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ON A POSTERIORI ERROR ESTIMATION STRATEGIES FOR ELLIPTIC PROBLEMS

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Abstract: This article summarizes the concepts of a posteriori error estimates of the finite element method for linear elliptic problems. Its aim is to show the main streams and to present fundamental ideas the estimates are based on. The different approaches are described very briefly, just to introduce clearly the main ideas.

Keywords: a posteriori error estimates, finite element method, elliptic problems.

1 Introduction

The term a posteriori error estimator, which was first used by Ostrowski [18] in 1940, is used in literature with different meanings. To cover as many meanings as possible we give quite general definition: *a posteriori error estimator* is a quantity which bounds or approximates the error and can be computed from the knowledge of numerical solution and input data.

Let us emphasize that the goal of numerical computation is not only to find an approximate solution but to solve the problem *with a prescribed tolerance*. The best tools to achieve this goal are a posteriori error estimates.

We distinguish between local error *indicator*, which estimates the error on some subdomain, i.e., on one element or on a patch of elements, and global error *estimator*, which estimates the error on the whole domain. The error estimator is often obtained as a sum of local indicators. Error estimator is *guaranteed*, if it is a lower or upper bound of the error. The terms *reliable* and *efficient* error estimator are sometimes used in literature in the same meaning as the guaranteed upper and guaranteed lower bound, respectively. An approximation of error is called *asymptotic* error estimator if it converges to the true error while the discretization parameter tends to zero. The quality of an error estimator is measured by its *effectivity index*, which is the ratio of the error estimator to the exact error.

A posteriori error estimators are utilized in two important ways connected with *adaptivity*. The first one is an indication which elements will be refined. Numerical examples (see, e.g., [2, 9]) show that the quality of numerical solution is not very sensitive to small changes of the mesh. Therefore, quite cheap and inaccurate error indicators can be employed in this situation. The second way is the *stopping criterion*. When guaranteed error estimator indicates that we are under a given tolerance, we stop the adaptive process.

Explicit and implicit residual based error estimators are derived in Sections 2.1 and 2.2. Hierarchical approach and estimates based on adjoint and dual problems are presented in Sections 2.3, 2.4 and 2.5. Finally, Section 2.6 shows recovery based estimators.

2 Error estimation for elliptic problems

We describe the main ideas on a simple linear elliptic problem: find $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ such that

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{2.1}$$

where $\Omega \subset \mathbb{R}^2$ is a polygonal domain with a Lipschitz continuous boundary $\partial\Omega$ and $f \in C^0(\overline{\Omega})$ is a given right-hand side. Weak formulation of (2.1) allows us to consider $u \in V = H_0^1(\Omega)$ and $f \in L^2(\Omega)$. It reads: find $u \in V$ such that

$$a(u, v) = (f, v) \quad \forall v \in V, \tag{2.2}$$

where $a(u, v) = \int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}x$ and $(f, v) = \int_\Omega f v \, \mathrm{d}x$. The finite element solution $u_h \in V_h$ of problem (2.2) satisfies

$$a(u_h, v_h) = (f, v_h) \quad \forall v_h \in V_h,$$

where $h > 0$ is the so-called discretization parameter, $V_h \subset V$ is a suitable finite element subspace of V based on a triangulation $\mathcal{T}_h$ of $\overline{\Omega}$. Denote by $\mathcal{Z}_h$ the set of all edges of elements in $\mathcal{T}_h$, which do not lie on $\partial\Omega$ and fix an arbitrary normal direction ν_ℓ to all edges $\ell \in \mathcal{Z}_h$.

We assume finite elements of order p, i.e., $\|e\| \le C(u)h^p$, where $e = u - u_h$ is the discretization error and $\|u\| = \sqrt{a(u,u)}$, $\|u\|_{0,\Omega} = \sqrt{(u,u)}$, and

$$\mathcal{R}(v) = (f, v) - a(u_h, v) \quad \forall v \in V, \tag{2.3}$$

denote the energy norm, the $L^2(\Omega)$ norm and the residuum, respectively.

