# Spectral line broadening in stellar atmospheres

- Motivation:

1 Line broadening theory is a key ingredient in the construction of model atmospheres and synthetic spectra.

2 Useful temperature, density, abundance diagnostics.

- This lecture will discuss

1 The quasi-static line broadening theory applied to hydrogen line profiles. In particular will show that in the line wings the line opacity is well described by:

$$\alpha(\Delta\nu) \propto \Delta\nu^{-5/2}$$

2 applications to

white dwarfs: $T_{\rm eff}$, $\log g$,

$\delta$ Scuti stars: $T_{\rm eff}$, [Fe/H], $v \sin i$.

- These lecture notes are based on:

1 Griem, H.R., *Spectral line broadening by plasmas*,

2 Sobel'man, I.I., Vainshtein, L.A., Yukov, E.A., *Excitation of atoms and broadening of spectral lines*,

3 Gray, D.F., *The observation and analysis of stellar photospheres.*

1. Spectral lines as a temperature, density and abundance diagnostic.

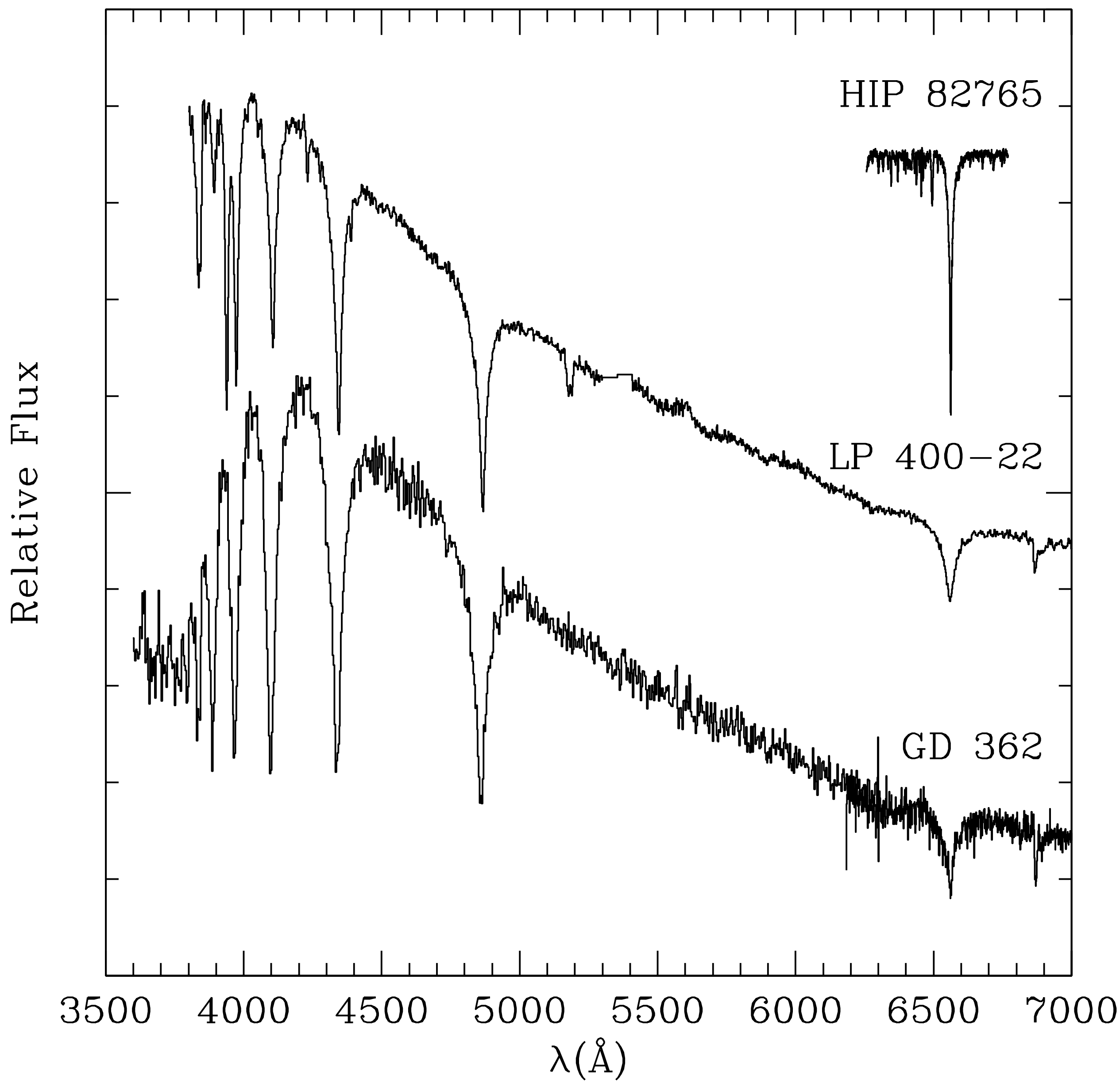



Figure 1: Spectral lines (H I Balmer, Ca II, Fe I, etc...) in white dwarfs and a giant F star ($\delta$Scuti variable).

