'Surprising' spectra of $\mathcal{PT}$-symmetric point interactions

Petr Siegl

Nuclear Physics Institute, Řež
Faculty of Nuclear Sciences and Physical Engineering, Prague
Laboratoire Astroparticules et Cosmologie, Université Paris 7, Paris

$\mathcal{PT}$-symmetry

Origins of $\mathcal{PT}$-symmetry

- Hamiltonian $-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \mathrm{i}x^3$ has real, positive, discrete spectrum

  Bender, Boettcher 1998

- original hypothesis - the reality of spectrum due to $\mathcal{PT}$-symmetry
  - $[\mathcal{PT}, H] = 0$
  - parity $\mathcal{P}$, $(\mathcal{P}\psi)(x) = \psi(-x)$
  - complex conjugation $\mathcal{T}$, $(\mathcal{T}\psi)(x) = \overline{\psi}(x)$
- $\mathcal{PT}$-symmetry is not sufficient for reality of the spectrum
- some $\mathcal{PT}$-symmetric operators are similar to the self-adjoint ones

  $h = \varrho^{-1} H \varrho = h^*$

Antilinear symmetry

Definition

Let $A \in \mathscr{C}(\mathcal{H})$. We say that A possesses an antilinear symmetry if there exists an antilinear bijective operator C and the relation

$$AC\psi = CA\psi$$

holds for all $\psi \in \mathrm{Dom}(A)$.

- $\lambda \in \sigma_{p,c,r}(A) \Longleftrightarrow \overline{\lambda} \in \sigma_{p,c,r}(A)$
- example: $\mathcal{C} = \mathcal{PT}$, $H = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V(x)$, $\overline{V(-x)} = V(x)$

Pseudo-Hermiticity

Definition

Let $A \in \mathscr{L}(\mathcal{H})$ be densely defined. A is called pseudo-Hermitian, if there exists an operator η with properties

(i) $\eta, \eta^{-1} \in \mathscr{B}(\mathcal{H})$,

(ii) $\eta = \eta^*$

(iii) $A = \eta^{-1} A^* \eta$.

- $\sigma_{p,c,r}(A) = \sigma_{p,c,r}(A^*)$
- example: $\eta = \mathcal{P}$, $H = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V(x)$, $\overline{V(-x)} = V(x)$
- A is a self-adjoint operator in a Krein space $[\cdot,\cdot]_J = \langle J\cdot,\cdot\rangle$ fundamental symmetry $J = \eta|\eta|^{-1}$

Relations between the operator classes

- finite dimension: antilinear symmetry $\Leftrightarrow$ pseudo-Hermiticity
  essential fact: C-symmetric operators
  C antilinear isometric involution,
  $C^2 = I, \ \langle Cx, Cy \rangle = \langle y, x \rangle, \ A = CA^*C$
- assumption of spectral decomposition (spectral operators of scalar and finite type): AS $\Leftrightarrow$ P-H

  2002 Scolarici, Solombrino, 2009 Siegl
- bounded operators AS is not equivalent to P-H !

  2009 Siegl

Antilinear symmetry without pseudo-Hermiticity

Example

- $\{e_n\}_{n=1}^{\infty}$ standard orthonormal basis of $\mathcal{H} = l_2(\mathbb{N})$, $e_n(m) = \delta_{mn}$
- $Te_n := e_{n-1}, \quad n \in \mathbb{N}, \quad e_0 := 0$
- $T^* e_n := e_{n+1}, \quad n \in \mathbb{N}$
- $T = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & 0 & \\ 0 & 0 & 0 & 1 & 0 & \\ 0 & 0 & 0 & 0 & 1 & \\ \vdots & & & & \ddots & \ddots \end{pmatrix}$

- antilinear symmetry $\mathcal{T}$
- for $|\lambda| < 1$, $\lambda \in \sigma_p(T)$ but $\lambda \in \sigma_r(T^*)$

Pseudo-Hermiticity without antilinear symmetry

Example

- $\{e_i\}_{-\infty}^{\infty}$ orthonormal basis of $\mathcal{H} = l^2(\mathbb{Z})$, $e_n(m) = \delta_{mn}$
- $$Te_i := \begin{cases} \lambda_0 e_i + e_{i+1}, & i \geq 1, \\ 0, & i = 0, \\ \overline{\lambda}_0 e_{-1}, & i = -1, \\ \overline{\lambda}_0 e_i + e_{i+1}, & i < -1, \end{cases}$$
- $\lambda_0 \in \mathbb{C}, \ \operatorname{Im} \lambda_0 > \frac{1}{2}$

