# Improving Conditioning of *hp*-FEM

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## 1 Introduction

The *hp* version of the finite element method (*hp*-FEM) gains its popularity thanks to its capability to converge exponentially fast even in presence of singularities. Recent results (see [3] and references therein) show that the performance of *hp*-FEM can be further improved by a suitable choice of higher-order basis functions. Proper basis functions can decrease the condition number of the corresponding stiffness matrix dramaticly.

For elliptic problems in 2D we distinguish three types of these basis functions: vertex, edge, and bubble functions. The vertex functions corresponds to vertices in the mesh, edge functions to edges, and bubble functions to interiors of elements. The bubble functions are characterized by the fact that they are supported in a single element only. The number of bubble functions grows quadraticly with the polynomial degree in contrast to the linear grows of the number of edge functions and constant number of vertex functions. Hence, the bubble functions form the major part of the basis for higher polynomial degrees.

Moreover, the construction of bubble functions is not restricted by conformity requirements and there is a freedom in their definition. Our idea is to use this freedom and find a set of basis functions with best conditioning properties.

We analyze the problem of optimal basis functions for the Poisson equation with homogeneous Dirichlet boundary conditions

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{1}$$

The optimal basis functions are known in the 1D case. The 1D bubble functions are defined as the so called Lobatto polynomials [4] which are defined as primitive functions to the Legendre polynomials. These Lobatto polynomials are orthogonal in the energetic inner product induced by the 1D Laplacian. Therefore, the corresponding bubble-bubble block in the stiffness matrix is diagonal.

The situation is much more difficult in higher dimension, where it is not possible to construct bubble functions that would give diagonal bubble-bubble block. Therefore various sets of bubble functions satisfying various criteria are constructed. The optimal choice is not known, yet.

In this work, we compare several popular sets of basis functions with respect to their conditioning properties and show an interesting result about orthonormal basis functions.

## 2 *hp*-FEM

From now on, we restrict ourselves to the 2D case. For simplicity, we assume $\Omega$ to be a polygonal domain.

The *hp*-FEM solution of the Poisson problem (1) is based on the weak formulation: find $u \in V = H_0^1(\Omega)$ such that

$$a(u,v) = F(v) \quad \forall v \in V,$$

where

$$a(u,v) = \int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x \quad \text{and} \quad F(v) = \int_\Omega f v \,\mathrm{d}x.$$

The *hp*-FEM formulation then reads: find $u_{hp} \in V_{hp}$ such that

$$a(u_{hp}, v_{hp}) = F(v_{hp}) \quad \forall v_{hp} \in V_{hp},$$

where $V_{hp}$ is a suitable finite dimensional subspace of $V = H_0^1(\Omega)$.

The space $V_{hp}$ is defined with the help of the triangulation $\mathcal{T}_{hp}$ of the domain $\Omega$. In *hp*-FEM, every element $K \in \mathcal{T}_{hp}$ is assigned an arbitrary polynomial degree $p_K > 1$. The space $V_{hp}$ is then given by

$$V_{hp} = \left\{ v_{hp} \in V : v_{hp}|_K \in P^{p_K}(K),\ K \in \mathcal{T}_{hp} \right\},$$

where $P^{p_K}(K)$ stands for the space of polynomials of degree at most $p_K$ in $K$.

Let $N = \dim V_{hp}$ and $\varphi_j$, $j = 1, 2, \ldots, N$, be a basis in $V_{hp}$. The finite element solution $u_{hp}$ is then given as

$$u_{hp}(x) = \sum_{j=1}^{N} y_j \varphi_j(x).$$

The coefficients $y_j$ solve the linear algebraic system $Ay = b$, where the stiffness matrix $A$ has entries $A_{ij} = a(\varphi_j, \varphi_i)$ and the load vector $b$ has the entries $b_i = F(\varphi_i)$, $i, j = 1, 2, \ldots, N$.

Clearly, the choice of the basis $\varphi_j$ is crucial for the properties of the stiffness matrix $A$.

## 3 Choice of the Basis

The basis $\varphi_j$ is defined in a standard finite element way. First, the so called shape functions are defined in a reference element. These shape functions are then mapped by an affine transformation from the reference element to the physical elements in the mesh. For simplicity, we consider triangular elements only.

