A posteriori error estimates in the finite element method

A survey of techniques

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Poisson problem

- Classical formulation: find $u \in C^2(\Omega) \cap C(\overline{\Omega})$:

$$-\Delta u = f \quad \text{in } \Omega \qquad u = 0 \quad \text{on } \partial\Omega$$

- Weak formulation: $V = H_0^1(\Omega)$

$$u \in V \ : \quad \underbrace{\mathcal{B}(u,v)}_{\int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}x} = \underbrace{\mathcal{F}(v)}_{\int_\Omega fv \, \mathrm{d}x} \quad \forall v \in V$$

- Galerkin method $\quad V_h \subset V \quad \dim V_h < \infty$

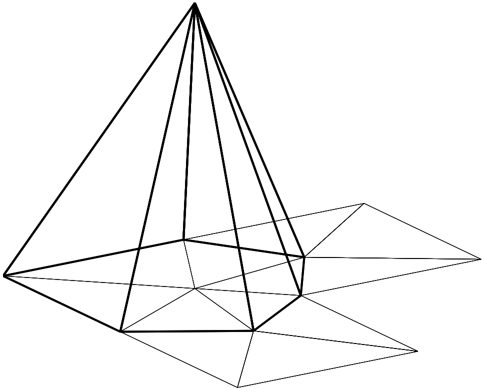
$$u_h \in V_h \ : \quad \mathcal{B}(u_h, v_h) = \mathcal{F}(v_h) \quad \forall v_h \in V_h \quad \Leftrightarrow \quad Ay = F$$

$$u_h(x) = \sum_{j=1}^{N} y_j \varphi_j(x) \qquad \sum_{j=1}^{N} y_j \underbrace{\mathcal{B}(\varphi_j, \varphi_i)}_{A_{ij}} = \underbrace{\mathcal{F}(\varphi_i)}_{F_i}$$

Finite element method (FEM)

- FEM $\quad V_h = \{v_h \in V : v_h|_K \in P^1(K)\ \forall K \in \mathcal{T}_h\}$

$\varphi_1, \ldots, \varphi_N$ . . . FEM basis functions $\quad \varphi_i(x_j) = \delta_{ij}$



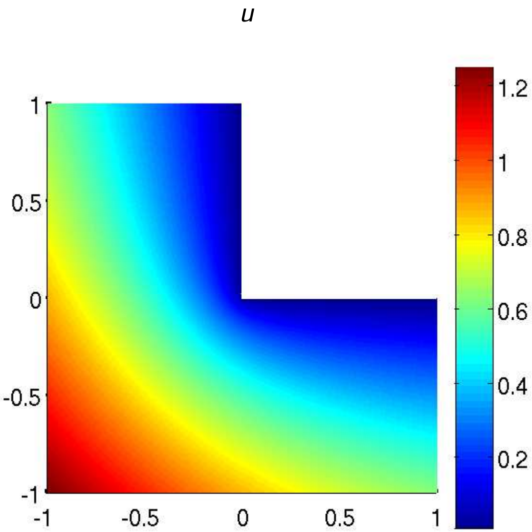
Motivation – adaptive algorithm

1. Initialize: Construct the initial mesh $\mathcal{T}_h$.
2. Solve: Find u_h on $\mathcal{T}_h$.
3. Estimate error: Compute η_K for all $K \in \mathcal{T}_h$.
4. Stopping criterion: If $\sum_{K \in \mathcal{T}_h} \eta_K^2 \leq \mathrm{TOL}^2 \quad \Rightarrow \quad \mathrm{STOP}.$
5. Mark: If $\eta_K \geq \Theta \max_{K \in \mathcal{T}_h} \eta_K \quad \Rightarrow \quad \text{mark } K. \qquad 0 < \Theta < 1$
6. Refine: Refine marked elements and build the new mesh $\mathcal{T}_h$.
7. GO TO 2.

Motivation – example

$$-\Delta u = 0 \quad \text{in } \Omega$$
$$u = g_D \quad \text{on } \partial\Omega$$

$$u = r^{\frac{2}{3}} \sin \frac{2\theta - \pi}{3}$$

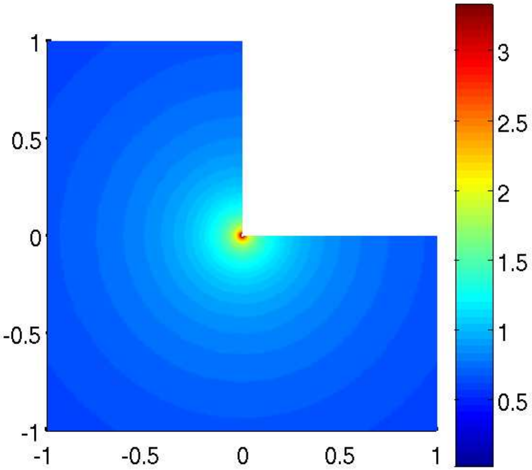


Motivation – example

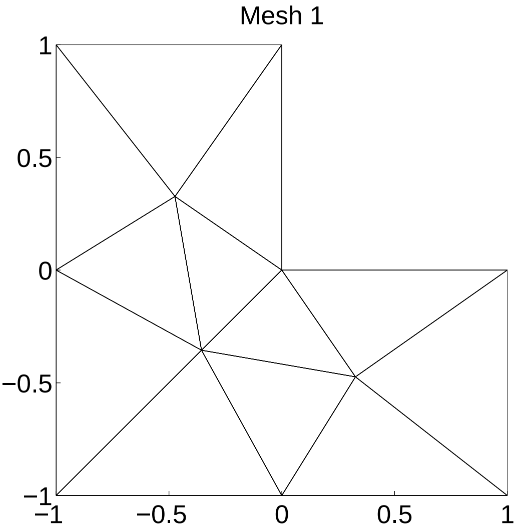
$$-\Delta u = 0 \quad \text{in } \Omega$$
$$u = g_D \quad \text{on } \partial\Omega$$

