frac{11}{2}}(x^2-1)(21x^4-14x^2+1),\\
l_7(x) &= \frac{1}{16}\sqrt{\frac{13}{2}}(x^2-1)(33x^4-30x^2+5)x,\\
l_8(x) &= \frac{1}{128}\sqrt{\frac{15}{2}}(x^2-1)(429x^6-495x^4+135x^2-5),\\
l_9(x) &= \frac{1}{128}\sqrt{\frac{17}{2}}(x^2-1)(715x^6-1001x^4+385x^2-35)x,\\
l_{10}(x) &= \frac{1}{256}\sqrt{\frac{19}{2}}(x^2-1)(2431x^8-4004x^6+2002x^4-308x^2+7).
\end{aligned}
$$

Proof: reference element



Proof: $p = 2$

$$u_{hp}(x) = y_1 l_2(x), \qquad l_2(x) \le 0$$

$$y_1 = \int_{-1}^{1} y_1 l_2'(z) l_2'(z)\,\mathrm{d}z$$

$$= \int_{-1}^{1} u_{hp}'(z) l_2'(z)\,\mathrm{d}z = \int_{-1}^{1} \underbrace{f_{hp}(z)}_{\ge 0}\,\underbrace{l_2(z)}_{\le 0}\,\mathrm{d}z \le 0$$

Thus $u_{hp}(x) = \underbrace{y_1}_{\le 0}\,\underbrace{l_2(x)}_{\le 0} \ge 0$ in $(-1, 1)$!

Proof: $p = 3$

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$$u'_{hp}(-1) \geq 0 \quad \& \quad u'_{hp}(1) \leq 0 \quad \Rightarrow \quad u_{hp} \geq 0 \text{ in } (-1, 1)$$

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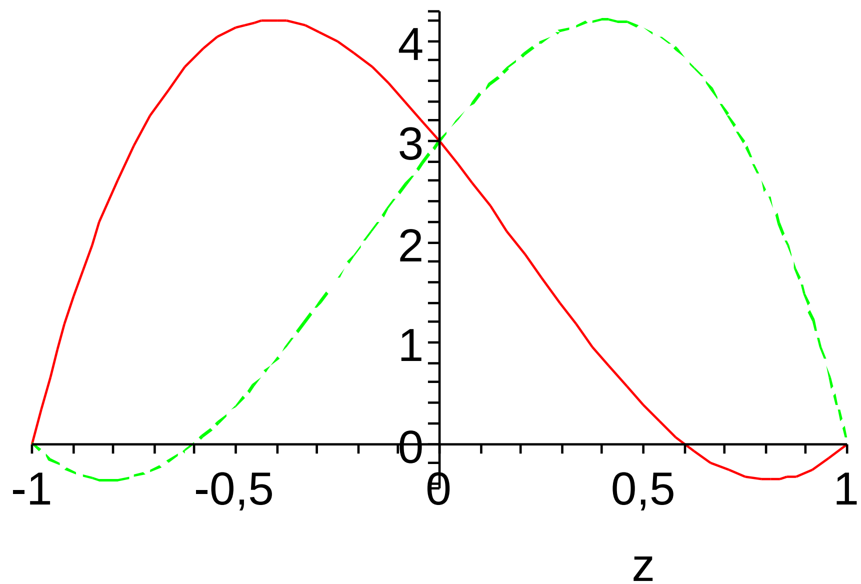
$$0 \leq u'_{hp}(-1) = y_1 l'_2(-1) + y_2 l'_3(-1) = \int_{-1}^{1} f_{hp}(z) \underbrace{\left[l'_2(-1) l_2(z) + l'_3(-1) l_3(z)\right]}_{g_a(z)=(z^2-1)(5z-3)} \mathsf{d}z$$

Proof: $p = 3$

$g_a(x)$, $g_b(x)$:



Show that $\int_{-1}^{1} f_{hp}(z) g_a(z)\,\mathbf{d}z \geq 0$ for all $0 \leq f_{hp} \in P^3(-1,1)$.

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3. f_{hp} is a quadratic function with
   (a) two complex-conjugate complex roots and positive leading term,
   (b) one real root of multiplicity two and positive leading term,
   (c) two roots in $(-\infty, -1]$ and positive leading term,
   (d) two roots in $[1, \infty)$ and positive leading term,
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4. f_{hp} is a cubic function with positive leading term and
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   (d) three different roots in $(-\infty, -1]$,
   (e) one root of multiplicity three in $(-\infty, -1]$,
5. f_{hp} is a cubic function with negative leading term and
   (a)–(e) symmetric conditions to the previous ones.

Eighteen cases.

Proof: $p = 3$, case 4(a)

(one single root in $(-\infty, -1]$ and one root of multiplicity two in $\mathbb{R}$)

It is $f_{hp}(z) = (z-c)^2(z+d)$, where $c \in \mathbb{R}$ and $d \geq 1$:

$$u'_{hp}(-1) = \int_{-1}^{1} f_{hp}(z) g_a(z) \, \mathrm{d}z = d \underbrace{\left(4c^2 + \frac{8}{3}c + \frac{4}{5}\right)}_{\geq 0 \text{ for all } c \in \mathbb{R}} - \frac{4}{7} - \frac{8}{5}c - \frac{4}{3}c^2$$

$$\geq \left(4c^2 + \frac{8}{3}c + \frac{4}{5}\right) - \frac{4}{7} - \frac{8}{5}c - \frac{4}{3}c^2 = \frac{8}{3}c^2 + \frac{16}{15}c + \frac{8}{35} \geq 0$$

Proof: $p = 3$, case 4(b)

(one root in $(-\infty, -1]$ and two real roots in $[1, \infty)$)

It is $f_{hp}(z) = (z - c)(z - d)(z + e)$, where $c, d \geq 1$ such that $d = c + \varepsilon$, $\varepsilon > 0$, and $e \geq 1$:

$$u'_{hp}(-1) = \int_{-1}^{1} f_{hp}(z) g_a(z) \, \mathrm{d}z = -\frac{4}{7} - \frac{4}{5}c + \frac{4}{5}e + \frac{4}{3}ce - \frac{4}{5}d - \frac{4}{3}cd + \frac{4}{3}de + 4cde$$

$$= \Bigg(\underbrace{\frac{4}{3}e - \frac{4}{5}}_{\geq 0} + \underbrace{\left(4e - \frac{4}{3}\right)}_{\geq 0} c \Bigg)\varepsilon + \underbrace{\left(4e - \frac{4}{3}\right)}_{\geq 0} c^2 + \underbrace{\left(\frac{8}{3}e - \frac{8}{5}\right)}_{\geq 0} c + \underbrace{\frac{4}{5}e - \frac{4}{7}}_{\geq 0} \geq 0$$

All 18 cases hold $\Rightarrow$ cubic case solved.