Note that $\mathcal{R}(v_h) = 0$ for all $v_h \in V_h$. Residuum $\mathcal{R}(v)$ can be rewritten in terms of interior and edge contributions if the integration on the right-hand side of (2.3) is split elementwise, the Green's formula is used and resulting sums are rearranged:

$$\mathcal{R}(v) = (f, v) - (\nabla u_h, \nabla v) = \sum_{K\in\mathcal{T}_h} \int_K (fv - \nabla u_h \cdot \nabla v)\,\mathrm{d}x$$
$$= \sum_{K\in\mathcal{T}_h} \int_K (fv + \Delta u_h v)\,\mathrm{d}x + \sum_{\ell\in\mathcal{Z}_h} \int_\ell J_\ell(\nabla u_h) v\,\mathrm{d}x. \tag{2.4}$$

Here, the jump in the gradient of u_h across the edge ℓ is denoted by $J_\ell(\nabla u_h) = (\nabla u_h^+ - \nabla u_h^-) \cdot \nu_\ell$, where the direction of the unit normal ν_ℓ to the edge ℓ is irrelevant and $\nabla u_h(x)^\pm = \lim\limits_{s\to 0_+} \nabla u_h(x \pm s\nu_\ell)$ for $x \in \ell$.

The error $e \in V$ satisfies the following residual equation

$$a(e, v) = \mathcal{R}(v) \quad \forall v \in V \tag{2.5}$$

and the Galerkin orthogonality

$$a(e, v_h) = \mathcal{R}(v_h) = 0 \quad \forall v_h \in V_h. \tag{2.6}$$

The norm of the linear functional $\mathcal{R}$ is equal to the energy norm of the error:

$$\|\mathcal{R}\|_{\mathcal{L}(V,\mathbb{R})} = \sup_{v\in V} \frac{|\mathcal{R}(v)|}{\|v\|} = \sup_{v\in V} \frac{|a(e,v)|}{\|v\|} = \|e\|\,. \tag{2.7}$$

2.1 Explicit residual estimates

Although residual explicit a posteriori error estimates were the first introduced (see [3]), they are still quite popular, cf. [9]. Let us illustrate the main idea: Take residual equation (2.5), test it by e and employ (2.6) to get

$$a(e, e) = \mathcal{R}(e - w_h) \quad \forall w_h \in V_h. \tag{2.8}$$

For each element $K \in \mathcal{T}_h$ we define the set $\mathcal{Y}_K$ of all its edges which do not lie on $\partial\Omega$. We will need the Clément interpolation $\Pi_h : V \mapsto V_h$ which has the following properties (see, e.g., [11])

$$\|v - \Pi_h v\|_{0,K} \le C\,|K|^{1/2}\,\|v\|_{1,\Delta K}\,,$$
$$\|v - \Pi_h v\|_{0,\ell} \le C\,|\ell|^{1/2}\,\|v\|_{1,\Delta K}\,,$$

where $v \in V$, $\Delta K = \operatorname{int}\left\{\cup K', K' \in \mathcal{T}_h : \overline{K'} \cap \overline{K} \neq \emptyset\right\}$ is the *patch* of elements associated with $K \in \mathcal{T}_h$, $\|\cdot\|_{1,\Omega}$ denotes $H^1(\Omega)$ norm, $|\cdot|$ stands for the Lebesgue measure, $\ell \in \mathcal{Y}_K$, and C depends only on the regularity of the triangulation. Substituting $w_h = \Pi_h e$ into (2.8) and writing the residuum according to (2.4), we obtain