$$u = r^{\frac{2}{3}} \sin \frac{2\theta - \pi}{3}$$

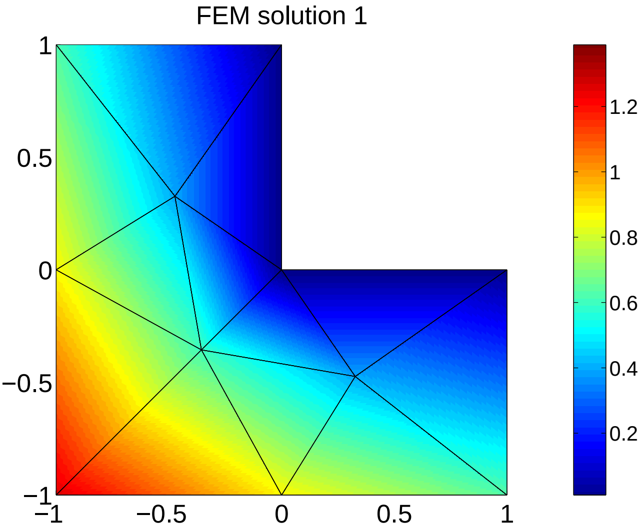
$|\nabla u|$



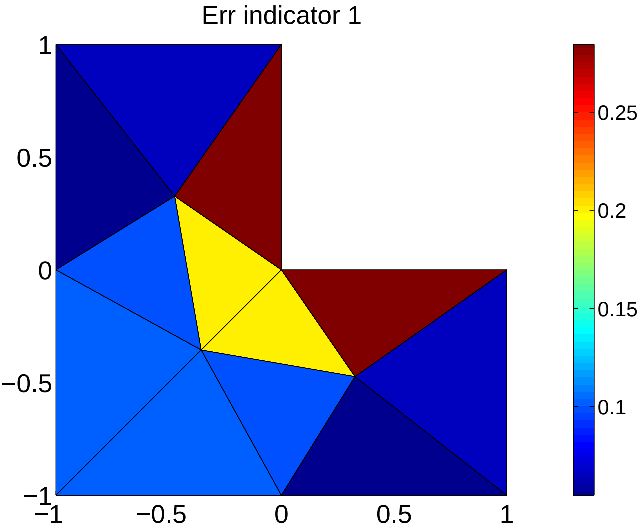
Adaptive algorithm



Adaptive algorithm



Adaptive algorithm

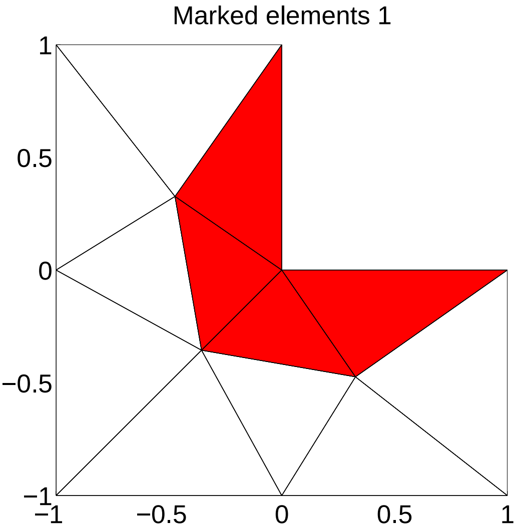


$\eta_{\mathrm{rel}} = 38.8\,\%$

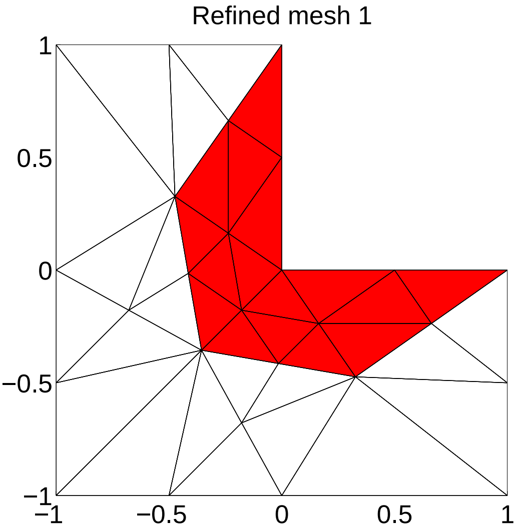
$e_{\mathrm{rel}} = 22.4\,\%$

$I_{\mathrm{eff}} = 1.73$

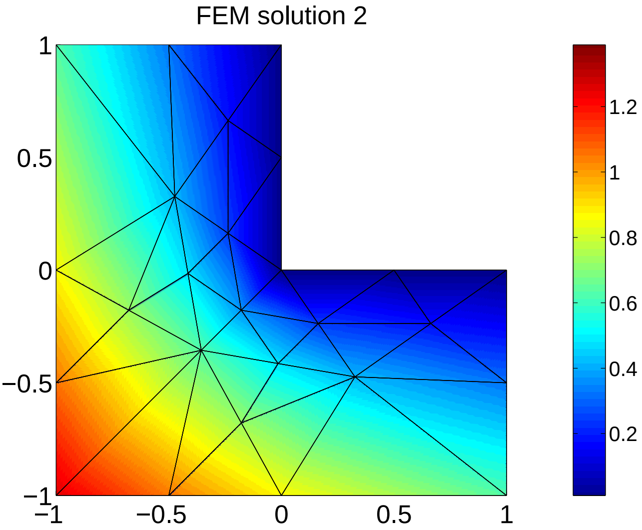
Adaptive algorithm



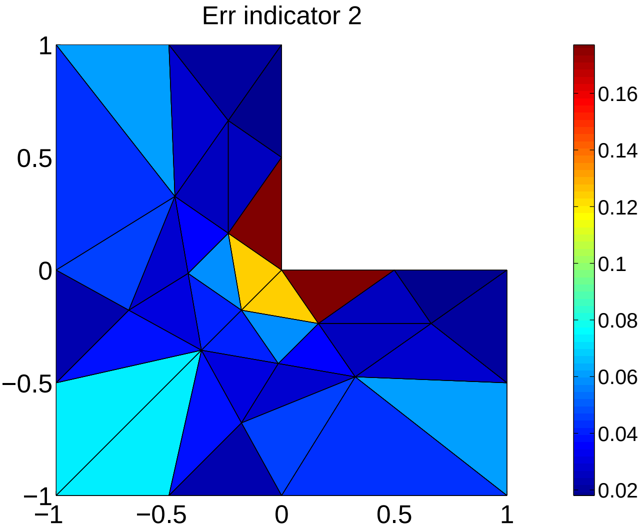
Adaptive algorithm



Adaptive algorithm



Adaptive algorithm

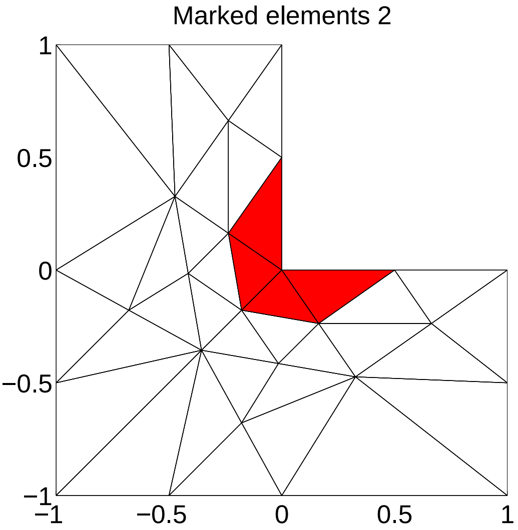


$\eta_{\text{rel}} = 27.5\,\%$

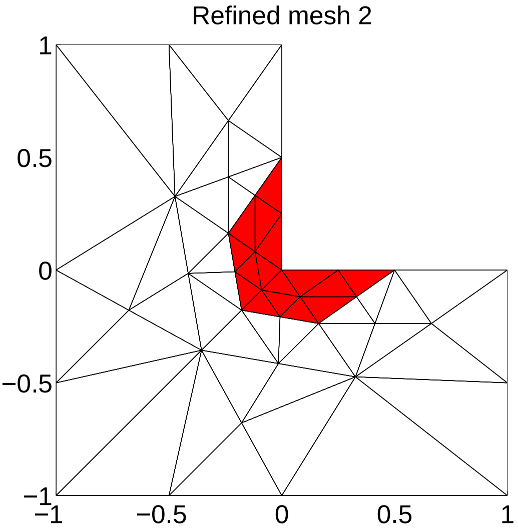
$e_{\text{rel}} = 15.9\,\%$

$I_{\text{eff}} = 1.73$

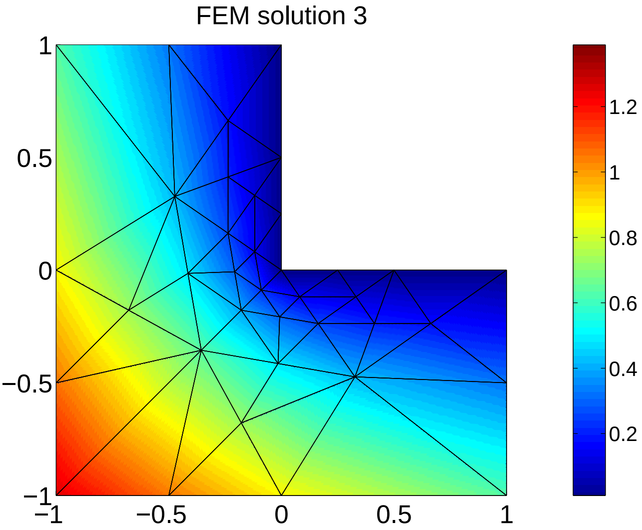
Adaptive algorithm



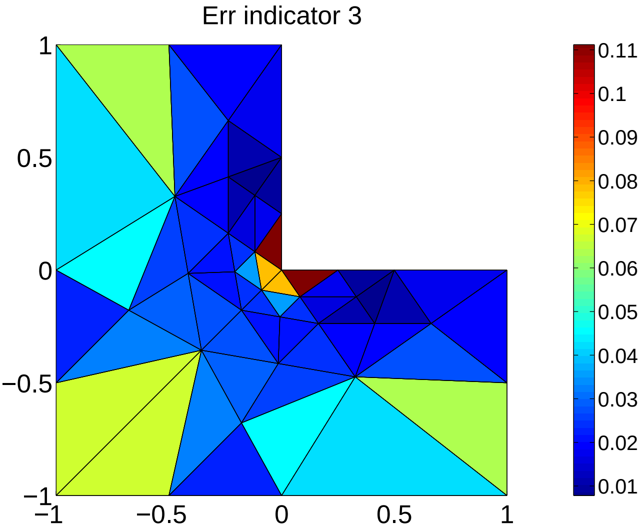
Adaptive algorithm



Adaptive algorithm



Adaptive algorithm

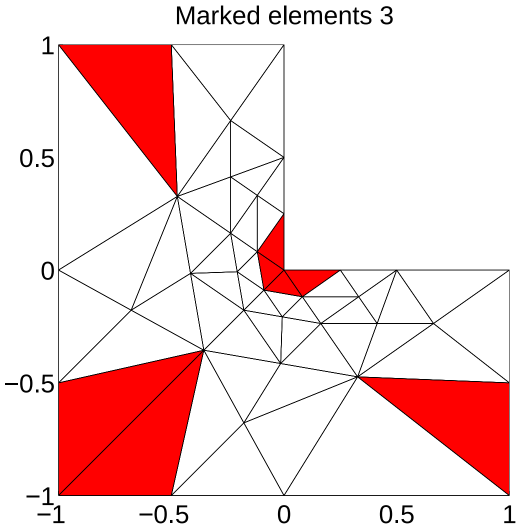


$$\eta_{\mathrm{rel}} = 20.9\,\%$$

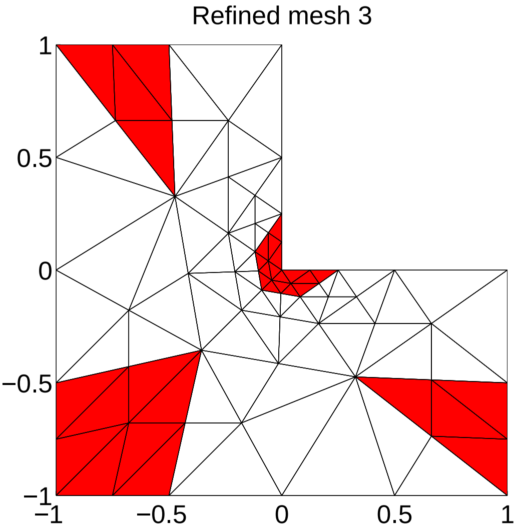
$$e_{\mathrm{rel}} = 12.7\,\%$$

$$I_{\mathrm{eff}} = 1.65$$

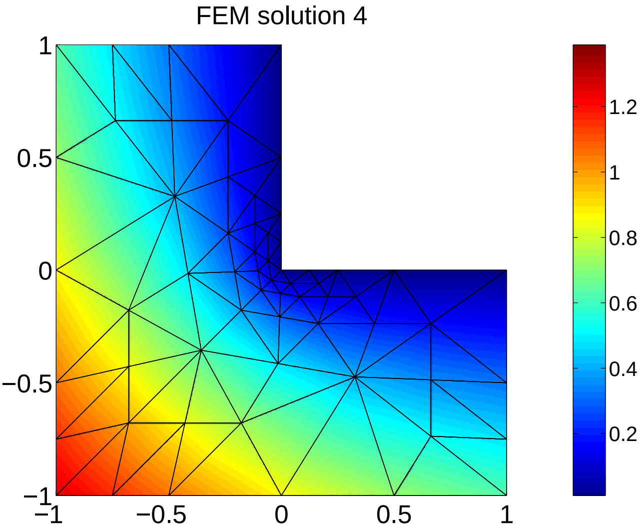
Adaptive algorithm



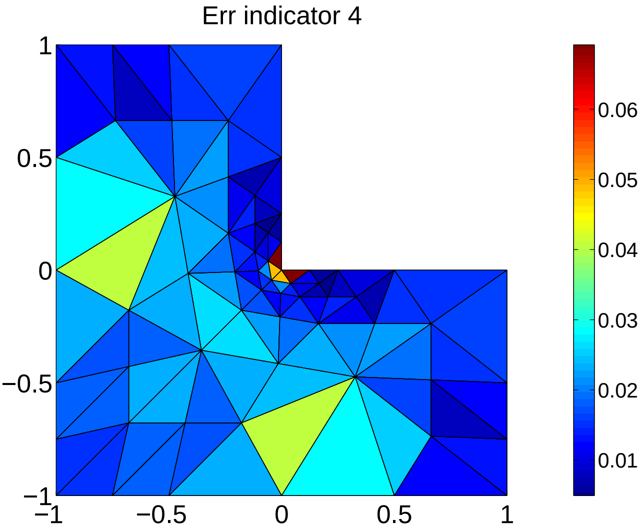
Adaptive algorithm



Adaptive algorithm



Adaptive algorithm