Proof: general p

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$$u_{hp}(x) = \sum_{i=1}^{p-1} \left(\int_{-1}^{1} f_{hp}(z) l_{i+1}(z)\,\mathsf{d}z \right) l_{i+1}(x) = \int_{-1}^{1} f_{hp}(z) \Phi_p(x,z)\,\mathsf{d}z,$$

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$$\Phi_p(x,z) = \sum_{i=1}^{p-1} l_{i+1}(x) l_{i+1}(z)$$

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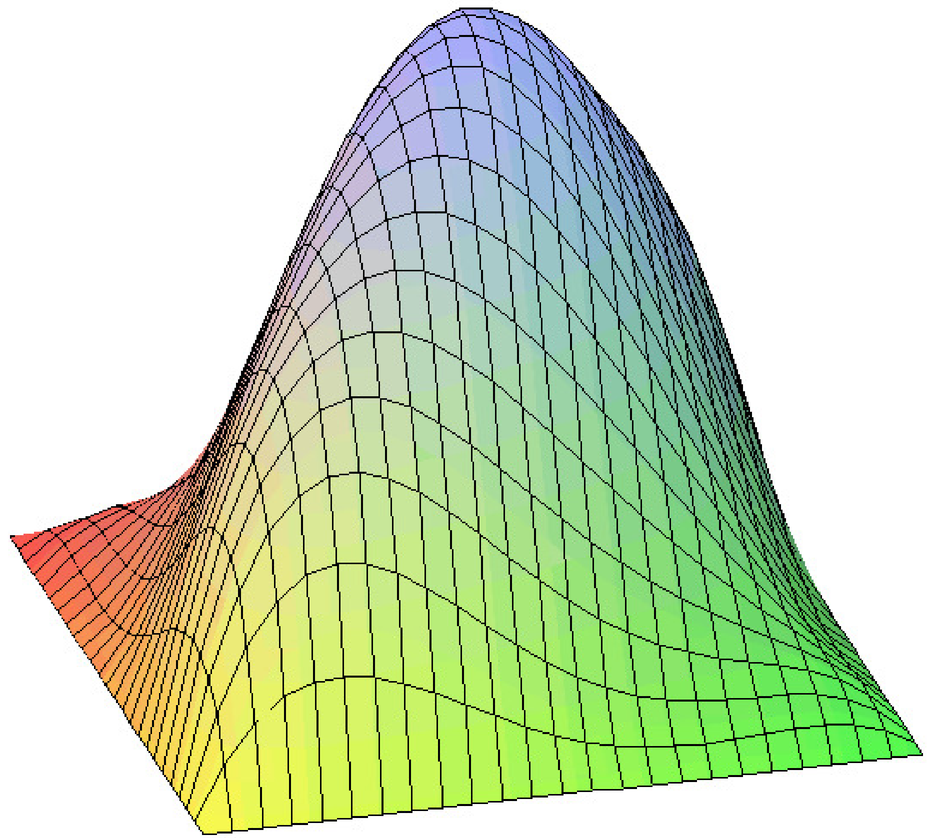
$$\Phi_p(x,z) = \sum_{i=1}^{p-1} l_{i+1}(x) l_{i+1}(z)$$

What can we say about $\Phi_p(x,z)$?

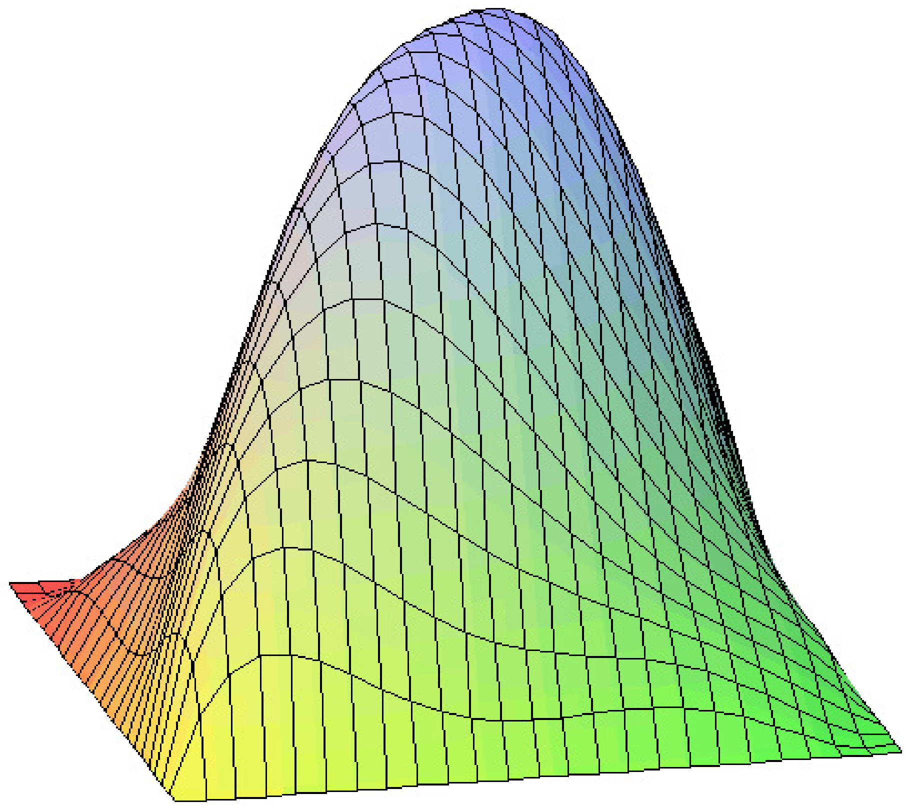
Proof: $p = 4$

$\Phi_4(x, z)$ **is nonnegative in** $(-1, 1)^2 \Rightarrow$ **quartic case holds!**

