Shape sensitivity analysis of time-dependent flows of shear-thickening fluids

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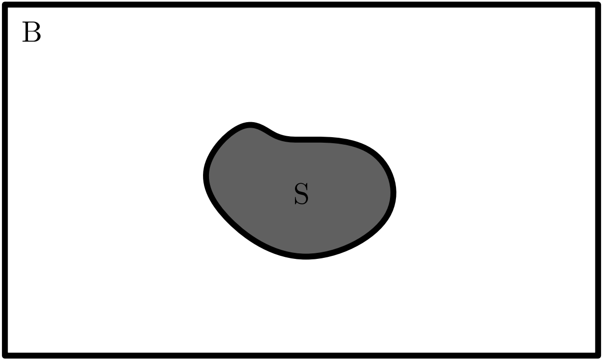
Introduction

We consider the flow of an incompressible fluid in a bounded domain $\Omega := B \setminus S \subset \mathbb{R}^d$, where B is a container, S is an obstacle whose shape is to be optimized and $d \in \{2, 3\}$.



Motion of the fluid is described by the system of equations

$$\operatorname{div} \mathbf{v} = 0, \tag{1a}$$

$$\mathbf{v}_{,t} + \operatorname{div}(\mathbf{v} \otimes \mathbf{v}) - \operatorname{div} \mathbb{S}(\mathbb{D}(\mathbf{v})) + \nabla p + \mathbb{C}\mathbf{v} = \mathbf{f} \tag{1b}$$

in $Q_T := (0, T) \times \Omega$, completed by the Navier slip boundary condition

$$(\mathbb{S}\mathbf{n})_\tau = -a\mathbf{v}_\tau, \ \mathbf{v} \cdot \mathbf{n} = 0 \tag{1c}$$

on $\Sigma_T := (0, T) \times \partial\Omega$.

$\mathbb{S}$	traceless part of the Cauchy stress
$\mathbb{D}(\mathbf{v})$	symmetric part of $\nabla\mathbf{v}$
$\mathbb{C}$	Coriolis force (skew symmetric matrix)
$\mathbf{n}$	unit outer normal vector to $\partial\Omega$
$\mathbf{v}_\tau$	tangent part of a vector: $\mathbf{v}_\tau := \mathbf{v} - (\mathbf{v} \cdot \mathbf{n})\mathbf{n}$

Cost function

The shape of the obstacle S is to be optimized subject to the work functional

$$J(\Omega) := \int_0^T \int_{\partial S} (p\mathbb{I} - \mathbb{S})\mathbf{n} \cdot \mathbf{v} = \int_0^T \int_{\partial S} |\mathbf{v}|^2.$$

Our aim is to show that

- there exists an optimal shape in a reasonable class of domains;
- J is differentiable;
- find the shape gradient of J.

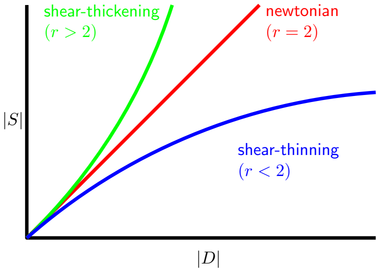
Fluids with shear rate-dependent viscosity

Non-newtonian fluids have applications in many areas of sciences and industry, e.g.:

- hemodynamics, biomechanics, mechanics of geomaterials;
- mechanical engineering, polymer chemistry, food industry...

Essentially, we deal with stress tensors of the type

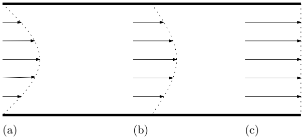
$$\mathbb{S}(\mathbb{D}(\mathbf{v})) \approx (\kappa + |\mathbb{D}(\mathbf{v})|^{r-2})\mathbb{D}(\mathbf{v}),\ \kappa \in \{0,1\},\ r > 1.$$



Boundary conditions for viscous fluids

In general it is not clear what is the right boundary condition for walls.

- In many situations it is reasonable to assume that the fluid adheres to the wall, i.e. to prescribe no slip: $\mathbf{v}_{|\partial\Omega} = 0$.
- In case of e.g. rough or chemically patterned surfaces some kind of slip condition is more suitable.