1. What is the microscopic situation ?

   1.1: the motion of the test particle (an atom) and the perturbing particles is "classical".

   1.2: trajectory is rectilinear.

   1.3: the perturbation is adiabatic ... the encounter does not change the state of the atom

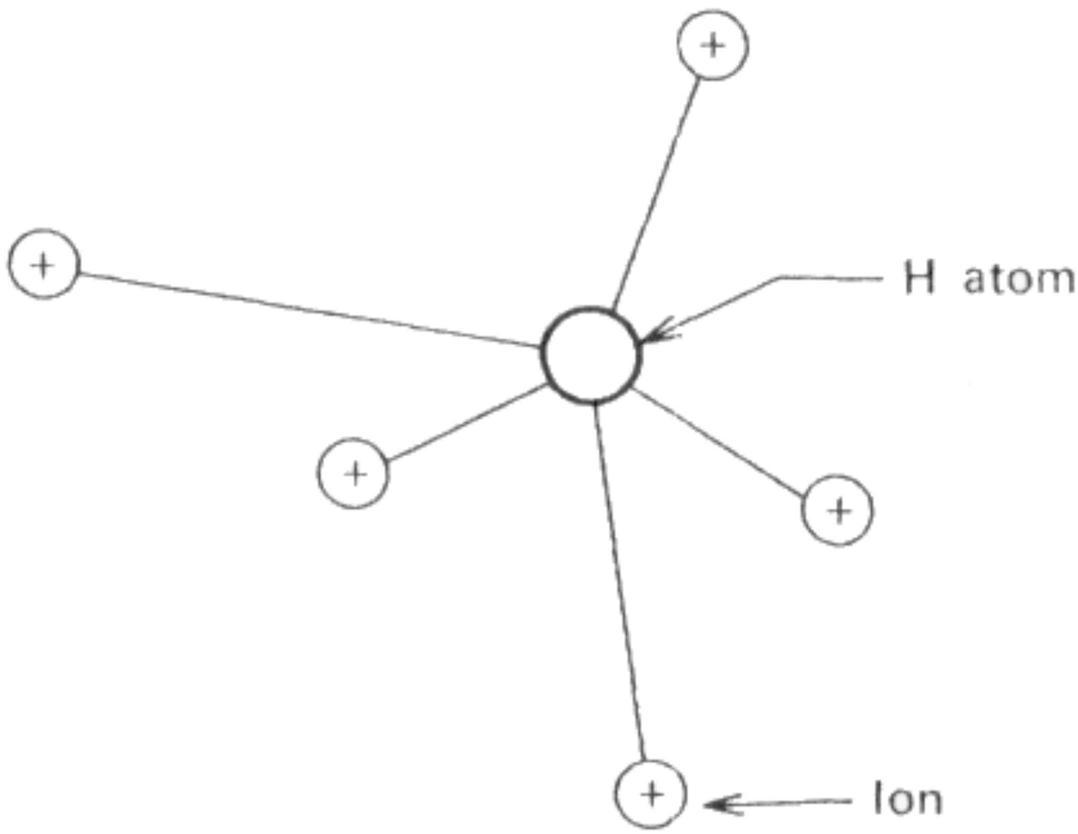
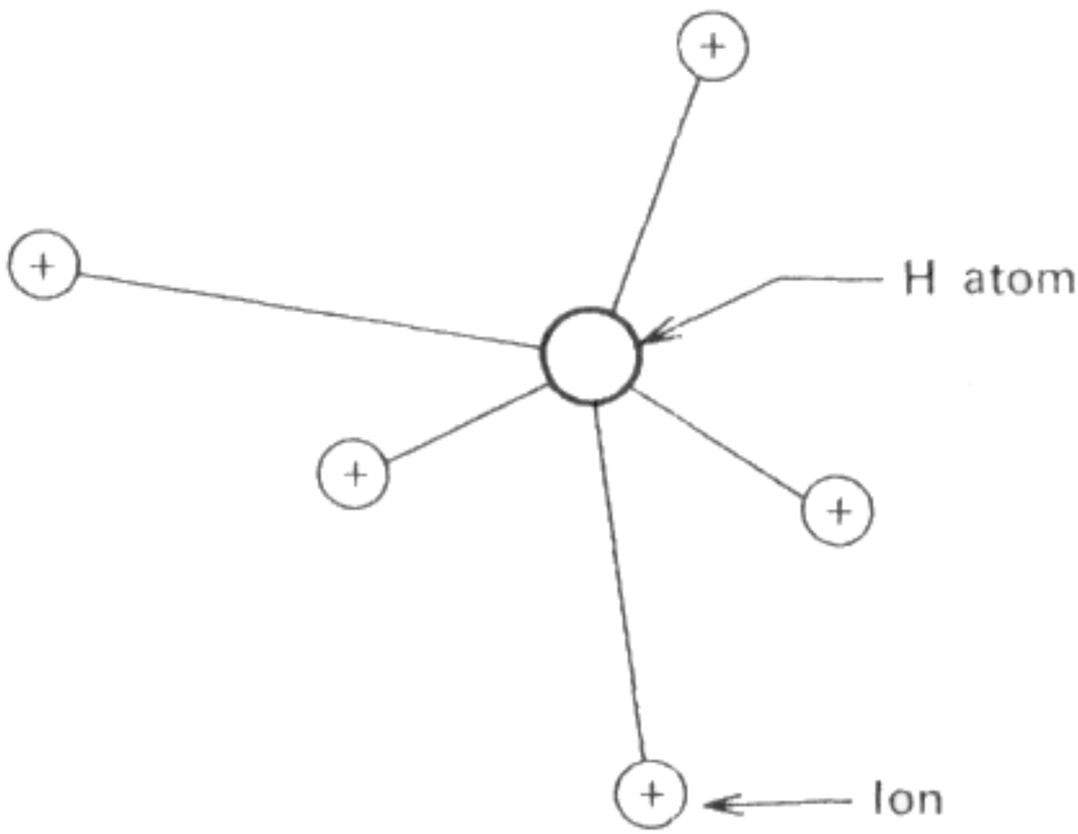