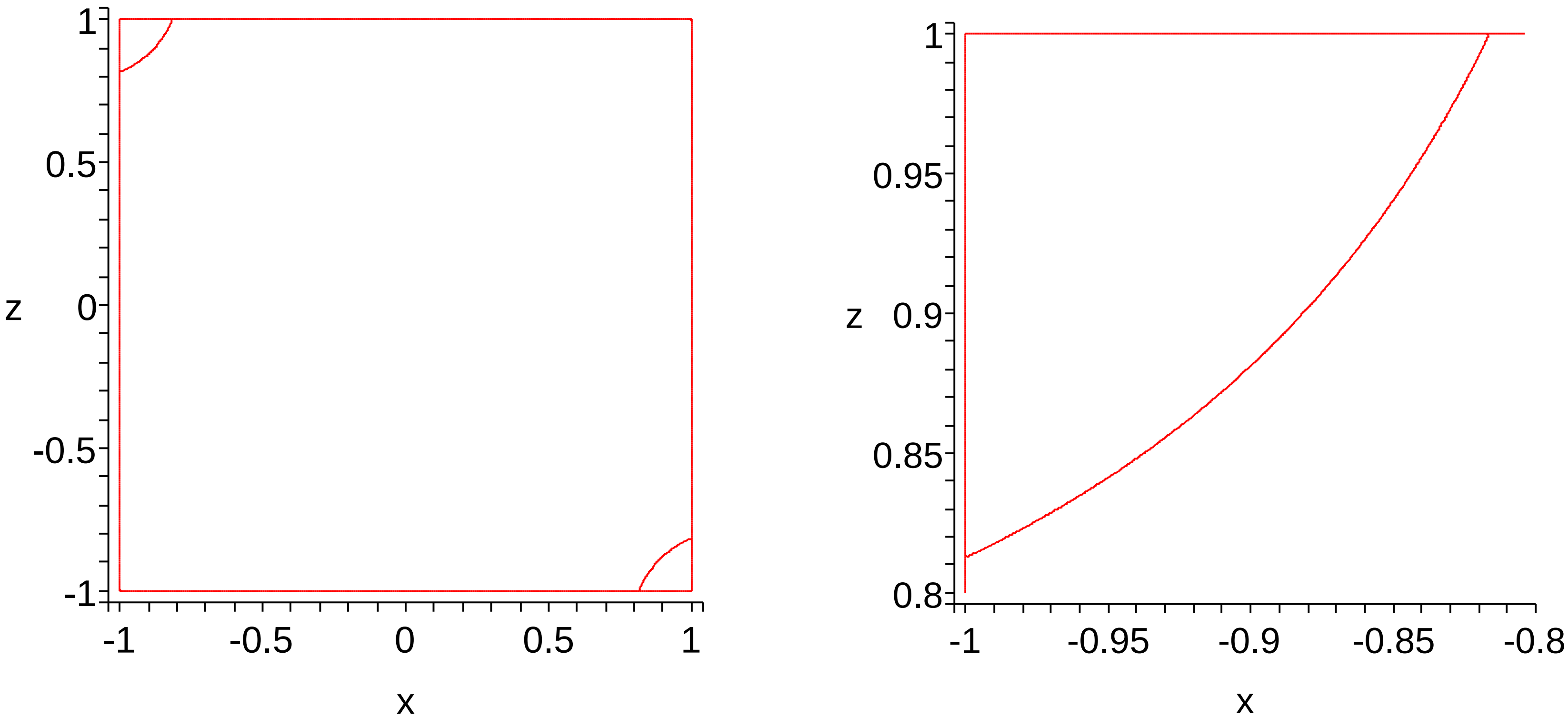


Proof: $p = 5$

$\Phi_5(x,z)$ **is not nonnegative in** $(-1,1)^2$



Proof: $p = 5$ continued



$$u_{hp}(x) = \int_{-1}^{1} f_{hp}(z) \Phi_5(x, z) \, \mathrm{d}z$$

Proof: $p = 5$ continued

Look for a 10th-order quadrature rule in $(-1, 1)$ with

- positive weights w_i,
- outside of the domains of negativity of Φ_5.

Then we will have

$$u_{hp}(x) = \int_{-1}^{1} f_{hp}(z)\Phi_5(x,z)\,\mathrm{d}z = \int_{-1}^{1} F_x^{(10)}(z)\,\mathrm{d}z$$

$$= \sum_{i=0}^{10} \underbrace{w_i}_{\geq 0} \underbrace{F_x^{(10)}(z_i)}_{\geq 0} \geq 0$$

for all $x \in (-1, 1)$.

Proof: $p = 5$ continued

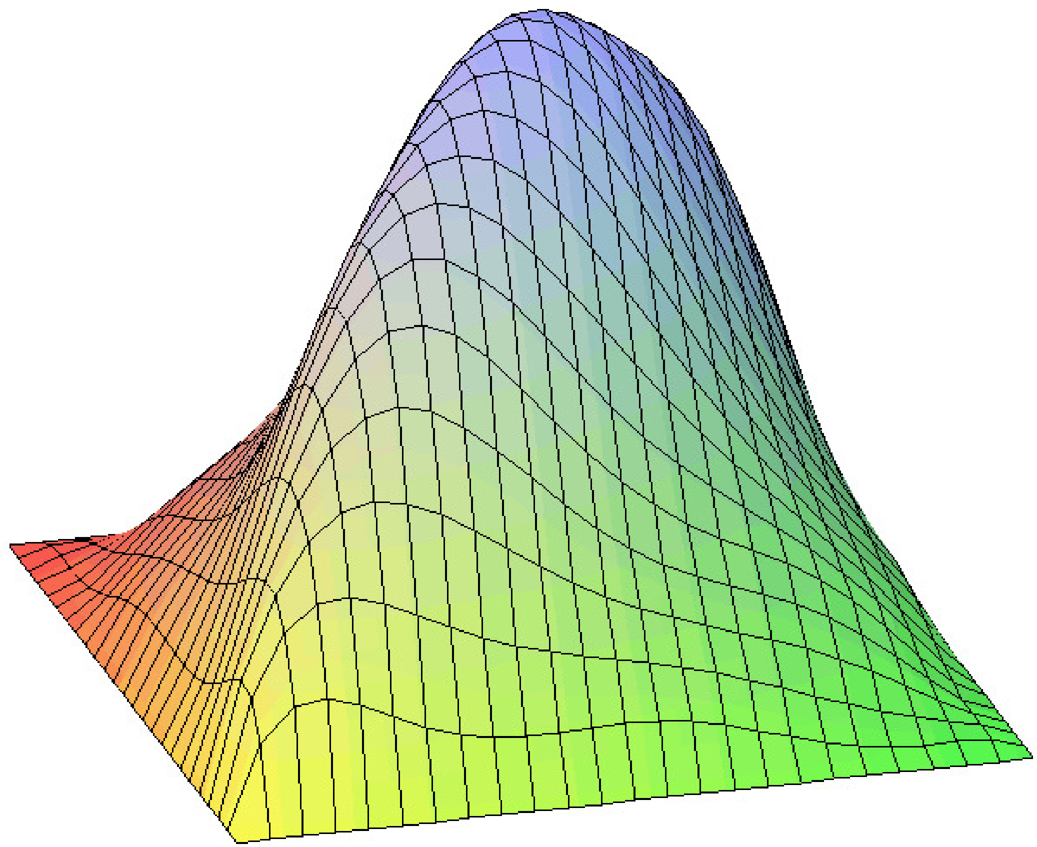
Point	Weight	Point	Weight
-1	0.0534286192	-0.811	0.3054087580
-0.59	0.0030544353	-0.42	0.4473230113
-0.2	0.0066984041	0	0.2760767276
0.2	0.2939694773	0.43	0.0149245373
0.6	0.3805105712	0.9	0.1999066353
1	0.0186988234		

Table 1: 10th-order quadrature rule in Ω with positive weights and points lying outside of $(-1, -0.811)$ – calculated by Maple.

This concludes the proof for $p = 5$.