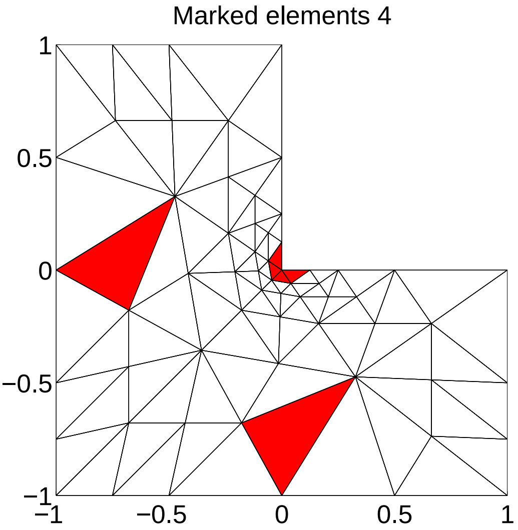


$\eta_{\text{rel}} = 15.4\,\%$

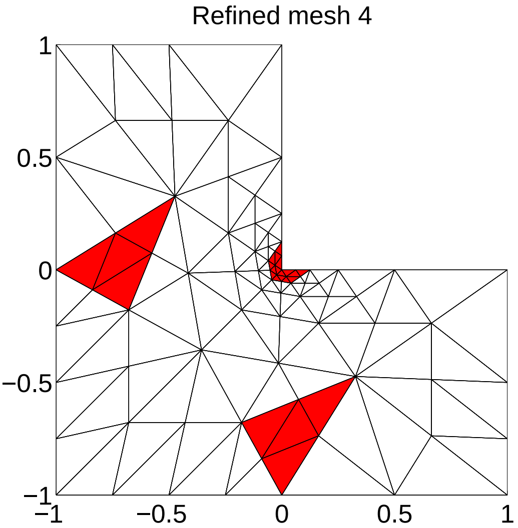
$e_{\text{rel}} = 9.55\,\%$

$I_{\text{eff}} = 1.61$

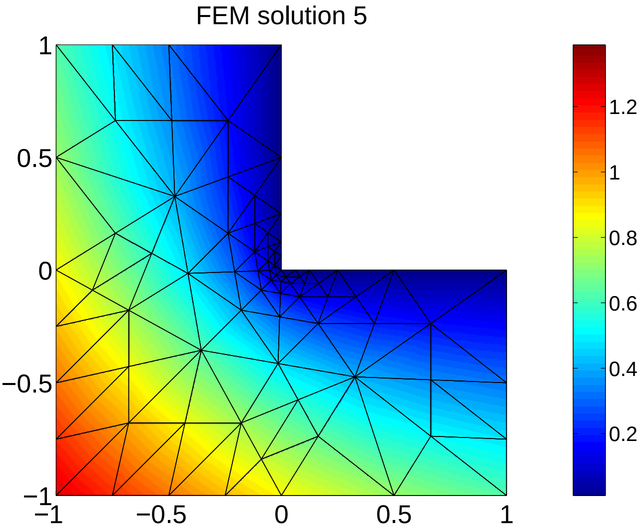
Adaptive algorithm



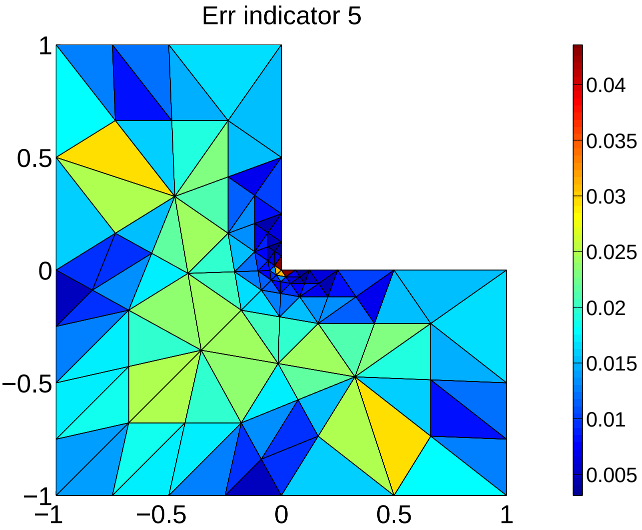
Adaptive algorithm



Adaptive algorithm



Adaptive algorithm



$$\eta_{\mathrm{rel}} = 13.4\,\%$$

$$e_{\mathrm{rel}} = 8.32\,\%$$

$$I_{\mathrm{eff}} = 1.61$$

A priori error estimates

- Discretization error: $e = u - u_h$
- Energy norm: $\|u\|^2 = \mathcal{B}(u,u) = |u|^2_{H^1(\Omega)}$
- Céa's lemma: $\|e\| = \inf_{v_h \in V_h} \|u - v_h\| = \mathrm{dist}(u, V_h)$
- $h = \max_{K \in \mathcal{T}_h} \mathrm{diam}(K)$
- Lagrange interpolation: $\pi_h^{\mathrm{Lag}} : C(\overline{\Omega}) \mapsto V_h$
  Theorem:
  $v \in H^2(\Omega) \cap C(\overline{\Omega}) \quad \Rightarrow \quad \|v - \pi_h^{\mathrm{Lag}} v\|_{H^1(\Omega)} \le C\, h\, |v|_{H^2(\Omega)}$
- Corollary (a priori estimate):
  $u \in H^2(\Omega) \cap C(\overline{\Omega}) \quad \Rightarrow \quad \|e\| \le C\, h\, |u|_{H^2(\Omega)}$

A posteriori error estimates I

Definition

- Estimates the error: $\|e\| \approx \eta \quad (\text{or } \|e\| \leq \eta, \quad \text{or } \eta \leq \|e\|)$