Figure 2: A test particle (H atom) is besieged by positively charged perturbers (Gray, *The Observation and Analysis of Stellar Photospheres*).




In addition, we may assume

1.3: that only the nearest particle contributes to line broadening ... binary interactions or **nearest-neighbor** hypothesis.

1.5: that the perturbers move slowly ... the **quasi-static** hypothesis.

1.6: or, in the **impact broadening** approximation, that spectral line shift are caused by instantaneous collisions.

2. What happens to a spectral line ?

2.1: assume that the energy levels involved in a line transition are perturbed (shifted) differently resulting in a net spectral line shift.

2.2: and describe that shift by:

$$\kappa = \Delta\omega = 2\pi\Delta\nu = C_n R^{-n}$$

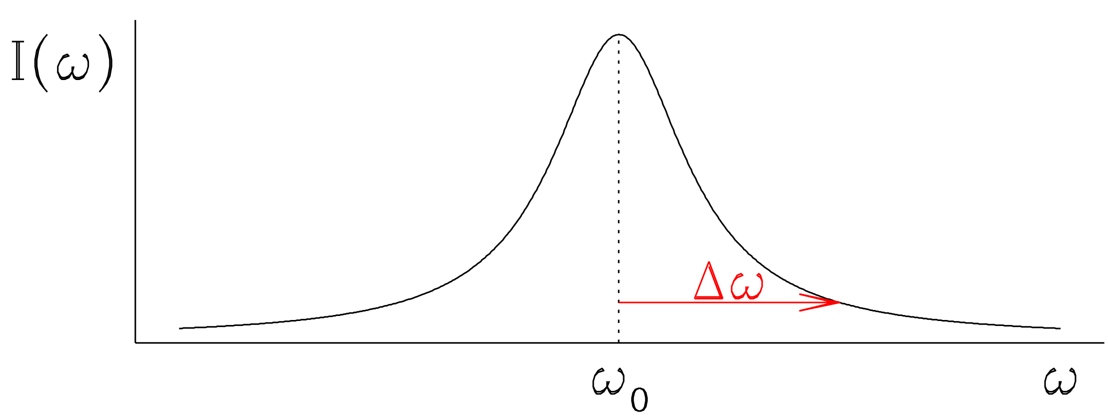
$n$ describes the form of the interaction field,
$C_n$ is an interaction constant, and
$R$ is the separation between the atom and a single perturber.

2.3: For example, in a simple electrostatic field $n = 2$:

$$\Delta\omega = C_2 R^{-2} = C_2 \frac{F}{Ze} \qquad (C_2 \approx 1\,\mathrm{cm}^2/\mathrm{s})$$

from which we can conclude that the **nearest neighbor** is responsible for the largest shift :