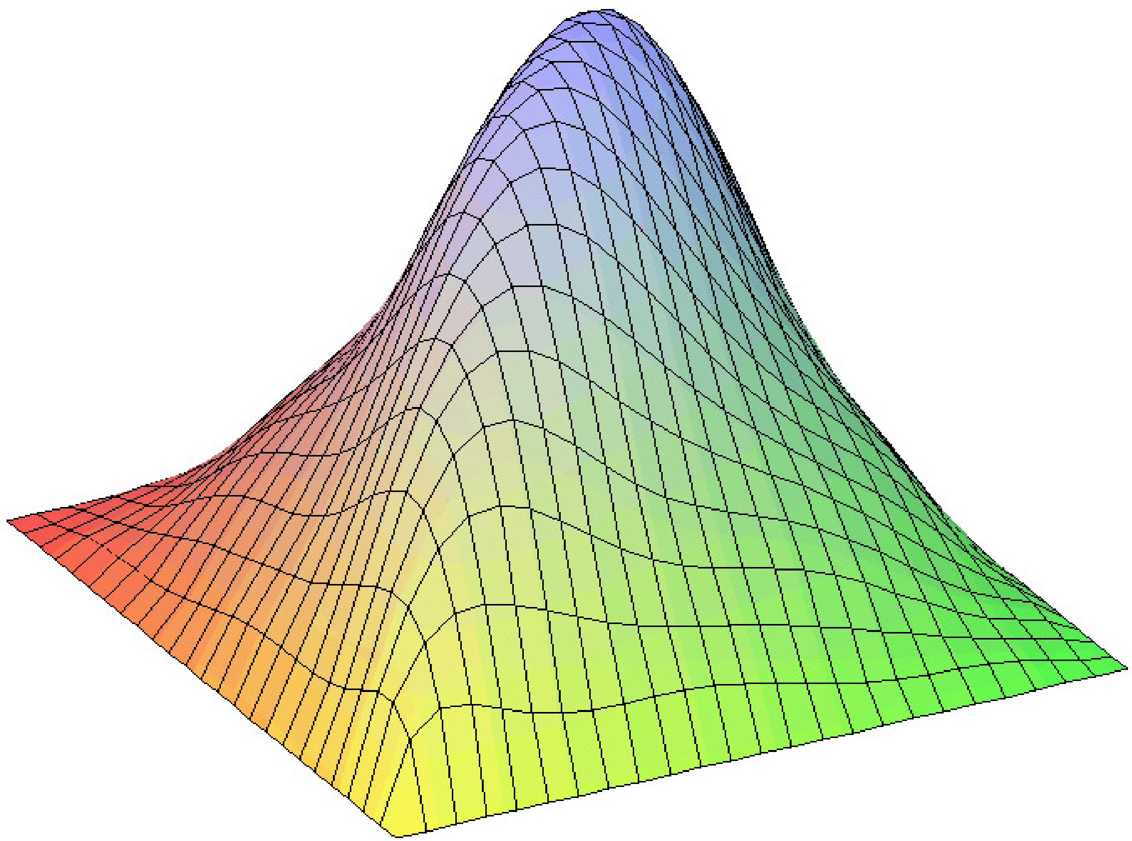
Proof: $p = 6$

$\Phi_6(x,z)$ **is nonnegative in** $(-1,1)^2 \Rightarrow$ **case** $p = 6$ **holds!**

