Discrete Maximum Principle for Prismatic Finite Elements

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Diffusion-Reaction Problem

- Classical $\quad -\Delta u + \kappa^2 u = f \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega$
- Weak

  $$u \in V = H_0^1(\Omega): \qquad \underbrace{\mathcal{B}(u,v)}_{\int_\Omega \nabla u \cdot \nabla v + \kappa^2 uv \,\mathrm{d}x} = \underbrace{(f,v)}_{\int_\Omega fv \,\mathrm{d}x} \quad \forall v \in V$$
- FEM

  $$u_h \in V_h \subset V: \quad \mathcal{B}(u_h, v_h) = (f, v_h) \quad \forall v_h \in V_h$$

Diffusion-Reaction Problem

- Classical $\qquad -\Delta u + \kappa^2 u = f \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega$
- Maximum Principle

$$f \geq 0 \text{ in } \Omega \quad \Rightarrow \quad u \geq 0 \text{ in } \Omega$$

- FEM

$$u_h \in V_h \subset V : \quad \mathcal{B}(u_h, v_h) = (f, v_h) \quad \forall v_h \in V_h$$

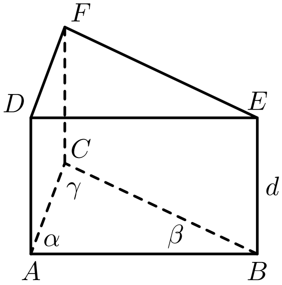
- Discrete Maximum Principle (DMP)

$$f \geq 0 \text{ in } \Omega \quad \Rightarrow \quad u_h \geq 0 \text{ in } \Omega$$

Discrete Maximum Principle – Lowest-Order FEM

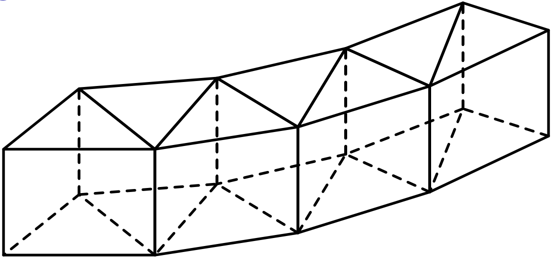
- 1D – interval: $\kappa^2 h^2 \leq 6$
- 2D – triangles: $\kappa^2 |T| \leq 6 \cot \alpha_{\mathrm{max}}$
- 2D – rectangles: $1/\sqrt{2} \leq a/b \leq \sqrt{2}$ (nonnarrow, $\kappa = 0$)
- 3D – tetrahedra: $\kappa^2 a_i a_j \leq 20 \cos(F_i, F_j)$
- 3D – bricks: cube and $\kappa = 0$
- 3D – prisms: today
- 3D – pyramids: future

Right Triangular Prismatic Element



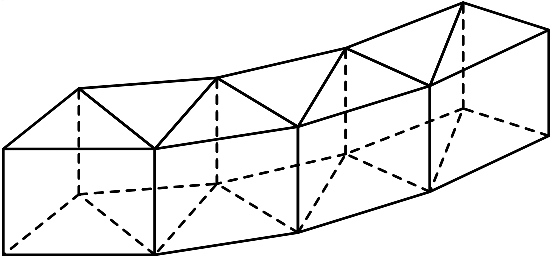
- $P = T \times I$

Right Triangular Prismatic Mesh



- $P = T \times I$
- $\mathcal{T}_h$ ... prismatic mesh

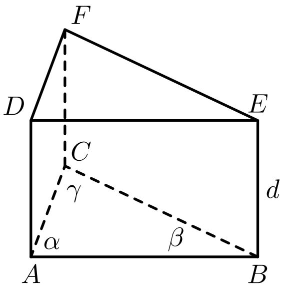
Right Triangular Prismatic Space



- $P = T \times I$
- $\mathcal{T}_h$ ... prismatic mesh
- $V_h = \Big\{ \varphi \in H_0^1(\Omega) : \ \varphi(x,y,z)|_P = \sum_{i=1}^{3} \sum_{j=1}^{2} \sigma_{ij} \lambda_i(x,y) \ell_j(z),$

  where $P \in \mathcal{T}_h, P = T \times I, \ \sigma_{ij} \in \mathbb{R}, \ \lambda_i \in \mathbb{P}^1(T), \ \ell_j \in \mathbb{P}^1(I) \Big\}$

Sufficient Condition for DMP



- $d_L^{(P)} = \left(\frac{2 \cot \alpha_{\max}^{(T)}}{|T|} - \frac{\kappa^2}{3} \right)^{-\frac{1}{2}}$
- $d_U^{(P)} = \left(\frac{\cot \alpha_{\text{med}}^{(T)} + \cot \alpha_{\min}^{(T)}}{2|T|} + \frac{\kappa^2}{6} \right)^{-\frac{1}{2}}$
- Condition $(\star)$: $d_L^{(P)} \le d^{(P)} \le d_U^{(P)}$ for all $P \in \mathcal{T}_h$
- Theorem: Condition $(\star)$ $\Rightarrow$ DMP

Angle Conditions

- Sufficient
  $\alpha_{\max}^{(T)} \leq \arctan\sqrt{7} \approx 69.2952^\circ$ and $\kappa = 0$
  $$\Rightarrow \quad \exists d^{(T)} : \text{ Condition } (\star)$$

- Necessary
  $$\text{Condition } (\star) \quad \Rightarrow \quad \alpha_{\max}^{(T)} \leq \arctan\sqrt{8} \approx 70.5288^\circ$$