Velocity profiles: (a) no-slip, (b) partial slip, (c) complete slip.

In this presentation we consider the partial slip

$$(\mathbb{S}\mathbf{n})_\tau = -a\mathbf{v}_\tau, \;\; \mathbf{v}\cdot\mathbf{n} = 0, \;\; a \equiv 1.$$

Structural assumptions

We impose the following assumptions on the data:

(A1) $\mathbb{S} \in \mathcal{C}^2(\mathbb{R}^{d\times d}_{sym}, \mathbb{R}^{d\times d}_{sym})$, $\mathbb{S}(0) = 0$;

(A2) There exist constants $C_1, C_2, C_3 > 0$, $\kappa \in \{0, 1\}$ and $r > 1$ s.t.

$$C_1(\kappa + |\mathbb{A}|^{r-2})|\mathbb{B}|^2 \leq \mathbb{S}'(\mathbb{A}) :: (\mathbb{B} \otimes \mathbb{B}) \leq C_2(\kappa + |\mathbb{A}|^{r-2})|\mathbb{B}|^2,$$

$$|\mathbb{S}''(\mathbb{A})| \leq C_3(\kappa + |\mathbb{A}|^{r-3})$$

for any $0 \neq \mathbb{A}, \mathbb{B} \in \mathbb{R}^{d\times d}$;

(A3) $\mathbb{C} \in L^{\frac{5r}{5r-8}}((0, T) \times B, \mathbb{R}^{d\times d})$, $\mathbf{f} \in \mathbf{L}^{r'}((0, T) \times B, \mathbb{R}^d)$;

(A4) $\mathbf{v}_0 \in \mathbf{W}^{1,2}(B)$, $\operatorname{div} \mathbf{v}_0 = 0$ a.e. in B.

Properties of $\mathbb{S}$

Monotonicity:

- If $r > 1$ then $\mathbb{S}$ is strictly monotone;
- If $r \geq 2$ then $\mathbb{S}$ is strongly monotone, i.e.

$$(\mathbb{S}(\mathbb{A}) - \mathbb{S}(\mathbb{B})) : (\mathbb{A} - \mathbb{B}) \geq C|\mathbb{A} - \mathbb{B}|^r.$$

Continuity of Nemytskiĭ mappings:

- The mapping
$$\mathbb{D} \mapsto \mathbb{S}(\mathbb{D})(t, \mathbf{x})$$
is continuous from $L^r(Q_T, \mathbb{R}^{d\times d})$ to $L^{r-1}(Q_T, \mathbb{R}^{d\times d})$;
- The mapping
$$\mathbb{D} \mapsto \mathbb{S}'(\mathbb{D})(t, \mathbf{x})$$
is continuous from$L^r(Q_T, \mathbb{R}^{d\times d})$ to $L^{r-2}(Q_T, \mathbb{R}^{d\times d\times d\times d})$.

Existence of weak solutions

Theorem (Bulíček, Málek, Rajagopal (2007))

Let $r \geq (d+2)/2$, $T > 0$ *and* $\Omega \in \mathcal{C}^{1,1}$. *Then problem* (1) *has a unique weak solution*

$$(\mathbf{v}, p) \in \left[L^\infty(0,T;\mathbf{L}^2(\Omega)) \cap L^r(0,T;\mathbf{W}^{1,r}_N(\Omega)) \right] \times L^{r'}(0,T;L^{r'}_0(\Omega))$$

that satisfies $\mathit{div}\,\mathbf{v} = 0$ *a.a. in* Q_T *and*

$$\int_0^T \Big[\langle \mathbf{v}_{,t}, \phi \rangle_\Omega - (\mathbf{v} \otimes \mathbf{v}, \nabla \phi)_\Omega + (\mathbb{S}(\mathbb{D}(\mathbf{v})), \mathbb{D}(\phi))_\Omega - (p, \mathit{div}\,\phi)_\Omega + (\mathbb{C}\mathbf{v}, \phi)_\Omega + \int_{\partial\Omega} \mathbf{v} \cdot \phi \Big] = \int_0^T (\mathbf{f}, \phi)_\Omega$$

for every $\phi \in L^r(0,T;\mathbf{W}^{1,r}_N(\Omega))$.