Pseudo-Hermiticity without antilinear symmetry

Example

- $$T = \begin{pmatrix} \ddots & & & & & & \\ \ddots & \overline{\lambda_0} & 0 & 0 & 0 & 0 & \\ & 1 & \overline{\lambda_0} & 0 & 0 & 0 & \\ & 0 & 0 & 0 & 0 & 0 & \\ & 0 & 0 & 0 & \lambda_0 & 0 & \\ & 0 & 0 & 0 & 1 & \lambda_0 & \\ & & & & & \ddots & \ddots \end{pmatrix}$$

- pseudo-Hermiticity $\eta = \mathcal{P},\ \mathcal{P}e_i = e_{-i}$
- $\lambda_0 \in \sigma_r(T)$ but $\overline{\lambda_0} \in \sigma_p(T)$
- $|\lambda - \lambda_0| < 1 \subset \sigma_p(T^*)$

Counterexamples

Counterexamples - properties

- both examples - not spectral - uncountable point spectrum
- AS+P-H $\Rightarrow$ C-symmetric operator $\Rightarrow$ $\sigma_r = \emptyset$

Related questions

? what are equivalent subclasses (AS, P-H)

? is AS+P-H related to existence of spectral decomposition?

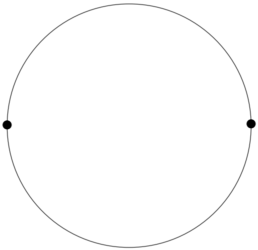
  ? at least for special classes of operators?

  ? point interactions?

$\mathcal{PT}$-symmetric point interactions

Definitions of operators

- line $L^2(\mathbb{R})$ or finite interval (circle) $L^2(a,b)$
- $H = -\frac{\mathrm{d}^2}{\mathrm{d}x^2}$
- $\mathrm{Dom}(H) = AC^1$ + boundary conditions at $x = 0$ or at $x = a, b$



$\mathcal{PT}$-symmetric point interactions

$\mathcal{PT}$-symmetric boundary conditions 2002 Albeverio, Fei, Kurasov

$$\begin{pmatrix} \psi(0+) \\ \psi'(0+) \end{pmatrix} = B \begin{pmatrix} \psi(0-) \\ \psi'(0-) \end{pmatrix}$$

$$B = \begin{pmatrix} \sqrt{1+bc}\, e^{i\phi} & b \\ c & \sqrt{1+bc}\, e^{-i\phi} \end{pmatrix}, \quad \begin{array}{l} b \geq 0, c \geq -1/b \\ \phi \in (-\pi, \pi] \end{array}$$

System on a line - interaction at $x = 0$

Symmetries

- $\mathcal{PT}$-symmetry: $\mathcal{PT}H\psi = H\mathcal{PT}\psi$, $\forall\psi \in \mathrm{Dom}(H)$
- $\mathcal{P}$-pseudo-Hermiticity: $H^* = \mathcal{P}H\mathcal{P}$
- $\mathcal{T}$-self-adjointness: $H^* = \mathcal{T}H\mathcal{T}$
- $\mathcal{T}$-complex conjugation, $\mathcal{P}$-parity

Spectrum

- residual part is empty $\sigma_r(H) = \emptyset$ 2008 Borisov, Krejčiřík
- continuous spectrum $\sigma_c(H) = [0, \infty)$
- $b \neq 0, c \neq 0$ point spectrum - at most two eigenvalues real if $bc\sin^2\phi \leq \cos^2\phi$ or $bc\sin^2\phi \geq \cos^2$ and $\cos\phi \geq 0$

  2002 Albeverio, Fei, Kurasov

Special case $b = 0, c = 0$

Boundary conditions

$\psi(0+) = e^{\mathrm{i}\phi}\psi(0-)$
$\psi'(0+) = e^{-\mathrm{i}\phi}\psi'(0-)$

Special cases

- $\phi = 0$ - self-adjoint operator, no interaction
  $\psi(0+) = \psi(0-)$, $\psi'(0+) = \psi'(0-)$
- $\phi \neq \pm\pi/2$ - continuous spectrum $[0, \infty)$, no eigenvalues, quasi-Hermitian
- $\phi = \pm\pi/2$ - 'surprising' case
  $\psi(0+) = \pm\mathrm{i}\psi(0-)$,
  $\psi'(0+) = \mp\mathrm{i}\psi'(0-)$

Special case $b = 0, c = 0, \phi \neq \pm\pi/2$

Quasi-Hermiticity

- H_ϕ is quasi-Hermitian:
  $\Theta H_\phi^* = H_\phi \Theta, \ \Theta, \Theta^{-1} \in \mathscr{B}(\mathcal{H}), \ \Theta > 0$
- $\Theta = I - \mathrm{i} \sin\phi P_{\mathrm{sign}} \mathcal{P}$
  $(P_{\mathrm{sign}} f)(x) = \mathrm{sign} x f(x)$, $\mathcal{P}$-parity
- similarity to s-a operator
  $\varrho = \sqrt{\Theta} = \cos\frac{\phi}{2} I - \frac{\mathrm{i}\sin\phi}{2\cos\frac{\phi}{2}} P_{\mathrm{sign}} \mathcal{P}$

Metric operator Θ

- spectrum - only two eigenvalues $1 - \sin\phi$, $1 + \sin\phi$
- $\Theta > 0$, $\Theta^{-1} \in \mathscr{B}(\mathcal{H})$
- $\Theta H_\phi^* = H_\phi \Theta$ is valid
- Θ is not invertible if $\phi = \pm\pi/2$!