- Computable: $\eta = \eta(u_h, f, \Omega, \mathcal{T}_h, \dots)$

Desirable properties

- Local: $$\eta^2 = \sum_{K \in \mathcal{T}_h} \eta_K^2$$
- Guaranteed upper (lower) bound: $\|e\| \leq \eta \quad (\eta \leq \|e\|)$
- Asymptotic exactness: $\lim_{h \to 0} I_{\text{eff}} = 1, \quad I_{\text{eff}} = \dfrac{\eta}{\|e\|}$
- Efficient and reliable: $C_1 \eta \leq \|e\| \leq C_2 \eta$
- Robust: C_1 and C_2 are independent from quantities like coefficients in the equation, mesh aspect ratio etc.

A posteriori error estimates II

Remarks:

- Locality
  $\Rightarrow$ fast evaluation of η

- Guaranteed upper bound
  $\Rightarrow$ adaptive algorithm guarantees $\|e\| \leq \mathrm{TOL}$

- Efficiency and reliability
  $\Rightarrow$ convergence of adaptive algorithm

- Assymptotic exactness
  $\Rightarrow$ efficiency and reliability (for sufficiently small h)
  Proof:

$$\forall \varepsilon > 0 \ \exists \delta > 0 \ \forall h < \delta : 1 - \varepsilon \leq \frac{\eta}{\|e\|} \leq 1 + \varepsilon$$

Residual estimates

$$u \in V: \quad \mathcal{B}(u,v) = \mathcal{F}(v) \quad \forall v \in V$$
$$u_h \in V_h: \quad \mathcal{B}(u_h,v_h) = \mathcal{F}(v_h) \quad \forall v_h \in V_h$$

- Residual: $\mathcal{R}(v) = \mathcal{F}(v) - \mathcal{B}(u_h,v) \quad \forall v \in V$