$$R \;<< \quad \rightarrow \quad \Delta\omega \;>>$$



3. The **nearest neighbor** approximation ...

3.1: the field caused by the nearest particle is known

$$F(R) \qquad (\text{or } \mathcal{E}(R))$$

3.2: what is the most probable distribution of this nearest perturber

$$W(R)dR$$

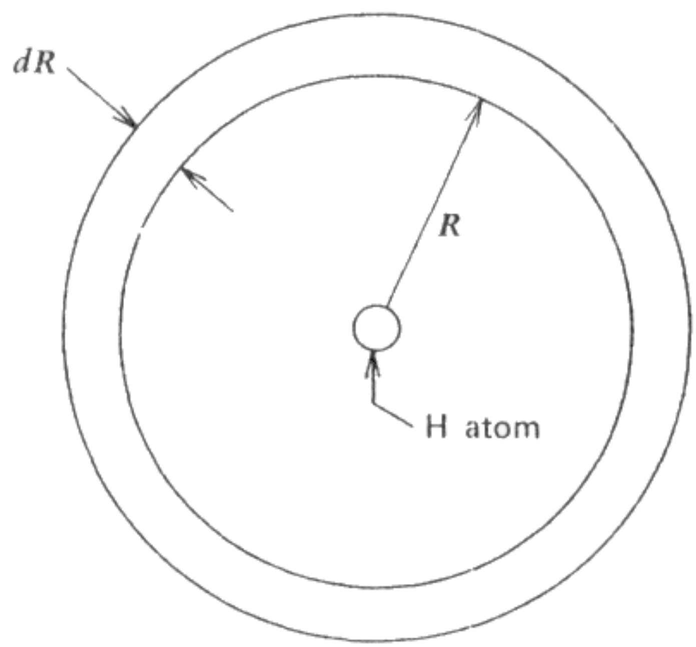
knowing that there are

$$N = \left(\frac{4\pi}{3} R_0^3\right)^{-1}$$

such perturber per unit volume ? *Find the probability that the nearest particle is in the range* $(R, R{+}dR)$:

$$\text{P(empty up to R)} \times \text{P(perturber inside dV)}$$

where $dV = 4\pi R^2 dR$ is the volume of a shell at $R$.



3.3: The probabilities that the perturber be inside $dV$ or not are simply:

$$\text{P(perturber inside dV)} = \frac{dV}{V_0} = N dV$$

$$\text{P(no perturber inside dV)} = 1 - N dV$$

where $V_0$ is the volume occupied by a single perturber

$$V_0 = \frac{4\pi}{3} R_0^3 = \frac{1}{N}$$

Next, we evaluate the probability that the perturber is **not** within $< R$:

$$\text{P(empty up to R+dR)}$$

$$= \text{ P(empty up to R)} \times \text{ P(no perturber inside dV)}$$

that is:

$$\text{P(empty up to R+dR)} = \text{ P(empty up to R)} \times (1 - N dV)$$

$$\frac{\text{P(empty up to R+dR)} - \text{P(empty up to R)}}{\text{P(empty up to R)}} = -N dV$$

$$\frac{dP}{P} = -N dV$$

$$\ln P' \Big|_1^P = -N V' \Big|_0^V$$

$$\text{P(empty up to R)} = e^{-NV} = e^{-V/V_0}$$

3.4 We now have the two ingredients required to describe the spatial distribution of a single perturber:

$$W(R)dR = \text{P(empty up to R)} \times \text{P(perturber inside dV)}$$

$$W(R)dR = e^{-V/V_0}\frac{dV}{V_0}$$

Is that probability distribution **normalized** ?

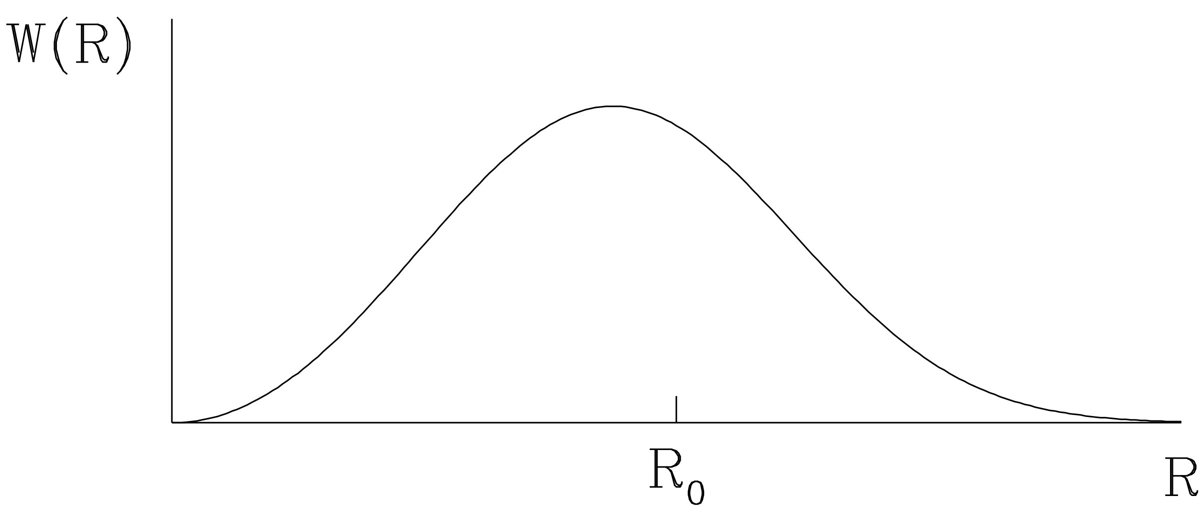
$$\int_0^\infty W(R)dR = \int_0^\infty e^{-V/V_0}\frac{dV}{V_0} = \int_0^\infty e^{-u}du = 1$$