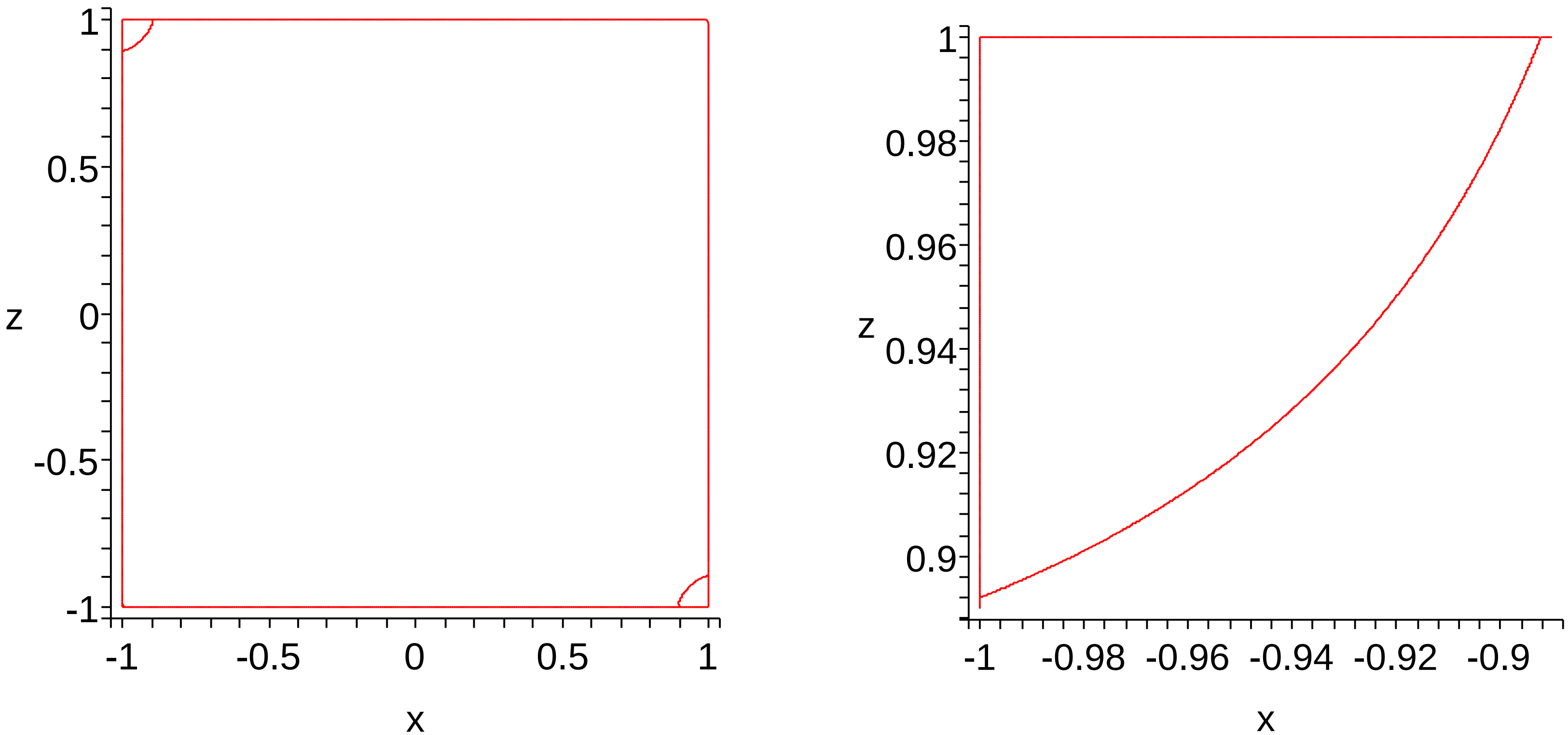


Proof: $p = 7$

$\Phi_7(x, z)$ **is not nonnegative in** $(-1, 1)^2$



Proof: $p = 7$ continued



$$u_{hp}(x) = \int_{-1}^{1} f_{hp}(z) \Phi_7(x, z) \, \mathrm{d}z$$

Proof: $p = 7$ continued

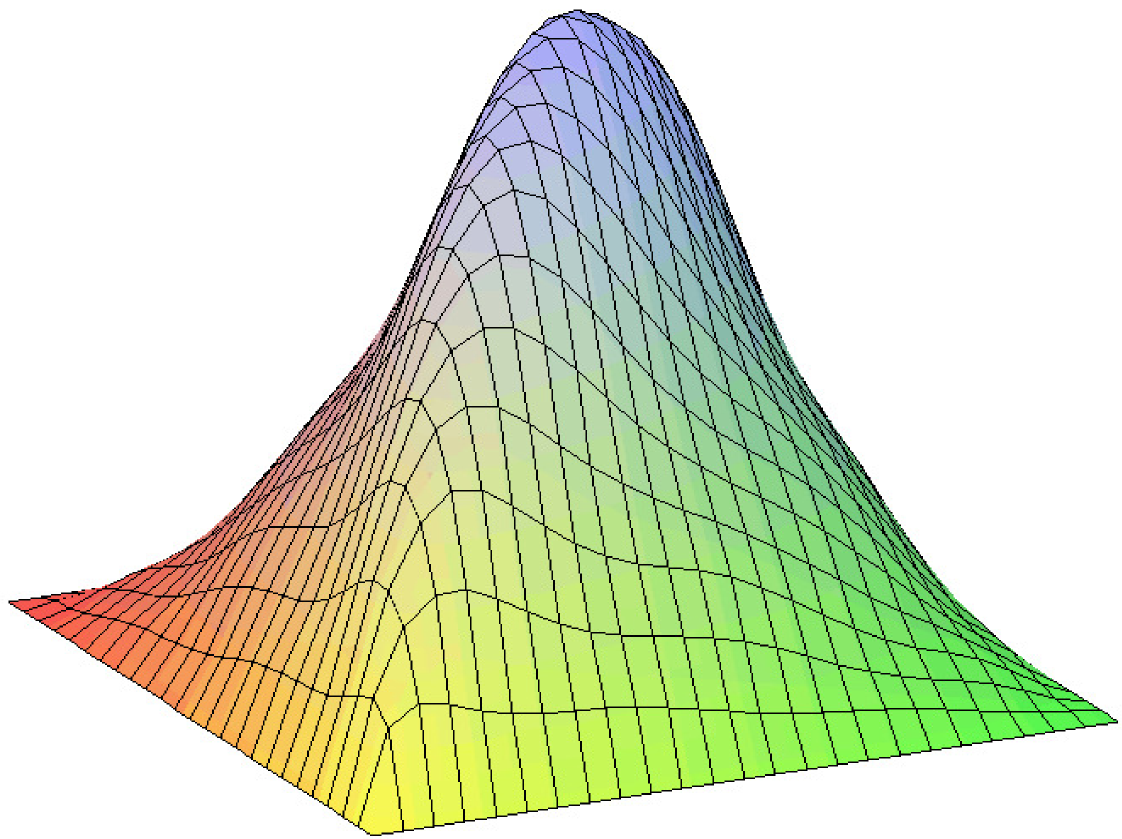
Point	Weight	Point	Weight
-1	0.0306200311	-0.89	0.1806438688
-0.75	0.0016558668	-0.65	0.2862680475
-0.45	0.0379885258	-0.31	0.2988638595
-0.16	0.0833146476	0.1	0.3554921618
0.16	0.0113639321	0.35	0.0204292124
0.47	0.3218682171	0.734	0.1289561668
0.80	0.1314089188	0.955	0.1093567805
1	0.0017697634		

Table 2: 14th-order quadrature rule in Ω with positive weights and points lying outside of $(-1, -0.89)$.

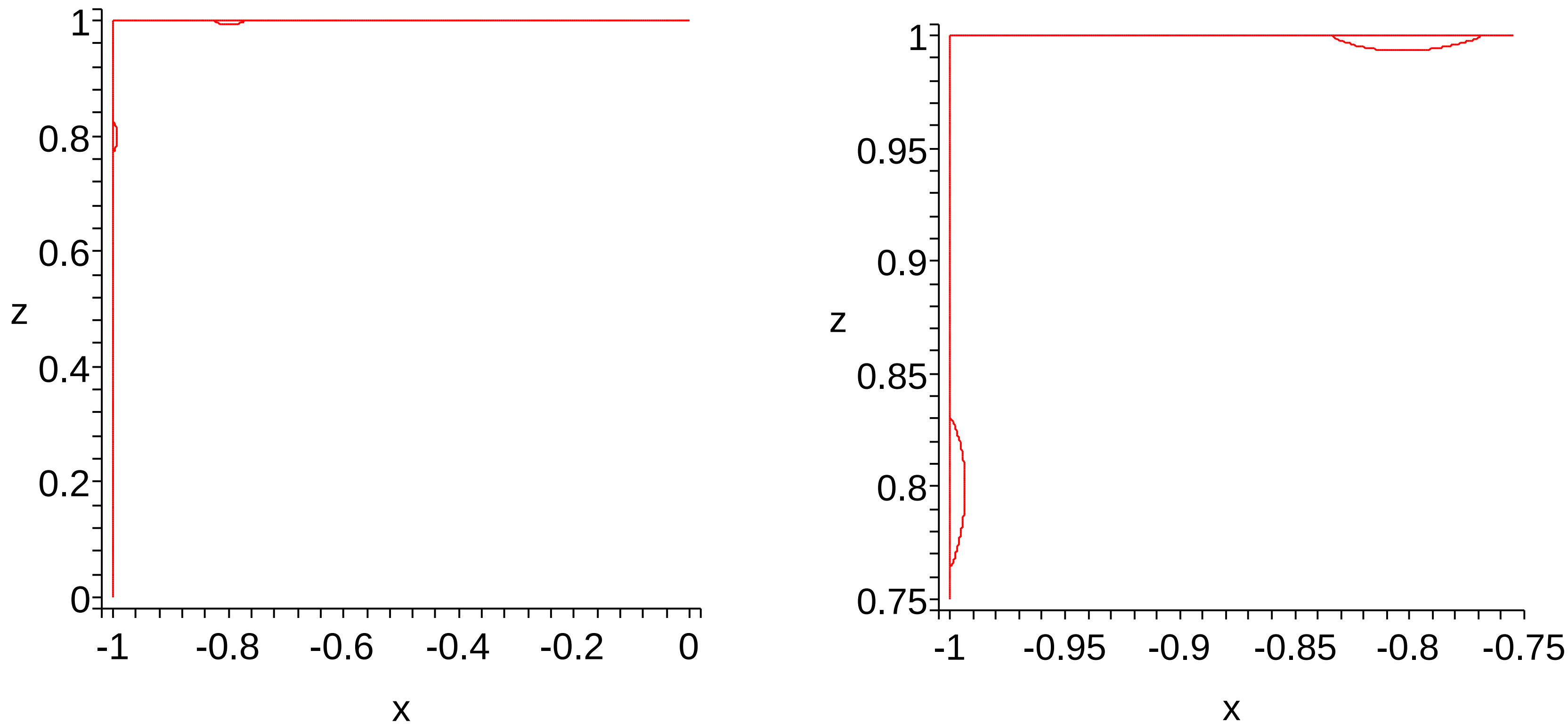
This conclues the proof for $p = 7$.