- Residual equation: $e \in V: \quad \mathcal{B}(e,v) = \mathcal{R}(v) \quad \forall v \in V$

  $$\left[\mathcal{B}(u,v) - \mathcal{B}(u_h,v) = \mathcal{F}(v) - \mathcal{B}(u_h,v) \quad \forall v \in V\right]$$
- Galerkin orthogonality: $\mathcal{B}(e,v_h) = 0 \quad \forall v_h \in V_h$
- $$\|e\| = \sup_{0 \neq v \in V} \frac{|\mathcal{B}(e,v)|}{\|v\|} = \sup_{0 \neq v \in V} \frac{|\mathcal{R}(v)|}{\|v\|} = \|\mathcal{R}\|_{V^*}$$

Explicit residual estimates I

- Residual splitting: $\mathcal{R}(v) = \sum_{K\in\mathcal{T}_h} \int_K rv \,\mathrm{d}x + \sum_{\ell} \int_\ell J_\ell v \,\mathrm{d}s$

  $r = f + \Delta u_h \qquad J_\ell = (\nabla u_h^+ - \nabla u_h^-)\cdot \nu_\ell \qquad \ell$ are edges in $\mathcal{T}_h$

Proof:

$$\mathcal{R}(v) = \mathcal{F}(v) - \mathcal{B}(u_h, v) = \sum_{K\in\mathcal{T}_h} \left(\int_K fv \,\mathrm{d}x - \int_K \nabla u_h \cdot \nabla v \,\mathrm{d}x \right)$$

$$= \sum_{K\in\mathcal{T}_h} \left(\int_K fv \,\mathrm{d}x + \int_K \Delta u_h v \,\mathrm{d}x - \int_{\partial K} \nabla u_h \cdot \nu_K v \,\mathrm{d}s \right)$$

$$= \sum_{K\in\mathcal{T}_h} \int_K rv \,\mathrm{d}x + \sum_{\ell} \int_\ell (\nabla u_h^+ - \nabla u_h^-)\cdot \nu_\ell v \,\mathrm{d}s$$

