Quasi-Hermitian Model with Point Interactions and Supersymmetry

Petr Siegl

Nuclear Physics Institute, Prague

Faculty of Nuclear Sciences and Physical Engineering

siegl@ujf.cas.cz

Introduction

SUPERSYMETRY in quantum mechanics with two Hermitian point interactions (at $x=0$ and $x=l$) is allowed only for two special classes of models [1]. Are the systems with two $\mathcal{PT}$-symmetric point interactions compatible with supersymmetry? What are the appropriate classes of boundary conditions? Are all energy levels real? Is it possible to construct positive metric operator Θ?

$\mathcal{PT}$-symmetric point interactions and SUSY

- General $\mathcal{PT}$-symmetric point interactions are characterized by boundary conditions [2]

$$\begin{pmatrix} \psi(0_+) \\ \psi'(0_+) \end{pmatrix} = e^{i\theta} \begin{pmatrix} \sqrt{1+bc}\, e^{i\phi} & b \\ c & \sqrt{1+bc}\, e^{-i\phi} \end{pmatrix} \begin{pmatrix} \psi(0_-) \\ \psi'(0_-) \end{pmatrix}$$

- Requirement of SUSY restricts the ranges of parameters b, c, θ, ϕ
- The only possible boundary conditions are given by matrices $B_\pm$

$$(B_\pm - I)\Psi + (B_\pm + I)\Psi' = 0$$

$$\Psi = \begin{pmatrix} \psi(0_+) \\ \psi(0_-) \end{pmatrix}, \quad \Psi' = \begin{pmatrix} \psi'(0_+) \\ -\psi'(0_-) \end{pmatrix}, \quad B_\pm = \pm \begin{pmatrix} i\tan\varphi & \frac{e^{i\theta}}{\cos\varphi} \\ \frac{e^{-i\theta}}{\cos\varphi} & -i\tan\varphi \end{pmatrix}$$

- Special structure of $B_\pm$

$$\beta_1 := -\frac{\cos\theta}{\cos\varphi}, \quad \beta_2 := \frac{\sin\theta}{\cos\varphi}, \quad \beta_3 := -i\tan\varphi, \quad \vec{\beta}^{\,2} = 1, \quad \beta_{1,2} \in \mathbb{R}, \quad \beta_3 \in i\mathbb{R}$$

$$B_\pm = \exp\left(i\tfrac{\pi}{2}(I \pm \vec{\beta}.\vec{\sigma})\right)$$

- Supercharges $Q_{a,b}$

$$\{Q_a, Q_b\} = H\delta_{ab}\,, \quad Q_a = i\frac{\sqrt{2}}{2}\mathcal{G}_a\,\mathcal{P}_3\frac{d}{dx}$$

$$\mathcal{G}_a = \vec{\gamma}_a.\vec{\mathcal{P}}, \quad \vec{\gamma}_a.\vec{\beta} = 0, \quad \vec{\gamma}_a.\vec{\gamma}_b = \delta_{ab}\,, \quad a, b \in \{1,2\}$$

$$\mathcal{P}_1 := \mathcal{P}, \ \mathcal{P}_2 := \mathcal{Q}, \ \mathcal{P}_3 := \mathcal{R},$$

$$(\mathcal{R}\psi)(x) := (\vartheta(x) - \vartheta(-x))\psi(x), \quad \mathcal{Q} := -i\mathcal{RP}$$

ϑ is a Heaviside step function

Two classes of models

Model of the type (+ +)

- Both interactions characterized by B_+ matrix
- Eigenvalues and eigenfunctions

$$E_n = \left(\frac{n\pi}{l}\right)^2, \quad n \in \mathbb{N}_0$$

$$\psi_{n+}(x) = C_n\left(\vartheta(x) - \vartheta(-x)\frac{\beta_1 + i\beta_2}{1+\beta_3}\right)\sin\frac{n\pi}{l}x, \ n \in \mathbb{N}$$

$$\psi_{n-}(x) = C_n\left(\vartheta(x) - \vartheta(-x)\frac{\beta_1 + i\beta_2}{1-\beta_3}\right)\cos\frac{n\pi}{l}x, \ n \in \mathbb{N}_0$$

- Supersymmetry is unbroken
- Metric operator

$$\Theta = I - \frac{\beta_3}{\beta_1 + i\beta_2}P^+\mathcal{P} + \frac{\beta_3}{\beta_1 - i\beta_2}P^-\mathcal{P}$$

$P^\pm$ are projectors

$$(P^\pm\psi)(x) = \vartheta(\pm x)\psi(x)$$

$$(P^\pm)^2 = P^\pm = (P^\pm)^*, \quad P^+P^- = P^-P^+ = 0$$

Model of the type (+ -)

- Interactions characterized by B_+ at $x=0$ and by B_- at $x=l$
- Eigenvalues and eigenfunctions

$$E_n = \left(\frac{(2n-1)\pi}{2l}\right)^2, \ n \in \mathbb{N}$$

$$\psi_{n+}(x) = C_n\left(\vartheta(x) - \vartheta(-x)\frac{\beta_1 + i\beta_2}{1+\beta_3}\right)\sin\frac{(n-1)\pi}{2l}x, \ n \in \mathbb{N}$$

$$\psi_{n-}(x) = C_n\left(\vartheta(x) - \vartheta(-x)\frac{\beta_1 + i\beta_2}{1-\beta_3}\right)\cos\frac{(n-1)\pi}{2l}x, \ n \in \mathbb{N}$$

- Supersymmetry is broken
- Metric operator

$$\Theta = P^+(O_1+O_2)P^+ + P^-(O_1+O_2)P^- - \frac{\beta_1 - i\beta_2}{1+\beta_3}P^+O_1P^- -$$

$$-\frac{\beta_1 + i\beta_2}{1-\beta_3}P^-O_1P^+ - \frac{\beta_1 - i\beta_2}{1-\beta_3}P^+O_2P^- - \frac{\beta_1 + i\beta_2}{1+\beta_3}P^-O_2P^+$$

$O_{1,2}$ are projectors

$$O_1e_{2k} = 0, \quad O_1e_{2k-1} = e_{2k-1}, \quad O_2f_{2k} = 0, \quad O_2f_{2k-1} = f_{2k-1}$$

$$e_0(x) = \frac{1}{\sqrt{2l}}, \quad e_{2k-1}(x) = \frac{1}{\sqrt{l}}\sin\frac{(2k-1)\pi}{2l}x, \quad e_{2k}(x) = \frac{1}{\sqrt{l}}\cos\frac{k\pi}{l}x$$

$$f_{2k-1}(x) = \frac{1}{\sqrt{l}}\cos\frac{(2k-1)\pi}{2l}x, \quad f_{2k}(x) = \frac{1}{\sqrt{l}}\sin\frac{k\pi}{l}x, \quad k \in \mathbb{N}$$

METRIC operator Θ is bounded, positive and requirement $\Theta H = H^*\Theta$ is fulfilled for all functions from the Dom(H) for both classes of models. The systems with two $\mathcal{PT}$-symmetric point interactions and supersymmetry are therefore quasi-Hermitian.

References

[1] T. Nagasawa, M. Sakamoto, K. Takenaga, Supersymmetry in Quantum Mechanics with Point Interactions, Phys. Lett. B 562, 2003, 358-364

[2] S. Albeverio, S.M. Fei, P. Kurasov, Point Interactions: PT-Hermiticity and Reality of the Spectrum, Lett. Math. Phys. 59, 2002, 227-242