The vertex shape functions coincide with the barycentric coordinates of the reference triangle. The edge shape functions vanish on all edges of the reference triangle except for one, where they coincide with Lobatto polynomials. The bubble functions can be defined in various ways. The simplest construction is based on products and powers of barycentric coordinates [2] and we call them monomial bubbles. These basis functions have very poor conditioning properties. If Lobatto polynomials are incorporated in the definition of bubbles then the condition number improves – see [1] for details. Another idea is to use the Gram-Schmidt orthogonalization to produce orthonormal bubble functions with respect to the energetic inner product on the reference element. Clearly, there is infinitely many of such orthonormal sets.

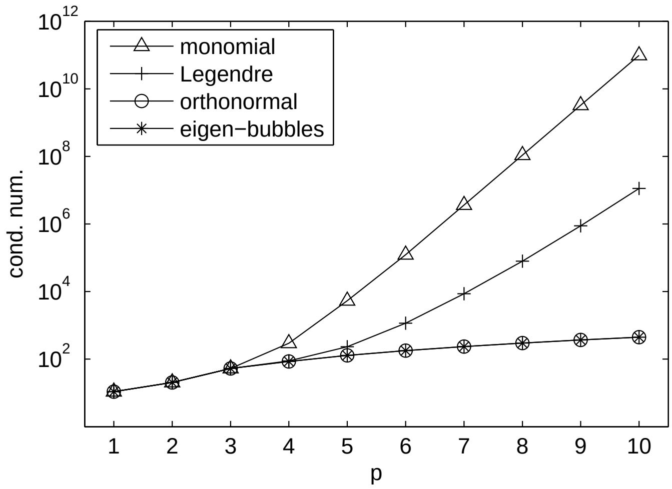


Figure 1: Comparison of the condition numbers for various sets of shape functions. Poisson problem, L-shape domain, a mesh refined towards the re-entrant corner, uniform distribution of polynomial degrees $p = 1, 2, \dots, 10$.

However, unique position among them have the generalized eigenfunctions of the discrete Laplacian on the reference element $\hat{K}$ because they are orthonormal in both energetic and $L^2$ inner products. They are defined by requirements $\hat{\varphi}_j \in P_0^p(\hat{K})$, $\hat{\varphi}_j \not\equiv 0$, and

$$\int_{\hat{K}} \nabla\hat{\varphi}_j \cdot \nabla\hat{v}\,\mathrm{d}\xi = \lambda_j \int_{\hat{K}} \hat{\varphi}_j \hat{v}\,\mathrm{d}\xi \quad \forall \hat{v} \in P_0^p(\hat{K}),$$

where $\lambda_j > 0$ are the eigenvalues and $P_0^p(\hat{K}) = \left\{\psi \in P^p(\hat{K}) : \psi|_{\partial\hat{K}} = 0\right\}$.

# 4 Numerical Results

We present results of computations in an L-shape domain using a mesh refined towards the re-entrant corner. Figure 1 compares the condition numbers of the stiffness matrices obtained by different sets of the basis functions. All the sets have identical vertex and edge functions. The first set includes monomial bubbles, the second have bubbles based on Lobatto polynomials, the bubbles in the third set are orthonormal with respect to the energetic inner product, and the fourth set has bubbles from the generalized eigenproblem. We can observe the exponential growth of the condition number for the first two sets and algebraic growth for the other sets. Interestingly enough all the orthonormal sets exhibit the same condition number of the stiffness matrix. This fact follows from the following lemma.

**Lemma 4.1** *Let $X$ be a finite dimensional vector space such that $X = V \oplus W$, where $V$ and $W$ be any subspaces of $X$ with $N = \dim X$, $M = \dim V$, and $M - N = \dim W$. Let $b : W \times W \mapsto \mathbb{R}$ be a symmetric bilinear form on $W$. Let $\{\varphi_1, \dots, \varphi_M\}$ be a basis of $V$. Let $\{\varphi_{M+1}, \dots, \varphi_N\}$ and $\{\psi_{M+1}, \dots, \psi_N\}$ be two bases of $W$ such that*

$$b(\varphi_i, \varphi_j) = \delta_{ij} \quad \text{for } i, j = M+1, M+2, \dots, N,$$
$$b(\psi_i, \psi_j) = \delta_{ij} \quad \text{for } i, j = M+1, M+2, \dots, N.$$