Recalling the density of perturber $N$

$$\frac{1}{V_0} = N$$

We now have the distance distribution for the **nearest neighbor**:

$$W(R)dR = e^{-\frac{4}{3}\pi R^3 N}\,4\pi R^2 N dR = 3\Big(\frac{R}{R_0}\Big)^2 e^{-(R/R_0)^3}\frac{dR}{R_0}$$



## 3.5 All distributions

| | |
|---|---|
| distance to nearest neighbor | $W(R)dR$ |
| resulting field at the atom | $W(F)dF$ |
| resulting line distribution | $I(\Delta\omega)d\Delta\omega$ |

follow the same distribution law:

$$I(\Delta\omega)d\Delta\omega = W(F)dF = W(R)dR$$

where $\Delta\omega$ is measured from the line center ...
So we may now calculate a normalized line distribution:

$$I(\Delta\omega) = W(R)\left|\frac{dR}{d\Delta\omega}\right|$$

For the frequency shift we have that in general:

$$\Delta\omega = C_n R^{-n} \qquad \rightarrow \qquad R = C_n^{1/n}\Delta\omega^{-1/n}$$

$$\overline{\Delta\omega} = C_n R_0^{-n} \qquad \rightarrow \qquad R_0 = C_n^{1/n}\overline{\Delta\omega}^{-1/n}$$

$$\left|\frac{dR}{d\Delta\omega}\right| = \frac{C_n^{1/n}}{n}\Delta\omega^{-(n+1)/n}$$

$$I(\Delta\omega) = \Big(\frac{3}{R_0}\Big(\frac{R}{R_0}\Big)^2 e^{-(R/R_0)^3}\Big)\Big(\frac{C_n^{1/n}}{n}\Delta\omega^{-(n+1)/n}\Big)$$

$$= \Big(\frac{3\overline{\Delta\omega}^{1/n}}{C_n^{1/n}}\Big(\frac{\Delta\omega}{\overline{\Delta\omega}}\Big)^{-2/n} e^{-(\Delta\omega/\overline{\Delta\omega})^{-3/n}}\Big)\Big(\frac{C_n^{1/n}}{n}\Delta\omega^{-(n+1)/n}\Big)$$

And re-grouping factors together and with a few cancellations, we have a more digestible:

$$I(\Delta\omega) = \frac{3}{n}\,\overline{\Delta\omega}^{3/n}\,\Delta\omega^{-(n+3)/n} e^{-(\Delta\omega/\overline{\Delta\omega})^{-3/n}}$$

The **nearest neighbor** approximation is only valid far in the line wing:

$$\Delta\omega >> \overline{\Delta\omega}$$

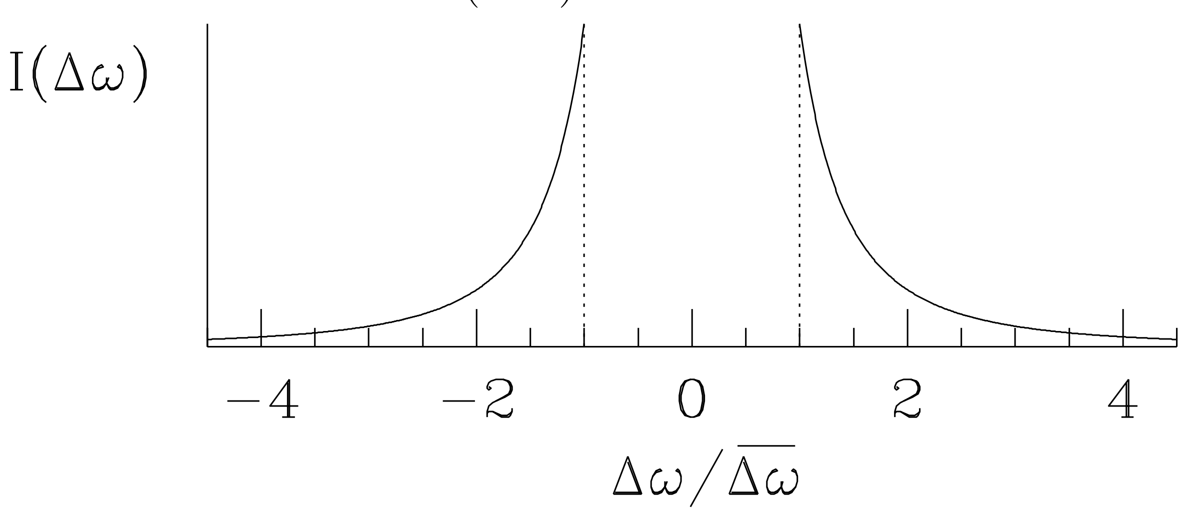
and the exponential factor $\approx 1$:

$$I(\Delta\omega) \approx \frac{3}{n}\,\overline{\Delta\omega}^{3/n}\,\Delta\omega^{-(n+3)/n}$$