Explicit residual estimates II

- Clément inter.: $\pi_h^{\mathrm{Cl}} : V \mapsto V_h$ $\quad \left\| v - \pi_h^{\mathrm{Cl}} v \right\|_{0,K} \leq C_1 |K|^{1/2} \left\| \nabla v \right\|_{0,\omega_K}$

$$\left\| v - \pi_h^{\mathrm{Cl}} v \right\|_{0,\ell} \leq C_2 |\ell|^{1/2} \left\| \nabla v \right\|_{0,\omega_\ell}$$

$$\begin{aligned}
\|e\|^2 &= \mathcal{B}(e,e) = \mathcal{R}(e) = \mathcal{R}(e - \pi_h^{\mathrm{Cl}} e) \\
&= \sum_{K \in \mathcal{T}_h} \int_K r(e - \pi_h^{\mathrm{Cl}} e)\,\mathrm{d}x + \sum_{\ell} \int_\ell J_\ell (e - \pi_h^{\mathrm{Cl}} e)\,\mathrm{d}s \\
&\leq \sum_{K \in \mathcal{T}_h} C_1 \|r\|_{0,K} |K|^{1/2} \|\nabla e\|_{0,\omega_K} + \sum_{\ell} C_2 \|J_\ell\|_{0,\ell} |\ell|^{1/2} \|\nabla e\|_{0,\omega_\ell} \\
&\leq C_3 \left(\sum_{K \in \mathcal{T}_h} |K| \, \|r\|_{0,K}^2 + \sum_{\ell} |\ell| \, \|J_\ell\|_{0,\ell}^2 \right) + \varepsilon \|e\|^2
\end{aligned}$$

$$\|e\|^2 \leq C_4 \underbrace{\left(\sum_{K \in \mathcal{T}_h} |K| \, \|r\|_{0,K}^2 + \sum_{\ell} |\ell| \, \|J_\ell\|_{0,\ell}^2 \right)}_{(\eta^{\mathrm{expl}})^2} \equiv C_4 (\eta^{\mathrm{expl}})^2$$

Implicit residual estimates – Dirichlet I

Construction:

- Local Dirichlet problems:

$$e_K^{\mathrm{Dir}} \in H_0^1(K): \quad \mathcal{B}_K(e_K^{\mathrm{Dir}}, v) = \mathcal{R}_K(v) \quad \forall v \in H_0^1(K)$$

- Approximate local problems: $V_{0,h}(K) \subset H_0^1(K)$

$$e_{K,h}^{\mathrm{Dir}} \in V_{0,h}(K): \quad \mathcal{B}_K(e_{K,h}^{\mathrm{Dir}}, v_h) = \mathcal{R}_K(v_h) \quad \forall v_h \in V_{0,h}(K)$$

- $\eta_K^{\mathrm{Dir}} = \|e_{K,h}^{\mathrm{Dir}}\|_K \quad (\eta^{\mathrm{Dir}})^2 = \sum_{K \in \mathcal{T}_h} (\eta_K^{\mathrm{Dir}})^2$

- Notation:

$$\mathcal{B}_K(u, v) = \int_K \nabla u \cdot \nabla v \,\mathrm{d}x$$

$$\mathcal{R}_K(v) = \int_K f v \,\mathrm{d}x - \int_K \nabla u_h \cdot \nabla v \,\mathrm{d}x$$

$$\|v\|_K^2 = \mathcal{B}_K(v, v)$$

Implicit residual estimates – Dirichlet II

Guaranteed lower bound:

- $e^{\text{Dir}}|_K = e_K^{\text{Dir}} \quad \forall K \in \mathcal{T}_h; \quad V_0 = \{v \in V : v|_K \in H_0^1(K)\} \subset V$
- $e_h^{\text{Dir}}|_K = e_{K,h}^{\text{Dir}} \quad \forall K \in \mathcal{T}_h; \quad V_{0,h} = \{v \in V : v|_K \in V_{0,h}(K)\} \subset V_0$
- Theorem: $\|e_h^{\text{Dir}}\| \leq \|e^{\text{Dir}}\| \leq \|e\|$
  Proof:
  - $\|e^{\text{Dir}}\| \leq \|e\|$:
    $e^{\text{Dir}} \in V_0 \quad v \in V_0$
    $\mathcal{B}(e - e^{\text{Dir}}, v) = \mathcal{R}(v) - \mathcal{R}(v) = 0$
    $\Rightarrow \quad \mathcal{B}(e, e^{\text{Dir}}) = \|e^{\text{Dir}}\|^2$
    $$\Rightarrow \quad \|e - e^{\text{Dir}}\|^2 = \|e\|^2 - 2\mathcal{B}(e, e^{\text{Dir}}) + \|e^{\text{Dir}}\|^2 = \|e\|^2 - \|e^{\text{Dir}}\|^2 \geq 0$$
  - $\|e_h^{\text{Dir}}\| \leq \|e^{\text{Dir}}\|$:
    $\mathcal{B}(e^{\text{Dir}} - e_h^{\text{Dir}}, v_h) = \mathcal{R}(v_h) - \mathcal{R}(v_h) = 0 \quad \forall v_h \in V_{0,h}$
    Similarly.