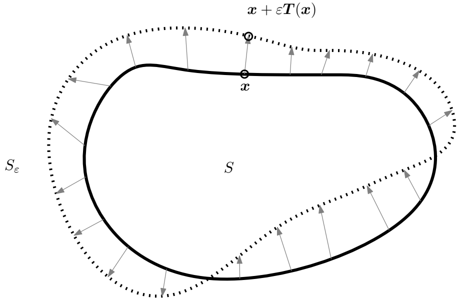
$$\mathbf{W}^{1,r}_N(\Omega) := \{\phi \in \mathbf{W}^{1,r}(\Omega);\ \phi \cdot \mathbf{n} = 0 \text{ on } \partial\Omega\}$$

Description of the shape of S

We choose a vector field $\mathbf{T} \in \mathcal{C}^2(\mathbb{R}^d, \mathbb{R}^d)$ vanishing in the vicinity of ∂B and define the mapping

$$\mathbf{y} = \mathbf{x} + \varepsilon \mathbf{T}(\mathbf{x}),$$

which describes the perturbation of the boundary ∂S. For small $\varepsilon > 0$ the mapping $\mathbf{x} \mapsto \mathbf{y}$ takes diffeomorphically the region Ω onto $\Omega_\varepsilon = B \setminus S_\varepsilon$ where $S_\varepsilon = \mathbf{y}(S)$.



Transformation of functions to Ω

Let $(\bar{\mathbf{v}}_\varepsilon, \bar{p}_\varepsilon)$ be the solution of problem (1) on $(0,T)\times\Omega_\varepsilon$. Introducing the transformations

$$\mathbf{v}_\varepsilon(t,\mathbf{x}) := \mathbb{N}^\top(\mathbf{x})\bar{\mathbf{v}}_\varepsilon(t,\mathbf{y}(\mathbf{x})), \qquad p_\varepsilon(t,\mathbf{x}) := \bar{p}_\varepsilon(t,\mathbf{y}(\mathbf{x})),$$

where

$$\mathbb{M}(\mathbf{x}) := \mathbb{I} + \varepsilon D\mathbf{T}(\mathbf{x}),\ \ \mathfrak{g}(\mathbf{x}) := \det\mathbb{M}(\mathbf{x}),\ \ \mathbb{N}(\mathbf{x}) := \mathfrak{g}(\mathbf{x})\mathbb{M}^{-1}(\mathbf{x}),$$

one can show that the new pair $(\mathbf{v}_\varepsilon, p_\varepsilon)$ is the weak solution of the problem

$$\mathbf{v}_{\varepsilon,t} + \operatorname{div}(\mathbf{v}_\varepsilon\otimes\mathbf{v}_\varepsilon) - \operatorname{div}\mathbb{S}(\mathbb{D}(\mathbf{v}_\varepsilon)) + \nabla p_\varepsilon + \mathbb{C}\mathbf{v}_\varepsilon = \mathbf{f} + \mathbf{A}_\varepsilon,$$
$$\operatorname{div}\mathbf{v}_\varepsilon = 0$$

in the fixed domain $Q_T := (0,T)\times\Omega$ with the same boundary conditions, where $\mathbf{A}_\varepsilon \in \left[L^r(0,T;\mathbf{W}^{1,r}_N(\Omega))\right]^*$ is certain term of order ε.

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Shape stability

Proposition

There is a constant $C > 0$ such that for sufficiently small $\varepsilon \geq 0$:

$$\sup_{t\in(0,T)} \|\mathbf{v}_\varepsilon(t)\|_2^2 + \int_0^T \left(\|\mathbf{v}_\varepsilon\|_{1,r}^r + \|\mathbf{v}_\varepsilon\|_{2,\partial\Omega}^2 + \|p_\varepsilon\|_{r'}^{r'} \right) \leq C.$$