Proof: $p = 8$

$\Phi_8(x, z)$ **is not nonnegative in** $(-1, 1)^2$



Proof: $p = 8$ continued



$$u_{hp}(x) = \int_{-1}^{1} f_{hp}(z) \Phi_8(x, z) \, \mathrm{d}z$$

Proof: $p = 8$ continued

Point	Weight	Point	Weight
-1	0.0137599529	-0.9564181650	0.0618586932
-0.8854980347	0.0892150513	-0.7582972896	0.1646935265
-0.5719162652	0.1875234174	-0.4628139806	0.0729252387
-0.2917166274	0.2435469772	-0.0811621291	0.0841621866
-0.0061521460	0.1800939083	0.1655560030	0.1320371771
0.3391628868	0.2286184297	0.5726348225	0.2184036287
0.75	0.1285378345	0.85	0.0908051678
0.9230637084	0.0427456544	0.9648584341	0.0509010934
1	0.0101720626		

Table 3: 16th-order quadrature rule in Ω with positive weights and points lying outside of $(0.75, 0.85)$.

Proof: $p = 8$ continued

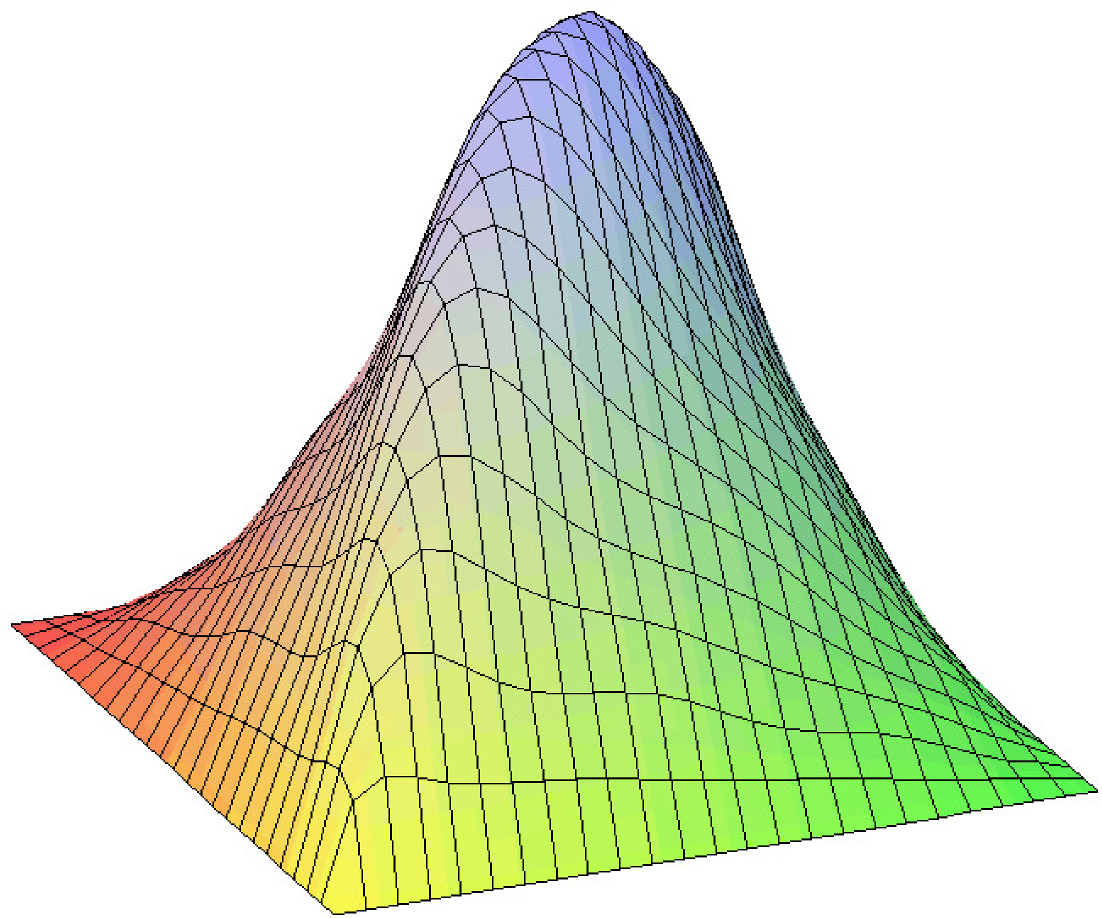
Point	Weight	Point	Weight
-1	0.0097495069	-0.9548248562	0.0857520162
-0.8409569422	0.1018591390	-0.7825414112	0.0149475627
-0.7708636219	0.0926211201	-0.5747624113	0.2476049720
-0.3937499257	0.0549434125	-0.3273530867	0.0276562411
-0.2532942335	0.2543287199	0.0382371812	0.2892622856
0.2837396038	0.1910189889	0.4501581170	0.1560300966
0.5808907063	0.1246581226	0.7443822112	0.1842879621
0.8927849373	0.0841645246	0.9421667341	0.0612885001
1	0.0198268291		

Table 4: 16th-order quadrature rule in Ω with positive weights and points lying outside of $(0.98, 1)$.