Implicit residual estimates – Neumann I

Construction:

- Weak f.: $e_K^{\mathrm{Neu}} \in H_E^1(K) = \{v \in H^1(K) : v = 0 \text{ on } \partial K \cap \partial\Omega\}$:

$$\mathcal{B}_K(e_K^{\mathrm{Neu}}, v) = \int_K f v \,\mathrm{d}x - \mathcal{B}_K(u_h, v) + \int_{\partial K} g_K v \,\mathrm{d}s \quad \forall v \in H_E^1(K)$$

- Classical f.:

$$\begin{aligned} -\Delta(e_K^{\mathrm{Neu}} + u_h) &= f && \text{in } K \\ \nabla(e_K^{\mathrm{Neu}} + u_h)\cdot \nu_K &= g_K && \text{on } \partial K \setminus \partial\Omega \\ e_K^{\mathrm{Neu}} + u_h &= 0 && \text{on } \partial K \cap \partial\Omega \end{aligned}$$

- $g_K|_\ell \in P^1(\ell)$, $\ell \subset \partial K$, $K \in \mathcal{T}_h$, $\quad g_K \approx \nabla u|_K \cdot \nu_K$ on ∂K
- Compatibility condition: $g_K|_\ell + g_{K^*}|_\ell = 0$ for $\ell = \partial K \cap \partial K^*$
- p-order equilibration condition ($p = 0, 1$):

$$\int_K f\varphi \,\mathrm{d}x - \mathcal{B}_K(u_h, \varphi) + \int_{\partial K} g_K \varphi \,\mathrm{d}s = 0 \quad \forall \varphi \in P^p(K)$$

- $\eta_K^{\mathrm{Neu}} = \|e_K^{\mathrm{Neu}}\|_K \quad (\eta^{\mathrm{Neu}})^2 = \sum_{K \in \mathcal{T}_h} (\eta_K^{\mathrm{Neu}})^2$

Implicit residual estimates – Neumann II

Guaranteed upper bound:

- Theorem: Compatibility cond. $\Rightarrow \quad \|e\| \leq \eta^{\mathrm{Neu}}$

  Proof: $v \in V$

$$\mathcal{B}(e,v) = \mathcal{R}(v) = \sum_{K \in \mathcal{T}_h} \left(\int_K f v \, \mathrm{d}x - \mathcal{B}_K(u_h, v) + \int_{\partial K} g_K v \, \mathrm{d}s \right)$$

$$= \sum_{K \in \mathcal{T}_h} \mathcal{B}_K(e_K^{\mathrm{Neu}}, v) \leq \sum_{K \in \mathcal{T}_h} \|e_K^{\mathrm{Neu}}\|_K \, \|v\|_K \leq \left(\sum_{K \in \mathcal{T}_h} \|e_K^{\mathrm{Neu}}\|_K^2 \right)^{\frac{1}{2}} \|v\|$$

- Remark: Approximate Neumann problems:

  $e_{K,h}^{\mathrm{Neu}} \in V_h^{\mathrm{Neu}} \subset H_E^1(K) \quad \Rightarrow \quad (\eta_h^{\mathrm{Neu}})^2 = \sum_{K \in \mathcal{T}_h} \|e_{K,h}^{\mathrm{Neu}}\|_K^2 \leq (\eta^{\mathrm{Neu}})^2$

- In general: $\|e\| \not\leq \eta_h^{\mathrm{Neu}}$

Hierarchic (residual) estimates

- $\widehat{V}_h = V_h \oplus Y_h, \quad Y_h \subset V, \quad V_h \cap Y_h = \{0\}$
- $\widehat{u}_h \in \widehat{V}_h : \quad \mathcal{B}(\widehat{u}_h, \widehat{v}_h) = \mathcal{F}(\widehat{v}_h) \quad \forall \widehat{v}_h \in \widehat{V}_h$
- $\|e\| \approx \|\widehat{u}_h - u_h\| \equiv \|\widehat{e}_h\|$
- $\overline{e}_h \in Y_h : \quad \mathcal{B}(\overline{e}_h, y_h) = \mathcal{R}(y_h) \quad \forall y_h \in Y_h$
- $\|e\| \approx \|\overline{e}_h\| \equiv \eta^{\text{Hie}}$
- Saturation assumption:
  $\exists \beta < 1: \quad \|u - \widehat{u}_h\| \leq \beta \, \|u - u_h\|$
- Strengthenned Cauchy-Schwarz inequality:
  $\exists \gamma < 1: \quad |\mathcal{B}(v_h, y_h)| \leq \gamma \, \|v_h\| \, \|y_h\| \quad \forall v_h \in V_h, \; y_h \in Y_h$
- $\|\overline{e}_h\| \leq \|\widehat{e}_h\| \leq \|e\| \leq \dfrac{1}{(1-\beta^2)^{\frac{1}{2}}} \|\widehat{e}_h\| \leq \dfrac{1}{(1-\beta^2)^{\frac{1}{2}}(1-\gamma^2)^{\frac{1}{2}}} \|\overline{e}_h\|$

Complementary estimates

- Friedrichs inequality: $\|v\|_{0,\Omega} \le C_\Omega \|\nabla v\|_{0,\Omega} \quad \forall v \in V = H_0^1(\Omega)$
- $\|e\| \le C_\Omega \|f + \operatorname{div} \mathbf{y}\|_{0,\Omega} + \|\mathbf{y} - \nabla u_h\|_{0,\Omega} \qquad \forall \mathbf{y} \in \mathbf{H}(\operatorname{div}, \Omega)$

  Proof:

  $v \in V \quad \mathbf{y} \in \mathbf{H}(\operatorname{div}, \Omega)$

$$
\begin{aligned}
\mathcal{B}(e, v) &= \mathcal{R}(v) + \int_\Omega \mathbf{y} \cdot \nabla v \,\mathrm{d}x + \int_\Omega \operatorname{div} \mathbf{y} v \,\mathrm{d}x - \int_{\partial\Omega} \mathbf{y} \cdot \nu v \,\mathrm{d}s \\
&= \int_\Omega (f + \operatorname{div} \mathbf{y}) v \,\mathrm{d}x + \int_\Omega (\mathbf{y} - \nabla u_h) \cdot \nabla v \,\mathrm{d}x \\
&\le \|f + \operatorname{div} \mathbf{y}\|_{0,\Omega} \|v\|_{0,\Omega} + \|\mathbf{y} - \nabla u_h\|_{0,\Omega} \|\nabla v\|_{0,\Omega} \\
&\le \left(C_\Omega \|f + \operatorname{div} \mathbf{y}\|_{0,\Omega} + \|\mathbf{y} - \nabla u_h\|_{0,\Omega} \right) \|\nabla v\|_{0,\Omega}
\end{aligned}
$$

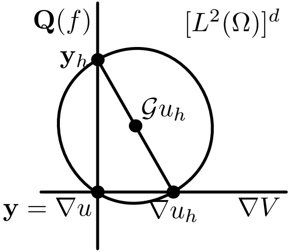
  Put $v = e$.