$$\begin{aligned}
\|e\|^2 &= \sum_{K \in \mathcal{T}_h} \left[\int_K (f + \Delta u_h)(e - \Pi_h e)\,\mathrm{d}x + \frac{1}{2} \sum_{\ell \in \mathcal{Y}_K} \int_\ell J_\ell(\nabla u_h)(e - \Pi_h e)\,\mathrm{d}S \right] \\
&\le \sum_{K \in \mathcal{T}_h} C_1 \left[|K|^{1/2} \|f + \Delta u_h\|_{0,K} + \frac{1}{2} \sum_{\ell \in \mathcal{Y}_K} |\ell|^{1/2} \|J_\ell(\nabla u_h)\|_{0,\ell} \right] \|e\|_{1,\Delta K} \\
&\le C_2 \left(\sum_{K \in \mathcal{T}_h} |K| \, \|f + \Delta u_h\|_{0,K}^2 + \sum_{\ell \in \mathcal{Z}_h} |\ell| \, \|J_\ell(\nabla u_h)\|_{0,\ell}^2 \right) + \varepsilon \|e\|_{1,\Omega}^2 ,
\end{aligned}$$

where we used Young's inequality in the last line and $0 < \varepsilon < 1$.

Using equivalence of the norm and seminorm on $H_0^1(\Omega)$ and absorbing the term $\varepsilon \|e\|_{1,\Omega}^2$ in the left-hand side, we arrive at

$$\|e\|^2 \le C_3 \left(\sum_{K \in \mathcal{T}_h} \eta_K^2 + \sum_{\ell \in \mathcal{Z}_h} \xi_\ell^2 \right), \tag{2.9}$$

where $\eta_K = |K|^{1/2} \|f + \Delta u_h\|_{0,K}$ and $\xi_\ell = |\ell|^{1/2} \|J_\ell(\nabla u_h)\|_{0,\ell}$.

The critical point is the evaluation of constant C_3, which depends on the domain Ω and on its triangulation. This constant can be evaluated either theoretically or from the numerical experiments. The theoretical approach, cf. [10], leads to gross overestimation, while the experimental approach does not guarantee the upper bound.

2.2 Implicit residual estimates

Implicit residual a posteriori error estimates are based on solving local Dirichlet or Neumann problem on each element. Error estimators of this type are very popular and theoretically quite well understood, see, e.g., [2, 1, 22].

Let e_K^{Dir} be the solution of local analogue of residual equation with homogeneous Dirichlet boundary conditions on each element, i.e., $e_K^{\mathrm{Dir}} \in H_0^1(K)$ and

$$a_K(e_K^{\mathrm{Dir}}, v) = \int_K (f + \Delta u_h) v \,\mathrm{d}x \quad \forall v \in H_0^1(K), \tag{2.10}$$

where $a_K(w,v) = \int_K \nabla w \cdot \nabla v \, \mathrm{d}x$. A global Dirichlet indicator function e^{Dir} is now defined as an element by element composition of e_K^{Dir}, i.e., $e^{\mathrm{Dir}}|_K = e_K^{\mathrm{Dir}}$ for all $K \in \mathcal{T}_h$ and $e^{\mathrm{Dir}}|_\ell = 0$ for all $\ell \in \mathcal{Z}_h$. The indicator function e^{Dir} is the elliptic projection, i.e., the projection in the energetic inner product $a(\cdot,\cdot)$ of e into $V_0 = \{v \in V : v|_\ell = 0, \ell \in \mathcal{Z}_h\}$, because

$$a(e^{\mathrm{Dir}}, v) = \sum_{K \in \mathcal{T}_h} \int_K (f + \Delta u_h) v \, \mathrm{d}x + \sum_{\ell \in \mathcal{Z}_h} \int_\ell J_\ell(\nabla u_h) v \, \mathrm{d}S = \mathcal{R}(v) = a(e, v)$$

holds for all $v \in V_0$. Hence, $\left\| e^{\mathrm{Dir}} \right\| \leq \|e\|$.

The estimator e^{Dir} is not computable. Its finite element approximation $e^{\mathrm{Dir},p+r}$ of order $p+r$ satisfies

$$\left\| e^{\mathrm{Dir},p+r} \right\| \leq \left\| e^{\mathrm{Dir}} \right\| \leq \|e\|, \tag{2.11}$$

because it is nothing else than elliptic projection into a finite element space. Thus, the computable Dirichlet estimator is a guaranteed lower bound of the error.