Special case $b = 0, c = 0$, $\phi \neq \pm\pi/2$

Quasi-Hermiticity

- H_ϕ is quasi-Hermitian:
  $\Theta H_\phi^* = H_\phi \Theta,\ \Theta, \Theta^{-1} \in \mathscr{B}(\mathcal{H}),\ \Theta > 0$
- $\Theta = I - \mathrm{i} \sin\phi P_{\text{sign}} \mathcal{P}$
  $(P_{\text{sign}} f)(x) = \operatorname{sign} x f(x)$, $\mathcal{P}$-parity
- similarity to s-a operator
  $\varrho = \sqrt{\Theta} = \cos\frac{\phi}{2} I - \frac{\mathrm{i}\sin\phi}{2\cos\frac{\phi}{2}} P_{\text{sign}} \mathcal{P}$

Metric operator Θ

- spectrum - only two eigenvalues $1 - \sin\phi$, $1 + \sin\phi$
- $\Theta > 0$, $\Theta^{-1} \in \mathscr{B}(\mathcal{H})$
- $\Theta H_\phi^* = H_\phi \Theta$ is valid
- Θ is not invertible if $\phi = \pm\pi/2$!

'Surprising' case $b = 0, c = 0, \phi = \pi/2$

Properties of $H_{\pi/2}$

- $\psi(0+) = \mathrm{i}\psi(0-)$, $\psi'(0+) = -\mathrm{i}\psi'(0-)$
- $H_{\pi/2}$ is $\mathcal{PT}$-symmetric, $\mathcal{P}$-pseudo-Hermitian, $\mathcal{T}$-self-adjoint
- $H^*_{\pi/2} = H_{-\pi/2}$, $H_{\pi/2}$ is closed
- $\Theta H^*_{\pi/2} = H_{\pi/2}\Theta$
- $\Theta = I - \mathrm{i}P_{\mathrm{sign}}\mathcal{P}$, $\Theta \geq 0$, Θ is not invertible !

Spectrum 2005 Albeverio and Kuzhel

- residual spectrum is empty
- continuous spectrum $[0, \infty)$
- point spectrum $\mathbb{C} \setminus [0, \infty)$

'Surprising' case $b = 0, c = 0, \phi = \pi/2$

Properties of $H_{\pi/2}$

- $\psi(0+) = \mathrm{i}\psi(0-)$, $\psi'(0+) = -\mathrm{i}\psi'(0-)$
- $H_{\pi/2}$ is $\mathcal{PT}$-symmetric, $\mathcal{P}$-pseudo-Hermitian, $\mathcal{T}$-self-adjoint
- $H^*_{\pi/2} = H_{-\pi/2}$, $H_{\pi/2}$ is closed
- $\Theta H^*_{\pi/2} = H_{\pi/2}\Theta$
- $\Theta = I - \mathrm{i}P_{\mathrm{sign}}\mathcal{P}$, $\Theta \geq 0$, Θ is not invertible !

Spectrum 2005 Albeverio and Kuzhel

- residual spectrum is empty
- continuous spectrum $[0, \infty)$
- point spectrum $\mathbb{C} \setminus [0, \infty)$

'Surprising' case $b = 0, c = 0$, $\phi = \pi/2$

Eigenfunctions of $H_{\pi/2}$

$$\psi_k(x) = \begin{cases} e^{kx}, & x < 0, \\ \mathrm{i}e^{-kx}, & x > 0, \end{cases} \qquad \varphi_k(x) = \begin{cases} e^{-kx}, & x < 0, \\ \mathrm{i}e^{kx}, & x > 0, \end{cases}$$

$$\zeta_k(x) = \begin{cases} e^{-\mathrm{i}kx}, & x < 0, \\ \mathrm{i}e^{\mathrm{i}kx}, & x > 0. \end{cases}$$

$\psi_k \in L^2(\mathbb{R})$ for $\operatorname{Re} k > 0$, $\varphi_k \in L^2(\mathbb{R})$ for $\operatorname{Re} k < 0$,
$\zeta_k \in L^2(\mathbb{R})$ for $\operatorname{Re} k = 0$ and $\operatorname{Im} k > 0$

Models on finite interval

Models on a finite interval $(-l, l)$

- $L^2(-l, l)$, $H = -\frac{\mathrm{d}^2}{\mathrm{d}x^2}$
- $\mathrm{Dom}(H) = AC^1(-l, l)$
- 2 interactions - at $x = 0$ and $x = \pm l$ - 2 BC
- at $x = 0$ - $\mathcal{PT}$-symmetric interaction $b = 0, c = 0$
- at $x = \pm l$ - general $\mathcal{PT}$-symmetric interactions

Compact resolvent guaranteed?