- Orthogonality: $\int_\Omega (\nabla u - \mathbf{y}) \cdot \nabla v \,\mathrm{d}x = 0 \quad \forall v \in V, \ \forall \mathbf{y} \in \mathbf{Q}(f)$

Complementary estimates

- Friedrichs inequality: $\|v\|_{0,\Omega} \leq C_\Omega \|\nabla v\|_{0,\Omega} \quad \forall v \in V = H_0^1(\Omega)$
- $\|e\| \leq \underbrace{C_\Omega \|f + \operatorname{div} \mathbf{y}\|_{0,\Omega} + \|\mathbf{y} - \nabla u_h\|_{0,\Omega}}_{\widehat{\eta}(u_h, \mathbf{y})} \quad \forall \mathbf{y} \in \mathbf{H}(\operatorname{div}, \Omega)$
- $\mathbf{Q}(f) = \left\{ \mathbf{y} \in \mathbf{H}(\operatorname{div}, \Omega) : \int_\Omega \mathbf{y} \cdot \nabla v \, \mathrm{d}x = \int_\Omega f v \, \mathrm{d}x \quad \forall v \in V \right\}$
- $\|e\| \leq \underbrace{\|\mathbf{y} - \nabla u_h\|_{0,\Omega}}_{\eta(u_h, \mathbf{y})} \quad \forall \mathbf{y} \in \mathbf{Q}(f)$
- Orthogonality: $\int_\Omega (\nabla u - \mathbf{y}) \cdot \nabla v \, \mathrm{d}x = 0 \quad \forall v \in V, \ \forall \mathbf{y} \in \mathbf{Q}(f)$

Method of hypercircle



Theorem: If

- $u \in V$ is primal solution
- $u_h \in V$, $\mathbf{y}_h \in \mathbf{Q}(f)$ arbitrary
- $\mathcal{G}u_h = (\mathbf{y}_h + \nabla u_h)/2$

Then

$$\|\nabla u - \mathcal{G}u_h\|_0 = \frac{1}{2}\eta(u_h, \mathbf{y}_h).$$

Proof:

$$\begin{aligned} 4\,\|\nabla u - \mathcal{G}u_h\|_0^2 &= \|\nabla u - \mathbf{y}_h + \nabla u - \nabla u_h\|_0^2 \\ &= \|\nabla u - \mathbf{y}_h\|_0^2 + \|\nabla u - \nabla u_h\|_0^2 = \|\nabla u_h - \mathbf{y}_h\|_0^2 \end{aligned}$$

Postprocessing

- Recovered gradient: $\nabla u_h \mapsto \mathcal{G}(u_h)$
- $\|e\| \approx \eta^{\text{post}} = \|\mathcal{G}(u_h) - \nabla u_h\|_{0,\Omega}$
- Superconvergence: $\|\nabla u - \mathcal{G}(u_h)\|_{0,\Omega} \leq C_1 h^{1+\epsilon}$
- Assumption: $\|e\| \geq C_2 h$
- Theorem (asymptotic exactness): $\lim_{h\to 0} \frac{\eta^{\text{post}}}{\|e\|} = 1$

Proof:

$$\frac{\eta^{\text{post}}}{\|e\|} \leq \frac{\|\nabla u - \nabla u_h\|_0}{\|e\|} + \frac{\|\nabla u - \mathcal{G}(u_h)\|_0}{\|e\|} \leq 1 + \frac{C_1 h^{1+\epsilon}}{C_2 h} \to 1$$

$$\frac{\eta^{\text{post}}}{\|e\|} \geq \frac{\|\nabla u - \nabla u_h\|_0}{\|e\|} - \frac{\|\nabla u - \mathcal{G}(u_h)\|_0}{\|e\|} \geq 1 - \frac{C_1 h^{1+\epsilon}}{C_2 h} \to 1$$

Quantity of interest

- Quantity of interest: $\Phi \in V^*$
- Adjoint problem: $z \in V: \quad \mathcal{B}(v, z) = \Phi(v) \quad \forall v \in V$
- Approx. adjoint prob.: $z_h \in V_h: \quad \mathcal{B}(v_h, z_h) = \Phi(v_h) \quad \forall v_h \in V_h$
- Error representation formula:

$$\Phi(e) = \mathcal{B}(e, z) = \mathcal{R}(z) = \mathcal{R}(z - z_h) = \mathcal{B}(u - u_h, z - z_h)$$

$$\begin{aligned} |\Phi(e)| &\le \|u - u_h\| \, \|z - z_h\| \\ &\le \eta^{\mathrm{pri}} \eta^{\mathrm{adj}} \end{aligned}$$

Summary

A posteriori error estimators

- Explicit residual – fast, simple, reliable, (efficient)
- Implicit residual
  - Dirichlet type – guaranteed lower bound, (reliable)
  - Neumann type – upper bound (not guaranteed), (efficient)
- Hierarchic (residual) – efficient and reliable
- Complementary – guaranteed upper bound, demanding
- Postprocessing – fast, simple, superconvergence $\Rightarrow$ asympt. exact
- Quantity of interest – if energy norm is not the goal

Recommended books

- M. Ainsworth, J. T. Oden, A posteriori error estimation in finite element analysis, Wiley, New York, 2000.
- I. Babuška, T. Strouboulis, The finite element method and its reliability, Clarendon Press, Oxford University Press, New York, 2001.
- W. Bangerth, R. Rannacher, Adaptive finite element methods for differential equations, Birkhäuser, Basel, 2003.
- P. Neittaanmäki, S. Repin, Reliable methods for computer simulation, error control and a posteriori estimates, Elsevier, Amsterdam, 2004.
- S. Repin, A posteriori estimates for partial differential equations, de Gruyter, Berlin, 2008.
- R. Verfürth, A review of a posteriori error estimation and adaptive mesh-refinement techniques., Wiley-Teubner, Chichester/Stuttgart, 1996.