The whole sequence $\{(\mathbf{v}_\varepsilon, p_\varepsilon)\}_{\varepsilon>0}$ tends to $(\mathbf{v}, p)$ as follows:

$$\begin{aligned}
\mathbf{v}_\varepsilon &\rightharpoonup^* \mathbf{v} && \text{in } L^\infty(0,T;\mathbf{L}^2(\Omega)),\\
\mathbf{v}_{\varepsilon,t} &\rightharpoonup \mathbf{v}_{,t} && \text{in } L^{r'}(0,T;\mathbf{W}_N^{-1,r'}(\Omega)),\\
p_\varepsilon &\rightharpoonup p && \text{in } L^{r'}(Q_T),\\
\mathbf{v}_\varepsilon &\to \mathbf{v} && \text{in } L^r(0,T;\mathbf{W}_N^{1,r}(\Omega)) \cap \mathbf{L}^{z(r)}(Q_T) \text{ and in } \mathbf{L}^2(\Sigma_T).
\end{aligned}$$

Here $z(r) := r\frac{d+2}{d}$.

Existence of optimal shapes

The cost function can be rewritten as follows:

$$J(\Omega_\varepsilon) = \int_0^T \int_{\partial S} |\mathbb{N}^{-\top}\mathbf{v}_\varepsilon|^2 |\mathbb{N}\mathbf{n}|,$$

from which we see that J is continuous w.r.t. strong convergence of $\mathbf{v}_\varepsilon$ in $\mathbf{L}^2(\Sigma_T)$.

This leads to the existence of a minimizing shape in a class of domains that are uniformly in $\mathcal{C}^{1,1}$.

Estimate of differences

We are going to estimate the differences

$$(\mathbf{u}_\varepsilon, q_\varepsilon) := (\frac{\mathbf{v}_\varepsilon - \mathbf{v}}{\varepsilon}, \frac{p_\varepsilon - p}{\varepsilon}).$$

which are weak solutions of the problem:

$$\mathbf{u}_{\varepsilon,t} + \operatorname{div}(\mathbf{u}_\varepsilon \otimes \mathbf{v} + \mathbf{v}_\varepsilon \otimes \mathbf{u}_\varepsilon) - \frac{1}{\varepsilon}\operatorname{div}(\mathbb{S}(\mathbb{D}(\mathbf{v}_\varepsilon)) - \mathbb{S}(\mathbb{D}(\mathbf{v}))) + \nabla q_\varepsilon + \mathbb{C}\mathbf{u}_\varepsilon = \frac{1}{\varepsilon}A_\varepsilon,$$

$$\operatorname{div}\mathbf{u}_\varepsilon = 0 \text{ in } Q_T.$$

Estimate of differences

Proposition

Let $\kappa = 1$ and $r \geq (d+2)/2$. Then there is a constant $C > 0$ independent of $\varepsilon > 0$ such that

$$\sup_{t \in (0,T)} \|\mathbf{u}_\varepsilon(t)\|_2^2 + \|\mathbb{D}(\mathbf{u}_\varepsilon)\|_{2,Q_T}^2 + \|\mathbf{u}_\varepsilon\|_{2,\Sigma_T}^2 \leq C.$$

If $(d+2)/2 \leq r < 4$ then we additionally have:

$$\int_0^T \|q_\varepsilon\|_{\frac{2r}{3r-4}}^{\frac{2r}{3r-4}} + \int_0^T \|\mathbf{u}_{\varepsilon,t}\|_{\mathbf{W}_N^{-1,\frac{2r}{3r-4}}(\Omega)}^{\frac{2r}{3r-4}} \leq C.$$

Regularity assumptions

We will need that given $\mathbf{A}_0' \in L^2(0,T;\mathbf{W}_N^{-1,2}(\Omega))$, the following problem has a unique weak solution $(\dot{\mathbf{v}},\dot{p})$:

$$\dot{\mathbf{v}}_{,t} + \operatorname{div}(\dot{\mathbf{v}}\otimes\mathbf{v} + \mathbf{v}\otimes\dot{\mathbf{v}}) - \operatorname{div}(\mathbb{S}'(\mathbb{D}(\mathbf{v}))\mathbb{D}(\dot{\mathbf{v}})) + \nabla\dot{p} + \mathbb{C}\dot{\mathbf{v}} = \mathbf{A}_0',$$
$$\operatorname{div}\dot{\mathbf{v}} = 0$$

in Q_T, with the boundary and initial conditions

$$\dot{\mathbf{v}}\cdot\mathbf{n} = 0, \quad \left[(\mathbb{S}'(\mathbb{D}(\mathbf{v}))\mathbb{D}(\dot{\mathbf{v}}))\mathbf{n}\right]_\tau = -\dot{\mathbf{v}}_\tau \text{ on } \Sigma_T,$$
$$\dot{\mathbf{v}}(0,\cdot) = \dot{\mathbf{v}}_0.$$