The upper bound can be obtained as the solution of a local problem with Neumann boundary conditions on each element. Homogeneous Neumann boundary conditions cannot be used in this situation and so-called equilibrated splitting of the residuum is needed for their definition. We split residuum (2.3) onto element contributions:

$$\mathcal{R}(v) = \sum_{K \in \mathcal{T}_h} \mathcal{R}_K^{EQ}(v|_K). \tag{2.12}$$

To do this we have to split the jumps $J_\ell(\nabla u_h)$, cf. (2.4), on two contributions from elements around the edge ℓ:

$$J_\ell(\nabla u_h) = J_\ell^K(\nabla u_h) + J_\ell^{K^*}(\nabla u_h),$$

where ℓ is a common edge of elements K and K^*, i.e., $\partial K \cap \partial K^* = \ell$. Then the element contribution $\mathcal{R}_K^{EQ}$ to the residuum can be defined as

$$\mathcal{R}_K^{EQ}(v) = \int_K (f + \Delta u_h) v \, \mathrm{d}x + \sum_{\ell \in \mathcal{Y}_K} \int_\ell J_\ell^K(\nabla u_h) v \, \mathrm{d}x.$$

Moreover, it is possible to construct split jumps $J_\ell^K(\nabla u_h)$ and $J_\ell^{K^*}(\nabla u_h)$ such that each $\mathcal{R}_K^{EQ}$ satisfies the orthogonality condition

$$\mathcal{R}_K^{EQ}(v) = 0 \quad \forall v \in P_0^p(K), \tag{2.13}$$

where $P_0^p(K)$ is the space of p degree polynomials vanishing on $\partial K \cap \partial\Omega$. Note that computation of split jumps is not trivial. Details can be found, e.g., in [2].

The Neumann element residual indicator function $e_K^{\mathrm{Neu}} \in V(K)$ is determined as the solution of the problem

$$a_K(e_K^{\mathrm{Neu}}, v) = \mathcal{R}_K^{EQ}(v) \quad \forall v \in V(K), \tag{2.14}$$

where $V(K) = \left\{v \in H^1(K) : v|_{\partial\Omega} = 0\right\}$. Problem (2.14) in interior elements corresponds to the classical elliptic problem with Neumann boundary conditions

$$\frac{\partial u}{\partial \nu_\ell} = \pm J_\ell^K(\nabla u_h) \quad \forall \ell \subset \partial K,$$

where the sign is positive if ν_ℓ is the outer normal to edge ℓ of element K and negative if ν_ℓ is the inner normal. These Neumann problems are well defined thanks to the orthogonality $\mathcal{R}_K^{EQ}(1) = 0$, see (2.13).

The following computation, which employs previous results and Cauchy-Schwarz inequality in $\mathbb{R}^N$, shows that $\mathcal{E}^{\mathrm{Neu}} = \sqrt{\sum_{K\in\mathcal{T}_h} \left\|\nabla e_K^{\mathrm{Neu}}\right\|_{0,K}^2}$ is a guaranteed upper bound of the error:

$$\begin{aligned}
\|e\| = \|\mathcal{R}\|_{\mathcal{L}(V,\mathbb{R})} &= \sup_{0\neq v\in V} \frac{\sum_{K\in\mathcal{T}_h} \mathcal{R}_K^{EQ}(v|_K)}{\|v\|} \leq \sup_{0\neq v\in V} \sum_{K\in\mathcal{T}_h} \frac{\left|\mathcal{R}_K^{EQ}(v|_K)\right|}{\|v\|_K} \frac{\|v\|_K}{\|v\|} \\
&\leq \sup_{0\neq v\in V} \sqrt{\sum_{K\in\mathcal{T}_h} \frac{\left|\mathcal{R}_K^{EQ}(v|_K)\right|^2}{\|v\|_K^2}} \sqrt{\sum_{K\in\mathcal{T}_h} \frac{\|v\|_K^2}{\|v\|^2}} \leq \sqrt{\sum_{K\in\mathcal{T}_h} \left(\sup_{0\neq v\in V} \frac{\mathcal{R}_K^{EQ}(v|_K)}{\|v\|_K}\right)^2} \\
&= \sqrt{\sum_{K\in\mathcal{T}_h} \left(\sup_{v\in V} \frac{a_K(e_K^{\mathrm{Neu}}, v|_K)}{\|v\|_K}\right)^2} = \sqrt{\sum_{K\in\mathcal{T}_h} \|e_K^{\mathrm{Neu}}\|_K^2},
\end{aligned}$$

where the local energy norm is defined by $\|v\|_K^2 = a_K(v,v)$.