*Moreover, let us set $\psi_i = \varphi_i$ for $i = 1, 2, \dots, M$.*

*Then for any other symmetric bilinear form $a : X \times X \mapsto \mathbb{R}$ the Gram matrices $A_\varphi = \{a(\varphi_j, \varphi_i)\}_{i,j=1}^N$ and $A_\psi = \{a(\psi_j, \psi_i)\}_{i,j=1}^N$ have identical eigenvalues.*

**Proof.** If the functions $\psi_i$ are expressed as linear combinations of functions $\varphi_k$ then we can show that the matrices $A_\varphi$ and $A_\psi$ are similar. Details can be found in [3]. $\square$

Let us show in detail how to use Lemma 4.1 to see that all sets of bubble functions that are orthogonal in energetic inner product on the reference element lead to stiffness matrices with identical spectral condition numbers. Let us put $X = V_{hp}$ and define $V$ as the span of all vertex and edge functions and $W$ as the span of all bubble functions. Let us consider two sets of basis functions $\{\varphi_i\}$ and $\{\psi_i\}$, $i = 1, 2, \dots, N$, such that their vertex and edge functions are identical, i.e. $\varphi_i = \psi_i$, $i = 1, 2, \dots, M < N$. The bubble functions in these two sets are defined as affine images of shape bubble functions that are orthonormal on the reference element in the energetic sense. More precisely, let $\{\hat{\varphi}_k(\xi)\}$ and $\{\hat{\psi}_k(\xi)\}$ be two sets the shape bubble functions defined on the reference element $\hat{K}$ that satisfy

$$\int_{\hat{K}} \nabla\hat{\varphi}_k \cdot \nabla\hat{\varphi}_\ell \, \mathrm{d}\xi = \int_{\hat{K}} \nabla\hat{\psi}_k \cdot \nabla\hat{\psi}_\ell \, \mathrm{d}\xi = \delta_{k\ell}. \tag{2}$$

The physical bubble functions in the mesh are defined with the aid of the bijective affine mapping between the reference and physical element $\mathbf{x}_K : \hat{K} \mapsto K$, $x = \mathbf{x}_K(\xi)$, as follows

$$\varphi_i(x) = \hat{\varphi}_k\big(\mathbf{x}_K^{-1}(x)\big) \quad \text{and} \quad \psi_i(x) = \hat{\psi}_k\big(\mathbf{x}_K^{-1}(x)\big).$$

Here $i = M+1, M+2, \dots, N$ stands for the index of the basis function in the mesh and $k$ stands for the index of the corresponding shape function on the reference element $\hat{K}$. The point is that the bubble functions defined in this way are orthonormal under the following inner product

$$b(u,v) = \begin{cases} 0 & \text{if } \operatorname{supp} u \neq \operatorname{supp} v, \\ \displaystyle\int_K \left(\frac{\mathrm{D}\mathbf{x}_K}{\mathrm{D}\xi}\right)^{\mathrm{T}} \nabla u \cdot \left(\frac{\mathrm{D}\mathbf{x}_K}{\mathrm{D}\xi}\right)^{\mathrm{T}} \nabla v \det \left(\frac{\mathrm{D}\mathbf{x}_K}{\mathrm{D}\xi}\right)^{-1} \mathrm{d}x & \text{if } \operatorname{supp} u = \operatorname{supp} v = K, \end{cases}$$

where $u, v \in W$ and $(\mathrm{D}\mathbf{x}_K/\mathrm{D}\xi)$ stands for the matrix of partial derivatives. The orthonormality of the bubble functions under this inner product follows from the substitution $x = \mathbf{x}_K(\xi)$ and from (2).

In this situation, finally, we can apply Lemma 4.1 to infer that the corresponding stiffness matrices $A_\varphi = \{\int_\Omega \nabla\varphi_j \cdot \nabla\varphi_i\}_{i,j=1}^N$ and $A_\psi = \{\int_\Omega \nabla\psi_j \cdot \nabla\psi_i\}_{i,j=1}^N$ have identical spectral condition numbers. In addition, notice that from the same reason the condition numbers of the mass matrices $M_\varphi = \{\int_\Omega \varphi_j\varphi_i\}_{i,j=1}^N$ and $M_\psi = \{\int_\Omega \psi_j\psi_i\}_{i,j=1}^N$ are equal as well.

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