Theorem (Kato)

Let $T_1, T_2 \in \mathscr{C}(\mathcal{H})$ have non-empty resolvent sets. Let T_1, T_2 be extensions of a common operator T_0, with order of extension for T_1 being finite. Then T_1 has compact resolvent if and only if T_2 has compact resolvent.

Two $\mathcal{PT}$-symmetric interactions

$\mathcal{PT}$-symmetric interactions

$$\psi(0+) = e^{\mathrm{i}\phi_1}\psi(0-)$$
$$\psi'(0+) = e^{-\mathrm{i}\phi_1}\psi'(0-)$$

$$\begin{pmatrix} \psi(l) \\ \psi'(l) \end{pmatrix} = B \begin{pmatrix} \psi(-l) \\ \psi'(-l) \end{pmatrix}$$

$$B = \begin{pmatrix} \sqrt{1+b_2c_2}\,e^{\mathrm{i}\phi_2} & b_2 \\ c_2 & \sqrt{1+b_2c_2}\,e^{-\mathrm{i}\phi_2} \end{pmatrix}$$

Two $\mathcal{PT}$-symmetric interactions

Spectrum

- discrete ($\lambda = k^2$) if
  - $\phi_1 \neq \pm\pi/2, \phi_2 \neq \pm\pi/2$
  - $\phi_1 \neq \pm\pi/2, \phi_2 = \pm\pi/2$ and $b_2 \neq 0$ or $c_2 \neq 0$

$$\cos\phi_1 \left(\left(b_2 k^2 - c_2\right) \sin 2kl + 2k\sqrt{1+b_2c_2}\cos\phi_2 \cos 2kl \right) + \\ +2k\left(\sqrt{1+b_2c_2}\sin\phi_1 \sin\phi_2 - 1\right) = 0.$$

- empty if $\phi_1 = \pm\pi/2$ and $\sqrt{1+b_2c_2}\sin\phi_2 - 1 \neq 0$
- entire $\mathbb{C}$ if $\phi_1 = \pm\pi/2$ and $\sqrt{1+b_2c_2}\sin\phi_2 - 1 = 0$
- $b_2 = c_2 = 0$
  - empty if $\phi_1 = \pm\pi/2$ and $\phi_2 \neq \pm\pi/2$
  - entire $\mathbb{C}$ if $\phi_1 = \phi_2 = \pm\pi/2$

Conclusions

Conclusions

- Antilinear symmetry is not equivalent to pseudo-Hermiticity
- Equivalent subclases are not determined
- $\mathcal{PT}$-symmetry, pseudo-Hermiticity, C-self-adjointness do not guarantee non-empty spectrum, countable point spectrum, spectral decomposition
- Examples of $\mathcal{PT}$-symmetric point interactions
  - line $\mathbb{R}$ - $\sigma = \mathbb{C}$, $\sigma_c = [0, \infty)$, $\sigma_p = \mathbb{C} \setminus [0, \infty)$
  - finite interval $(-l, l)$ - $\sigma = \emptyset$ versus $\sigma = \sigma_p = \mathbb{C}$

References

- 2002 Albeverio, Fei, Kurasov, LMP, Point Interactions: PT-Hermiticity and Reality of the Spectrum
- 2002 Solombrino, Weak pseudo-Hermiticity and antilinear commutant, quant-ph/0203101
- 2002 Scolarici, Solombrino, On the pseudo-Hermitian nondiagonalizable Hamiltonians, quant-ph/0211161
- 2005 Albeverio, Kuzhel, LMP, One-dimensional Schrdinger operators with $\mathcal{P}$-symmetric zero-range potentials
- 2008 Siegl, JPA, Supersymmetric quasi-Hermitian Hamiltonians with point interactions on a loop
- 2009 Albeverio, Gunther, Kuzhel, JPA, J-self-adjoint operators with $\mathcal{C}$-symmetries: an extension theory approach
- 2009 Siegl, PRAMANA, The non-equivalence of pseudo-Hermiticity and presence of antilinear symmetry
- 2009 Siegl, 'Surprising' spectra of PT-symmetric point interactions, arXiv:0906.0226v1
- 2009 Kuzhel, Shapovalova, Vavrykovych, On J-self-adjoint extensions of the Phillips symmetric operator, arXiv:0907.3280v1