Again, e_K^{Neu} cannot be computed exactly and, therefore, we use its finite element approximation $e_K^{\mathrm{Neu},p+r}$ of degree $p+r$, which is an elliptic projection, cf. (2.11). Consequently, $\|e_K^{\mathrm{Neu},p+r}\|_K \leq \left\|e_K^{\mathrm{Neu}}\right\|_K$. Hence, $\mathcal{E}^{\mathrm{Neu},p+r} = \sqrt{\sum_{K\in\mathcal{T}_h} \|e_K^{\mathrm{Neu},p+r}\|_K^2}$ is not a guaranteed upper estimator. Anyway, the guaranteed upper bound can be inferred from the triangle inequality

$$\|e\| \leq \sqrt{\sum_{K \in \mathcal{T}_h} \left(\|e_K^{\mathrm{Neu},p+r}\|_K^2 + \|e_K^{\mathrm{Neu}} - e_K^{\mathrm{Neu},p+r}\|_K^2 \right)} \equiv \hat{\mathcal{E}}^{\mathrm{Neu}}$$

and estimating the norm $\|e_K^{\mathrm{Neu}} - e_K^{\mathrm{Neu},p+r}\|_K$ by the explicit estimator.

Let us remark that better results for both lower and upper bounds are obtained when solving the local problems on larger subdomains, e.g., on *patches of elements* with a common vertex, for details see [2].

2.3 Hierarchical approach

A similar idea to residual implicit estimation is based on hierarchical basis functions and is described in survey article [4], which has its origin in work [5]. The general idea in a posteriori error estimation is to compute a higher order solution $\hat{u}_h$ and use the difference $\|e_h^{\mathrm{hie}}\| = \|\hat{u}_h - u_h\|$ as an error estimator. Recall that $u_h \in V_h$. Let $\hat{V}_h$ denote higher order finite element space such that $\hat{u}_h \in \hat{V}_h$ and let $e_h^{\mathrm{hie}} \in W_h$, where W_h is the so-called hierarchical surplus which may consist of the so-called bubble functions or of functions based on finer grid, such that the space $\hat{V}_h$ is a direct sum of V_h and W_h, i.e., $\hat{V}_h = V_h \oplus W_h$. The error indicator function $e_h^{\mathrm{hie}} \in W_h$ is then computed as an approximate solution of the residual equation

$$a(e_h^{\mathrm{hie}}, w_h) = \mathcal{R}(w_h) = (f, w_h) - a(u_h, w_h) \quad \forall w_h \in W_h. \tag{2.15}$$

The analysis of error estimator given by (2.15) is based on the so-called *strengthened Cauchy-Schwarz inequality*:

$$|a(v_h, w_h)| \leq \gamma \|v_h\| \|w_h\|, \quad \forall v_h \in V_h, \forall w_h \in W_h,$$

where $0 < \gamma < 1$, (for proof see [4]), and on the so-called *saturation assumption*:

$$\|u - \hat{u}_h\| \leq \beta \|u - u_h\|$$

with $\beta < 1$. Under these considerations it is not difficult to prove that

$$\left\|e_h^{\mathrm{hie}}\right\| \leq \|e\| \leq (1-\beta^2)^{-1/2}(1-\gamma^2)^{-1/2} \left\|e_h^{\mathrm{hie}}\right\|.$$

The dimension of W_h is smaller than $\hat{V}_h$ and, therefore, it is better to compute $e_h^{\mathrm{hie}} \in W_h$ than $\hat{u}_h \in \hat{V}_h$. However, hierarchical approach leads to ill posed problems and, moreover, the dimension of W_h is typically much larger than that of V_h and the computation of $e_h^{\mathrm{hie}} \in W_h$ as a solution of global problem is quite expensive. Problem (2.15) is, therefore, further simplified.