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  $\alpha^{(T)}_{\max} \leq \arctan\sqrt{7} \approx 69.2952°$ and $\kappa = 0$
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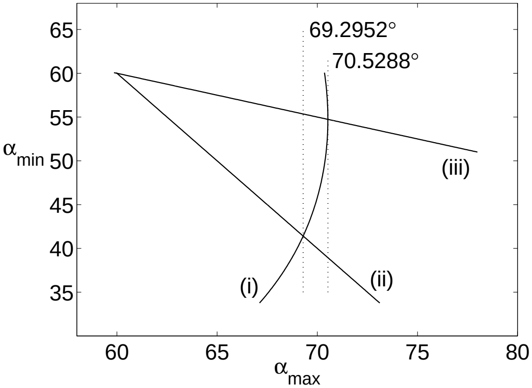

$$\alpha^{(T)}_{\text{med}} = \pi - \alpha^{(T)}_{\max} - \alpha^{(T)}_{\min}$$

(i) $d^{(P)}_L \leq d^{(P)}_U$

(ii) $\alpha^{(T)}_{\text{med}} \leq \alpha^{(T)}_{\max}$

(iii) $\alpha^{(T)}_{\min} \leq \alpha^{(T)}_{\text{med}}$

Angle Conditions

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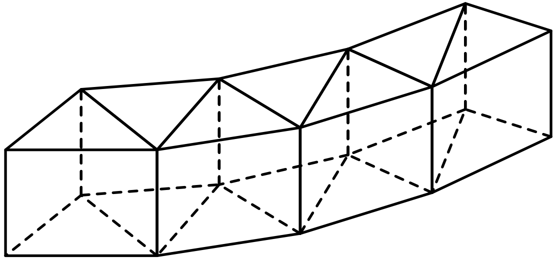
  $\Rightarrow \quad \exists d^{(T)} :$ Condition $(\star)$

- Necessary
  Condition $(\star) \quad \Rightarrow \quad \alpha_{\max}^{(T)} \leq \arctan\sqrt{8} \approx 70.5288^\circ$

- Remark
  $\alpha_{\max}^{(T)} \leq \arctan\sqrt{7} \approx 69.2952^\circ$ and $\kappa \neq 0$
  - $\exists \mathcal{T}_h^1 :$ Condition $(\star)$ with $\kappa = 0$
  - $M_P = \min\left\{6\left(\frac{|T|}{d^2} - \frac{\cot\beta + \cot\gamma}{2}\right), \; 3\left(2\cot\alpha - \frac{|T|}{d^2}\right)\right\}$
  - $m^2 \geq \max_{P \in \mathcal{T}_h^1} \kappa^2 |T| / M_P$
  - $\mathcal{T}_h^m$ ... m-fold uniform refinement of $\mathcal{T}_h^1$
  - prisms in $\mathcal{T}_h^m$ satisfy Condition $(\star)$ with $\kappa \neq 0$

Another Sufficient Condition



- Condition (†): $\frac{1}{2}|T_{\max}| \tan \alpha_{\max}^{\mathcal{T}_h^{\mathcal{G}}} \leq d_i^2 \leq |T_{\min}| \tan \alpha_{\min}^{\mathcal{T}_h^{\mathcal{G}}}$
- Condition (†) $\Rightarrow$ Condition ($\star$)
- Condition (†) $\Rightarrow$ $\frac{|T_{\max}|}{|T_{\min}|} \leq 2$

Idea of the proof

- Theorem: DMP $\quad\Leftrightarrow\quad A^{-1} \geq 0$
  Proof:
  - $u_h(y) = \int_\Omega G_h(x,y) f(x) \,\mathrm{d}x$
  - $G_h(x,y) = \sum_{i=1}^{N} \sum_{j=1}^{N} (A^{-1})_{ij} \varphi_i(x) \varphi_j(y)$ $\qquad\square$

- off-diag$(A^P) \leq 0$

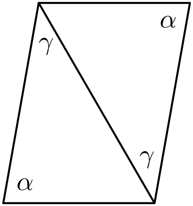
$$\Rightarrow \quad \text{off-diag}(A) = \text{off-diag}\left(\sum_{P \in \mathcal{T}_h} A^P \right) \leq 0$$

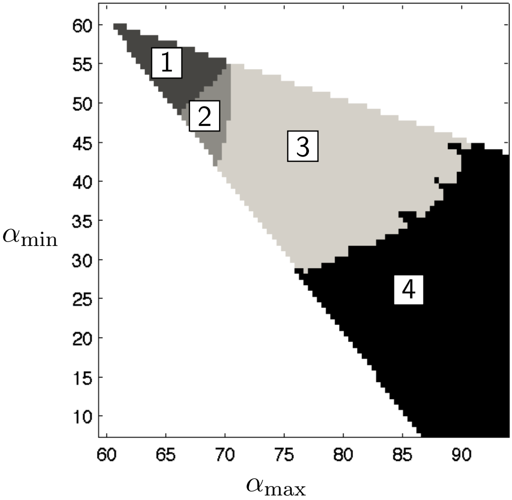
$$\Rightarrow \quad A \text{ is M-matrix}$$

$$\Rightarrow \quad A^{-1} \geq 0$$

Numerical Experiments

$$-\Delta u = f \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega$$





1. Condition (†)
2. Condition ($\star$)
3. $A^{-1} \geq 0$
4. no DMP

References

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[2] T. Vejchodský, S. Korotov, A. Hannukainen: *Discrete maximum principle for parabolic problems solved on prismatic meshes*, submitted to Math. Comput. Simul., 2008.

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Thank you for your attention

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