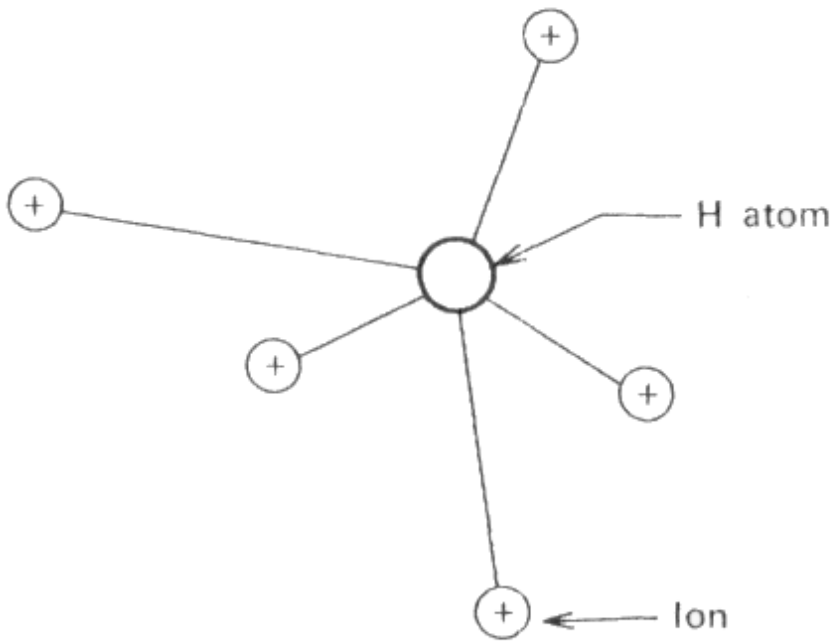
Adopt a simple Coulomb interaction $n = 2$, and the result is a classic line-wing approximation for the calculation of line opacities in stellar atmospheres:

$$I(\Delta\omega) \approx \frac{3}{2}\,\overline{\Delta\omega}^{3/2}\,\Delta\omega^{-5/2}$$

$$I(\Delta\omega) \propto \Delta\omega^{-5/2}$$



4. Holtsmark lifted the **nearest neighbor** restriction because it neglects the line centers.




The **nearest neighbor** at

$$R << R_0$$

interacts strongly with the atomic energy levels (the "classical oscillator") and dominates the line wings at

$$\Delta\omega >> \overline{\Delta\omega}$$

**Holtsmark theory** includes a large number of distant neighbors

$$R \gtrsim R_0$$

which dominate low fields $F$ and the corresponding line center

$$\Delta\omega \lesssim \overline{\Delta\omega}$$

4.1 **Holtsmark theory** ... the field distribution in three-dimension is the integral over a volume $V$ of

$$W_0(\mathbf{F}) = \int ... \int \int \delta(\mathbf{F} - \sum_{j=1}^{n} \mathbf{F}_j) P(\mathbf{r}_1, \mathbf{r}_2, ...\mathbf{r}_n) d^3\mathbf{r}_1 d^3\mathbf{r}_2 ... d^3\mathbf{r}_n$$

The integral weighs in the many possibilities for $\mathbf{r}_1, \mathbf{r}_2, ...\mathbf{r}_n$ leading to the desired value of $\mathbf{F}$:

$$\mathbf{F} = \sum_{j=1}^{n} \mathbf{F}_j$$

since the particles are uncorrelated each set $\mathbf{r}_1, \mathbf{r}_2, ...\mathbf{r}_n$ has a probability:

$$P(\mathbf{r}_1, \mathbf{r}_2, ...\mathbf{r}_n) = \frac{1}{V} \cdot \frac{1}{V} ... \cdot \frac{1}{V} = \frac{1}{V^n}$$

where $V$ is the volume containing all $n$ particles, so

$$W_0(\mathbf{F}) = \frac{1}{V^n} \int ... \int \int \delta(\mathbf{F} - \sum_{j=1}^{n} \mathbf{F}_j)\, d^3\mathbf{r}_1 d^3\mathbf{r}_2 ... d^3\mathbf{r}_n$$

Necessarily , the vector integral of the probability distribution over all field strengths is normalized

$$\int W_0(\mathbf{F}) d\mathbf{F} = 1$$

The Fourier transform $A(\mathbf{k})$ of this challenging integral is more amenable (after you're done, take the inverse transform to obtain $W_0(\mathbf{F})$).