One can notice that the space $V_h \oplus W_h$ corresponds to finite element approximation of degree $p + r$ in case of local Dirichlet and Neumann estimators. Thus, the global hierarchical approach and local approaches from Section 2.2 are really quite similar.

2.4 Estimates based on adjoint problem

The strategy of a posteriori error estimation based on adjoint problem is quite popular. There is a big amount of literature to this subject, e.g., [12, 15, 19, 6]. Our short description of the main idea is based on the work [7]. The greatest advantage of this approach is its suitability for those problems, where the quantity of interest is not the energy norm.

Let our aim is to compute some quantity, which can be described by a linear and continuous functional Φ from the dual space to V, i.e., $\Phi \in V^* = \mathcal{L}(V, \mathbb{R})$. It is useful, for example, if we are interested in a value of solution in one point or in some average value across certain region. The functional Φ is on the right-hand side of adjoint problem, whose weak formulation reads: find $z \in V$ such that

$$a(\varphi, z) = \Phi(\varphi) \quad \forall \varphi \in V. \tag{2.16}$$

From (2.16), (2.2), (2.3) and Galerkin orthogonality (2.6) we easily obtain the following equalities

$$\Phi(e) = a(u - u_h, z) = (f, z) - a(u_h, z) = \mathcal{R}(z) = \mathcal{R}(z - \varphi_h),$$

where $\varphi_h \in V_h$. Using (2.4) here we obtain the equality

$$\Phi(e) = \sum_{K \in \mathcal{T}_h} \int_K (f + \Delta u_h)(z - \varphi_h) \,\mathrm{d}x + \sum_{\ell \in \mathcal{Z}_h} \int_\ell J_\ell(\nabla u_h)(z - \varphi_h) \,\mathrm{d}x, \tag{2.17}$$

which is called the error representation formula. If the exact solution z of the adjoint problem was known, we would use the above formula as (exact) error estimate. The elementwise splitting of residuum in (2.17) leads to the local error indicator. For example, equilibrated splitting (2.12) gives

$$\Phi(e) \le \sum_{K \in \mathcal{T}_h} \left(\|f + \Delta u_h\|_{0,K} + \left\| J_\ell^K(\nabla u_h) \right\|_{0,\partial K} \right) \left(\|z - \varphi_h\|_{0,K} + \|z - \varphi_h\|_{0,\partial K} \right), \tag{2.18}$$

where the unknown function z is replaced by a higher order approximation of adjoint problem (2.16). In fact, error representation formula (2.17) is directly used for error estimation in practice. Error estimate (2.18) is presented here only to show more clearly the structure of the estimator and for comparison with (2.9).

2.5 Estimators based on complementary energy

This approach is based on a dual problem in the sense of calculus of variations. It is widely known that problem (2.2) can be equivalently formulated as the following minimization problem: find $u \in V$ such that

$$\mathcal{J}(u) \leq \mathcal{J}(v) \quad \forall v \in V, \tag{2.19}$$

where $\mathcal{J}(v) = \frac{1}{2}a(v,v) - (f,v)$. This is the so-called primal formulation which motivates the definition of the *dual variational problem*: find $p \in Q(f)$ such that

$$\mathcal{I}(p) \leq \mathcal{I}(q) \quad \forall q \in Q(f), \tag{2.20}$$

where $\mathcal{I}(q) = \frac{1}{2}(q,q)$ and $Q(f) = \left\{ q \in \left[L^2(\Omega)\right]^2 : (q, \nabla v) = (f,v)\ \forall v \in V \right\}$.