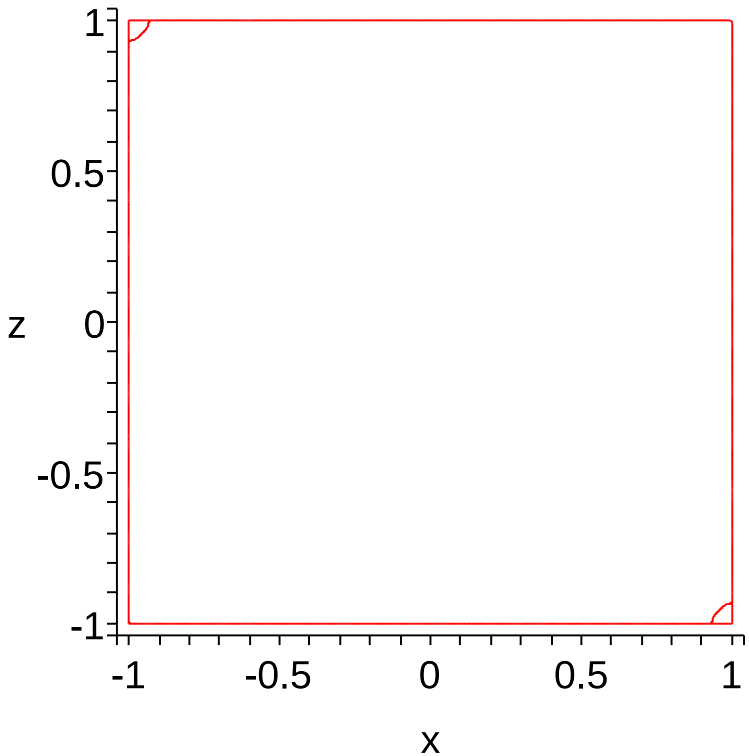
This conclues the proof for $p = 8$.

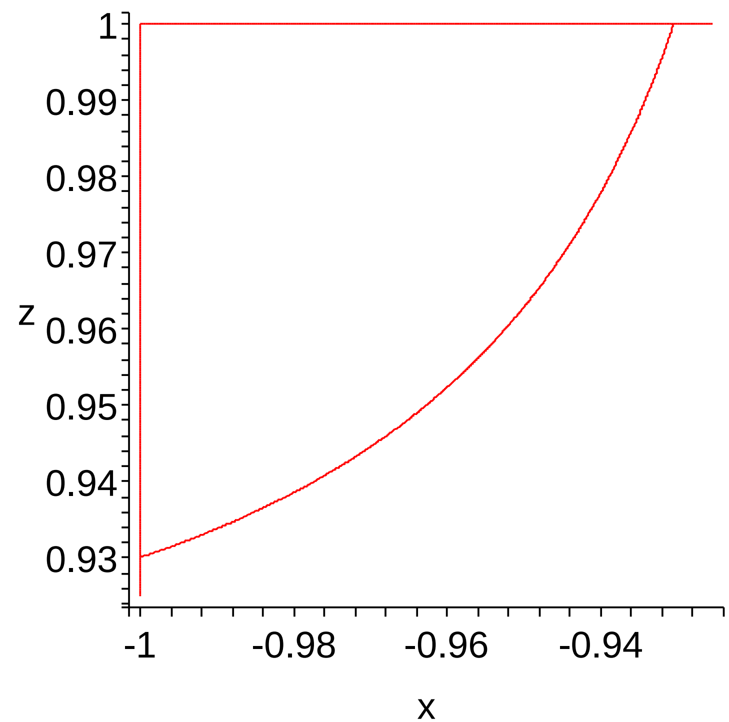
Proof: $p = 9$

$\Phi_9(x, z)$ **is not nonnegative in** $(-1, 1)^2$



Proof: $p = 9$ continued





$$u_{hp}(x) = \int_{-1}^{1} f_{hp}(z)\Phi_9(x, z)\,\mathrm{d}z$$

Proof: $p = 9$ continued

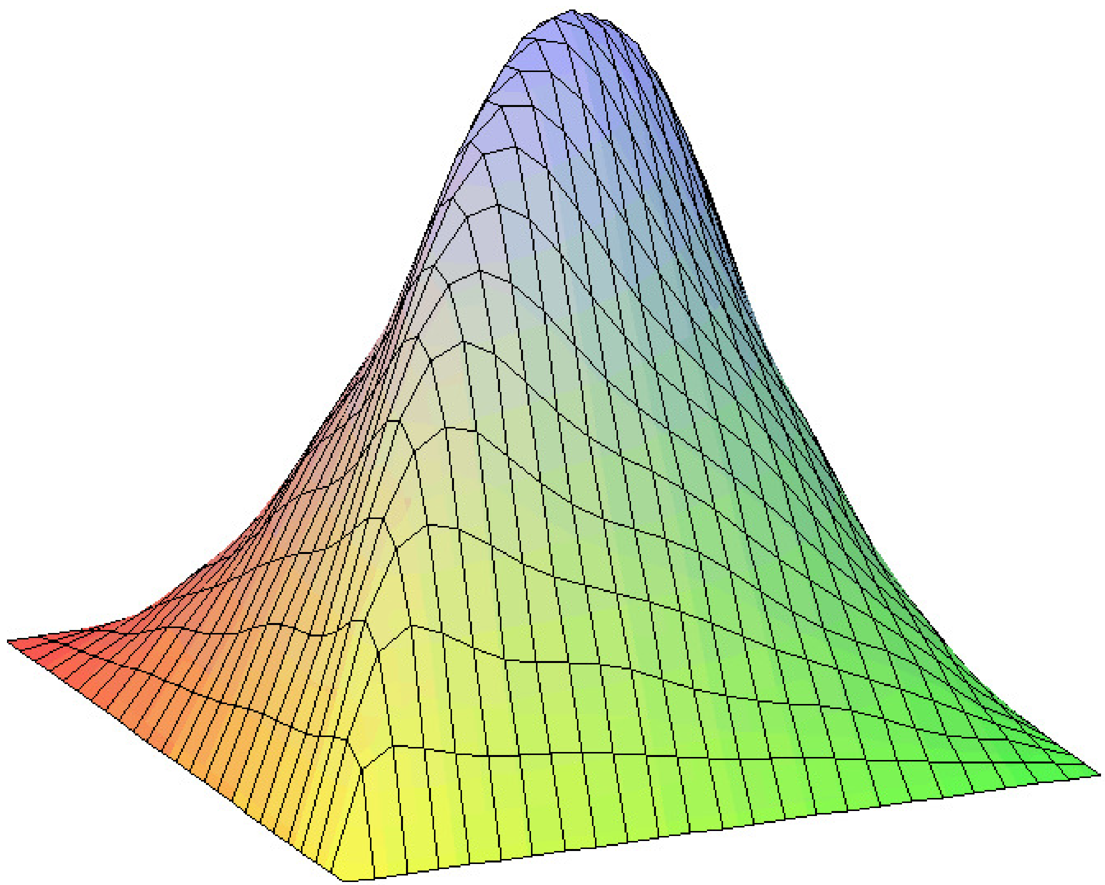
Point	Weight	Point	Weight
-1	0.01937406240	-0.93	0.1153128270
-0.885	0.00157968340	-0.772	0.1947443595
-0.65	0.00126499680	-0.55	0.2341166464
-0.4	0.06286669339	-0.25	0.2438572426
-0.08	0.08588496537	0.08	0.2395820916
0.19	0.04691799156	0.38	0.2665159766
0.6	0.00216030838	0.625	0.2029738760
0.73	0.04687189997	0.83	0.1072052560
0.89	0.06009091818	0.97	0.0648680095
1	0.00381219535		

Table 5: 18th-order quadrature rule in Ω with positive weights and points lying outside of $(-1, -0.93)$.