4.2 Fourier transform of the field distribution $W_0(\mathbf{F})$.

$$A(\mathbf{k}) = \int e^{i\mathbf{k}\cdot\mathbf{F}} W_0(\mathbf{F}) d\mathbf{F}$$

Inserting our definition for the probability distribution $W_0(\mathbf{F})$:

$$A(\mathbf{k}) = \frac{1}{V^n} \int e^{i\mathbf{k}\cdot\mathbf{F}} \Big[ \int ... \int \int \delta(\mathbf{F} - \sum_{j=1}^{n} \mathbf{F}_j)\, d^3\mathbf{r}_1 d^3\mathbf{r}_2 ... d^3\mathbf{r}_n \Big] d\mathbf{F}$$

and invert the order of integration:

$$A(\mathbf{k}) = \frac{1}{V^n} \int ... \int \int \Big[ \int e^{i\mathbf{k}\cdot\mathbf{F}} \delta(\mathbf{F} - \sum_{j=1}^{n} \mathbf{F}_j) d\mathbf{F} \Big]\, d^3\mathbf{r}_1 d^3\mathbf{r}_2 ... d^3\mathbf{r}_n$$

after integrating over $d\mathbf{F}$

$$A(\mathbf{k}) = \frac{1}{V^n} \int ... \int \int e^{i\mathbf{k}\cdot \sum_{j=1}^{n} \mathbf{F}_j}\, d^3\mathbf{r}_1 d^3\mathbf{r}_2 ... d^3\mathbf{r}_n$$

and each integral is independent and identical

$$A(\mathbf{k}) = \frac{1}{V^n} \int e^{i\mathbf{k}\cdot\,\mathbf{F}_1}\, d^3\mathbf{r}_1 \int e^{i\mathbf{k}\cdot\,\mathbf{F}_2}\, d^3\mathbf{r}_2 \ldots \int e^{i\mathbf{k}\cdot\,\mathbf{F}_n}\, d^3\mathbf{r}_n$$

$$A(\mathbf{k}) = \Big[\frac{1}{V} \int e^{i\mathbf{k}\cdot\,\mathbf{F}_1}\, d^3\mathbf{r}_1\Big]^n = \Big[\frac{1}{V} \int e^{i\mathbf{k}\cdot\,\mathbf{F}}\, d^3\mathbf{r}\Big]^n$$

after dropping the subscript $j$.

4.3 Solving the Fourier transform $A(\mathbf{k})$:

$$A(\mathbf{k}) = \left[\frac{1}{V}\int e^{i\mathbf{k}\cdot\,\mathbf{F}}\,d^3\mathbf{r}\right]^n = I^n(\mathbf{k})$$

where

$$I(\mathbf{k}) = \frac{1}{V}\int e^{i\mathbf{k}\cdot\,\mathbf{F}}\,d^3\mathbf{r}$$

Lets define a few things in spherical geometry:

(1) the dot product:

$$\mathbf{k}\cdot\,\mathbf{F} = kF\cos\theta$$

(2) the volume element:

$$d^3\mathbf{r} = r^2\sin\theta dr d\theta d\phi$$

(3) and happily integrate:

$$I(\mathbf{k}) = \frac{1}{V}\int r^2 dr\int_0^{2\pi} d\phi\int_0^{\pi}\sin\theta d\theta e^{ikF\cos\theta}$$

$F = F(r)$ so wait before integrating $r$, go ahead with $\phi = [0, 2\pi]$, and substitute $u = kF\cos\theta$:

$$I(\mathbf{k}) = \frac{2\pi}{V}\int r^2 dr\int_{-kF}^{kF}\frac{du}{kF}\,e^{iu}$$

(in $e^{iu} = \cos u + i\sin u$, $\cos u$ is an "even" function and the integral vanishes, but $\sin u$ is "odd")

$$I(\mathbf{k}) = \frac{2\pi}{V}\int r^2 dr\left(\frac{\sin u}{kF}\right)\Big|_{-kF}^{kF} = \frac{4\pi}{V}\int r^2 dr\frac{\sin kF}{kF}$$

We complete the calculation of $I(\mathbf{k})$ with the integration over $r$:

$$I(\mathbf{k}) = \frac{4\pi}{V}\int r^2 dr \frac{\sin kF}{kF}$$

$$I(\mathbf{k}) = \frac{4\pi}{V}\int r^2 dr - \frac{4\pi}{V}\int r^2 dr + \frac{4\pi}{V}\int r^2 dr \frac{\sin kF}{kF}$$

$V$ was taken large enough to include all $n$ perturber that affect the line profile, so $r$ is integrated over $V$:

$$I(\mathbf{k}) = 1 - \frac{4\pi}{V}\int r^2 dr + \frac{4\pi}{V}\int r^2 dr \frac{\sin kF}{kF}$$

$$I(\mathbf{k}) = 1 - \frac{4\pi}{V}\int \left[1 - \frac{\sin kF}{kF}\right] r^2 dr$$

Recall that the electric field $F$ is given by:

$$F = \frac{Ze}{r^2}$$

so introduce the variable $Y$

$$Y \equiv kF = \frac{kZe}{r^2} \longrightarrow r^2 = \frac{kZe}{Y} \longrightarrow dr = \frac{1}{2}\frac{(kZe)^{1/2}}{Y^{3/2}}$$

$$I(\mathbf{k}) = 1 - \frac{4\pi}{V}\frac{1}{2}(kZe)^{3/2}\int_0^\infty \left[1 - \frac{\sin Y}{Y}\right]\frac{dY}{Y^{5/2}}$$