It can be proved (see, e.g., [16]) that the exact solutions $u \in V$ and $p \in Q(f)$ satisfy $p = \nabla u$ and $\mathcal{J}(u) + \mathcal{I}(p) = 0$. Therefore, $\mathcal{I}$ is sometimes called the *functional of complementary energy.*

Let $p_h \in Q_h \subset Q(f)$ be an approximate solution to problem (2.20). Choosing arbitrary $q \in Q(f)$, we see that

$$(\nabla u - q, \nabla v) = (\nabla u, \nabla v) - (q, \nabla v) = (f,v) - (f,v) = 0$$

holds for any $v \in V$, e.g., for $v = u_h - u$. Thus,

$$\|\nabla u_h - q\|_{0,\Omega}^2 = \|\nabla u_h - \nabla u + \nabla u - q\|_{0,\Omega}^2 = \|\nabla u_h - \nabla u\|_{0,\Omega}^2 + \|\nabla u - q\|_{0,\Omega}^2 .$$

Putting now $q = p_h$, the previous equality gives

$$\|e\|^2 = \|\nabla u_h - \nabla u\|_{0,\Omega}^2 \leq \|\nabla u_h - p_h\|_{0,\Omega}^2 .$$

The main conclusion of these simple calculations is that $\mathcal{E}^{\mathrm{ce}} = \|\nabla u_h - p_h\|_{0,\Omega}$ can be employed as a guaranteed a posteriori error estimator.

This approach is quite old, see, e.g., [14], but it is also studied and generalized in recent publications, e.g., in [20, 21], where much wider class of problems is considered and where the author solves the drawback of this method, which is finding of $p_h \in Q_h$ as an element of $Q(f)$, i.e., as exact solution of $(p_h, \nabla v) = (f,v)\ \forall v \in V$. Note that $\mathcal{E}^{\mathrm{ce}}$ is called the *duality error majorant* in works [20, 21].

2.6 Recovery based error estimates

The idea of recovery based a posteriori error estimation is simple. Take a numerical solution, produce smoother recovery solution by some postprocessing and then use the difference of original and recovered solution as an error estimator. The strength and also the weakness of this approach is its independence of solved problem. It is easy to implement error estimators of this kind on computer. Moreover, they are quite accurate and cheap, therefore, they are very popular in practice. We should emphasize that recovery techniques work only if the error has an oscillatory character.

It is not numerical solution u_h which is usually recovered. It is its gradient ∇u_h, which is often of greater interest than u_h itself. Let us denote recovered gradient by $\mathcal{G}(u_h)$. One possibility how $\mathcal{G}(u_h)$ is constructed is given bellow. A posteriori error estimator is then given by

$$\mathcal{E}^{\text{rec}} = \|\mathcal{G}(u_h) - \nabla u_h\|_{0,\Omega} . \tag{2.21}$$

Note that if the operator $\mathcal{G}$ produce superconvergent approximation then the estimator $\mathcal{E}^{\text{rec}}$ is asymptotically exact, for more details see [1].

Superconvergent patch recovery procedure was developed by Zienkiewicz and Zhu [23]. We will describe here their postprocessing procedure on the easiest case of piecewise linear approximation in two dimensions. The recovered gradient $\mathcal{G}(u_h)$ will be a piecewise linear function based on triangulation $\mathcal{T}_h$. Thus, it will be given by values at vertices x_k. For each vertex x_k in the triangulation $\mathcal{T}_h$ define the patch $\widetilde{\Omega}_k$ as a union of elements $K \in \mathcal{T}_h$ with x_k as a common vertex. We inlay a plane $\tilde{p}_k$ through the values of the derivative $\partial u_h/\partial x$ sampled in centroids of elements K from the patch $\widetilde{\Omega}_k$ in the sense of least squares fitting. Similarly we inlay a plane $\tilde{q}_k$ through the values of $\partial u_h/\partial y$. Then we define two components of $\mathcal{G}(u_h)(x_k)$ as $\tilde{p}_k(x_k)$ and $\tilde{q}_k(x_k)$, i.e., as values of the planes $\tilde{p}_k$ and $\tilde{q}_k$ at the vertex x_k.

The postprocessing of the numerical solution is usually based either on superconvergence, see, e.g., [17], or on averaging technique, see, e.g., [9]. An interesting analysis of a posteriori error estimators based on recovery techniques can be found in [8].

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