For this reason we require an additional assumption on the regularity of solutions, namely that

$$\mathbf{v}_\varepsilon, \mathbf{v} \in L^\infty(0,T;\mathbf{W}_N^{1,\infty}(\Omega)) \cap \mathbf{W}^{1,2}(0,T;\mathbf{W}_N^{-1,2}(\Omega)) \qquad \text{(R)}$$

uniformly w.r.t. $\varepsilon > 0$.
The assumption (R) can be guaranteed in terms of the data at least in the case $d = 2$ (see e.g. Kaplický (2005)).

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The assumption (R) can be guaranteed in terms of the data at least in the case $d = 2$ (see e.g. Kaplický (2005)).

Existence of material derivatives

Theorem

Let the assumptions of the previous proposition hold and (R) *be satisfied. Then*

$$
\begin{aligned}
\mathbf{u}_\varepsilon &\rightharpoonup^* \dot{\mathbf{v}} && \text{in } L^\infty(0,T;\mathbf{L}^2(\Omega)),\\
\mathbf{u}_\varepsilon &\rightharpoonup \dot{\mathbf{v}} && \text{in } L^2(0,T;\mathbf{W}^{1,2}_N(\Omega)),\\
\mathbf{u}_\varepsilon &\to \dot{\mathbf{v}} && \text{strongly in } L^{z(2)}(Q_T) \text{ and in } L^2(\Sigma_T),\\
q_\varepsilon &\rightharpoonup \dot{p} && \text{in } L^2(Q_T),\\
\mathbf{u}_{\varepsilon,t} &\rightharpoonup \dot{\mathbf{v}}_{,t} && \text{in } L^2(\mathbf{W}^{-1,2}_N(\Omega)),\\
\frac{\mathbf{A}_\varepsilon}{\varepsilon} &\rightharpoonup \mathbf{A}'_0 && \text{in } L^2(0,T;\mathbf{W}^{-1,2}_N(\Omega)).
\end{aligned}
$$

Here $(\dot{\mathbf{v}}, \dot{p})$ *is the material derivative of* $(\mathbf{v}, p)$.

Shape derivative of cost function

Recall that the cost function can be rewritten as follows:

$$J(\Omega_\varepsilon) := \int_0^T \int_{\partial S_\varepsilon} |\bar{\mathbf{v}}_\varepsilon|^2 = \mathcal{J}(\varepsilon, \mathbf{v}_\varepsilon),$$

where

$$\mathcal{J}(\varepsilon, \mathbf{w}) := \int_0^T \int_{\partial S} |\mathbb{N}(\varepsilon)^{-\top}\mathbf{w}|^2 |\mathbb{N}(\varepsilon)\mathbf{n}|.$$

Theorem

The shape derivative of the cost function is given by

$$\left.\frac{\mathrm{d}J}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} = J_e(\mathbf{T}) + J_{\mathbf{v}}(\dot{\mathbf{v}}),$$

where

$$J_e(\mathbf{T}) := \int_0^T \int_{\partial S} ((\mathbf{n}\cdot\mathbb{N}'(\mathbf{T})\mathbf{n})|\mathbf{v}|^2 - 2\mathbb{N}'(\mathbf{T})\mathbf{v}\cdot\mathbf{v}),$$

$$J_{\mathbf{v}}(\dot{\mathbf{v}}) := 2\int_0^T \int_{\partial S} \mathbf{v}\cdot\dot{\mathbf{v}},$$

$$\mathbb{N}'(\mathbf{T}) := \left.\frac{\mathrm{d}\mathbb{N}}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} = (div\,\mathbf{T})\mathbb{I} - D\mathbf{T}.$$

Conclusion

We have shown:

- existence of material derivatives for a class of incompressible fluids,
- existence of optimal shapes,
- differentiability of the work functional.

The result:

- is not restricted to short time interval or small data,
- depends on the available regularity, in particular it is restricted to 2D.

Thank you for attention!

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We have shown:

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