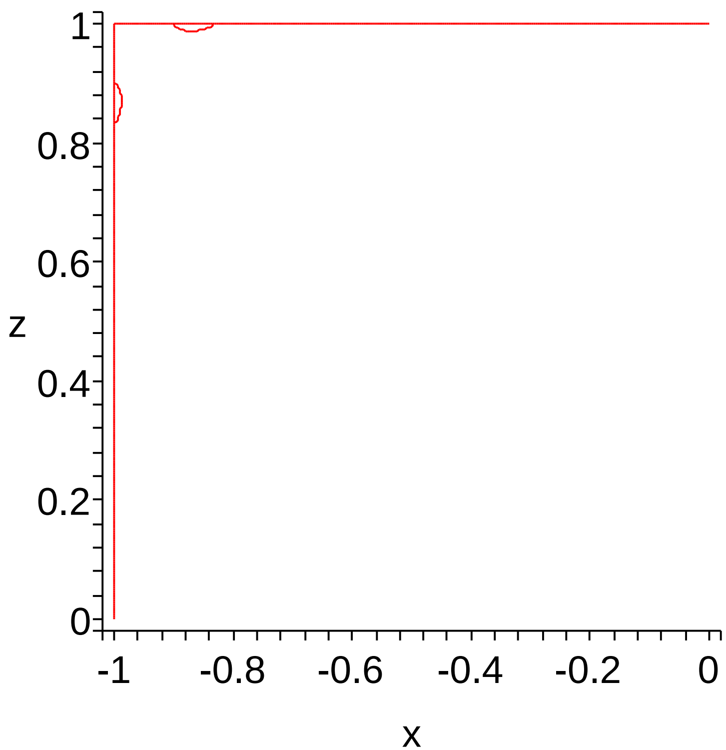
This concludes the proof for $p = 9$.

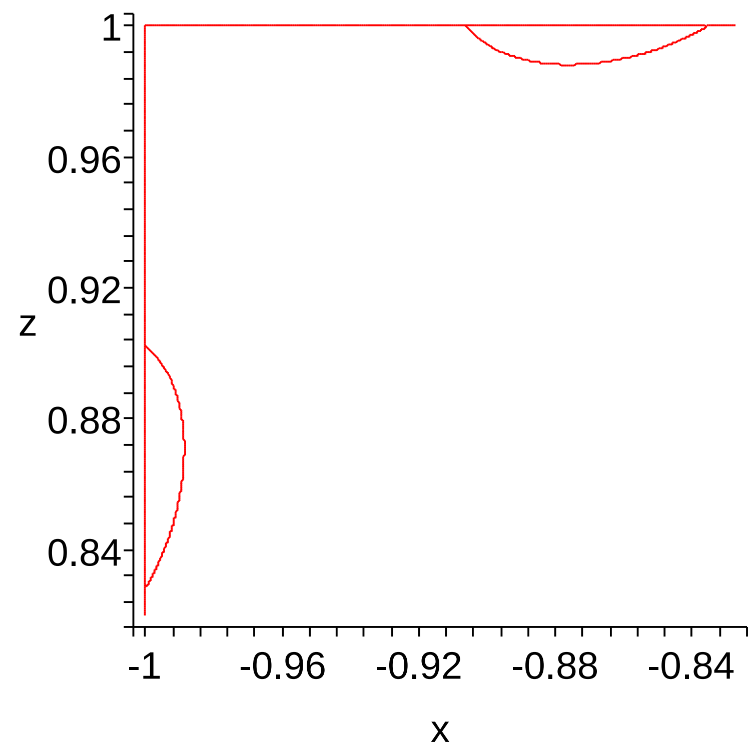
Proof: $p = 10$

$\Phi_{10}(x, z)$ **is not nonnegative in** $(-1, 1)^2$



Proof: $p = 10$ continued





$$u_{hp}(x) = \int_{-1}^{1} f_{hp}(z) \Phi_{10}(x, z) \, \mathrm{d}z$$

Proof: $p = 10$ continued

Point	Weight	Point	Weight
-1	0.0127411726	-0.9569019461	0.0603200758
-0.9344466123	0.0183508422	-0.8574545411	0.1032513172
-0.7530104489	0.1106942630	-0.6362178184	0.0412386636
-0.6061244531	0.1295220930	-0.4275824090	0.1937516842
-0.2340018112	0.1916905139	-0.0454114485	0.1774661870
0.0754465671	0.0755419308	0.1672504233	0.0745275871
0.2516247645	0.1488965177	0.3707975798	0.0207086237
0.4366736344	0.1397170181	0.5306011976	0.0924918512
0.6745457042	0.1639628301	0.82	0.1200387168
0.91	0.0649445615	0.9667274132	0.0502362251
1	0.0099073255		

Table 6: Case $p = 10$; 20th-order quadrature rule in Ω with positive weights and points lying outside of $(0.82, 0.91)$.

Proof: $p = 10$ continued

Point	Weight	Point	Weight
-1	0.0129961117	-0.9609467424	0.0393058650
-0.9366001558	0.0472129994	-0.8686571459	0.0307704321
-0.8222969304	0.1127110155	-0.6830858117	0.1442049485
-0.5515874908	0.1263749495	-0.4070028385	0.1615584597
-0.2391731402	0.1767071143	-0.0805321378	0.0223802647
-0.0404112041	0.1755155830	0.0382998004	0.0409103698
0.2054285570	0.2302298514	0.4168373782	0.1495405342
0.4862170553	0.0877842194	0.6284448676	0.0980645550
0.6932595712	0.1047143177	0.83041757281	0.1311485592
0.93562906418	0.0774056021	0.986	0.0267375743
1	0.0037266735		

Table 7: Case $p = 10$; 20th-order quadrature rule in Ω with positive weights and points lying outside of $(0.986, 1)$.

This concludes the proof for $p = 10$.

Summary

DMP in 1D on arbitrary hp-mesh

$$-u'' = f \quad \text{in } (a,b); \quad u(a) = u(b) = 0$$

- (strong) DMP: $u_{hp} \geq 0$ for all $f \geq 0$
- weak DMP: $u_{hp} \geq 0$ if L^2-projection of f is ≥ 0

Summary

DMP in 1D on arbitrary hp-mesh

Degree	DMP	Proof
$p = 1$	strong	easy
$p = 2$	strong	trivial
$p = 3$	weak	brute force, tedious
$p = 4$	strong	(computer aided) interval arithmetics*
$p = 5$	weak	computer aided
$p = 6$	strong	computer aided
$p = 7$	weak	computer aided
$p = 8$	weak	computer aided
$p = 9$	weak	computer aided
$p = 10$	weak	computer aided

* Roberto Araiza, Vladik Kreinovich, UTEP.

Outlook

- Bad news: weak DMP in 2D is not valid.
- Good news: Strong DMP in 1D is valid for meshes with two or more elements.

Thank you for your attention.

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