This last definite integral is known (look it up in Abramowitz and Stegun!):

$$\int_0^\infty \left[1 - \frac{\sin Y}{Y}\right]\frac{dY}{Y^{5/2}} = \frac{4}{15}(2\pi)^{1/2}$$

And we finally have for $I(\mathbf{k})$:

$$I(\mathbf{k}) = 1 - \frac{4\pi}{V}\frac{1}{2}(kZe)^{3/2}\frac{4}{15}(2\pi)^{1/2}$$

simplify and regroup factors:

$$I(\mathbf{k}) = 1 - \frac{1}{V}\frac{4}{15}(2\pi)^{3/2}(Ze)^{3/2}k^{3/2}$$

The volume $V$ is the volume chosen to contain all $n$ perturber ... so the density of perturber is

$$N_p = \frac{n}{V} \qquad \rightarrow \qquad \frac{1}{V} = \frac{N_p}{n}$$

$$I(\mathbf{k}) = 1 - \frac{1}{n}\frac{4}{15}(2\pi)^{3/2}(Ze)^{3/2}N_p k^{3/2}$$

The quantity $F_0$ is now defined as the "normal field strength":

$$\frac{4}{15}(2\pi)^{3/2}(Ze)^{3/2}N_p \equiv F_0^{3/2} \quad \rightarrow \quad F_0 = 2\pi(4/15)^{2/3}ZeN_p^{2/3}$$

$$F_0 = 2.603ZeN_p^{2/3}$$

Note that you could directly estimate $F_0 = Ze/R_0^2$ using $N_p = (4\pi/3)R_0^3$ and obtain

$$F_0 = 2.595ZeN_p^{2/3}$$

Close enough, but not exact.

Anyway we can now complete our estimate of the Fourier transform of $W_0(\mathbf{F})$:

$$A(\mathbf{k}) = I^n(\mathbf{k}) = \left[1 - \frac{1}{n} F_0^{3/2} k^{3/2}\right]^n$$

Take $n$ as large as you want! the limit is very useful:

$$\lim_{n\to\infty} \left[1 - \frac{1}{n} x\right]^n = e^{-x}$$

So that $A(\mathbf{k})$ is isotropic and a simple exponential:

$$A(\mathbf{k}) = e^{-F_0^{3/2} k^{3/2}} = e^{-(kF_0)^{3/2}}$$

4.4 We are now ready to determine the field distribution $W_0(\mathbf{F})$ by taking the inverse transform (in three-dimensions):

$$W_0(\mathbf{F}) = \frac{1}{(2\pi)^3} \int e^{-i\mathbf{k}\cdot\mathbf{F}} A(\mathbf{k}) d\mathbf{k}$$

We're again integrating in 3D so we wont repeat everything, but note that:

$$\mathbf{k} \cdot \mathbf{F} = kF \cos\theta$$

$$d\mathbf{k} = 2\pi k^2 dk \sin\theta d\theta$$

and the integration over $\theta$ is quickly performed, leaving the integration over $k$

$$W_0(\mathbf{F}) = \frac{1}{(2\pi)^2} \int A(\mathbf{k}) \frac{2\sin(kF)}{kF} k^2 dk$$

And simplifying a little:

$$W_0(\mathbf{F}) = \frac{1}{(2\pi)^2}\frac{2}{F}\int A(\mathbf{k})\sin{(kF)}kdk$$

We do expect the distribution to be isotropic (no preferred directions in space), so we can now estimate the probability $W(F)$ inside the shell $4\pi F^2$:

$$W(F) = 4\pi F^2 W_0(\mathbf{F}) = 4\pi F^2 \frac{1}{(2\pi)^2}\frac{2}{F}\int A(\mathbf{k})\sin{(kF)}kdk$$

$$W(F) = \frac{2}{\pi}F\int A(\mathbf{k})\sin{(kF)}kdk$$

And recall our result for the Fourier transform:

$$A(\mathbf{k}) = e^{-(kF_0)^{3/2}}$$

$$W(F) = \frac{2}{\pi}F\int e^{-(kF_0)^{3/2}}\sin{(kF)}kdk$$

With one last change of variable:

$$x = kF_0$$

$$W(F) = \frac{2}{\pi}F\int e^{-x^{3/2}}\sin{(x\frac{F}{F_0})}\frac{x}{F_0}\frac{dx}{F_0}$$

and define the dimensionless variable $\beta = F/F_0$:

$$W(F) = \frac{2}{\pi}\frac{\beta}{F_0}\int e^{-x^{3/2}}\sin{(x\beta)}x\,dx$$

4.5 And FINALLY, introduce the **Holtsmark distribution** $H(\beta)$:

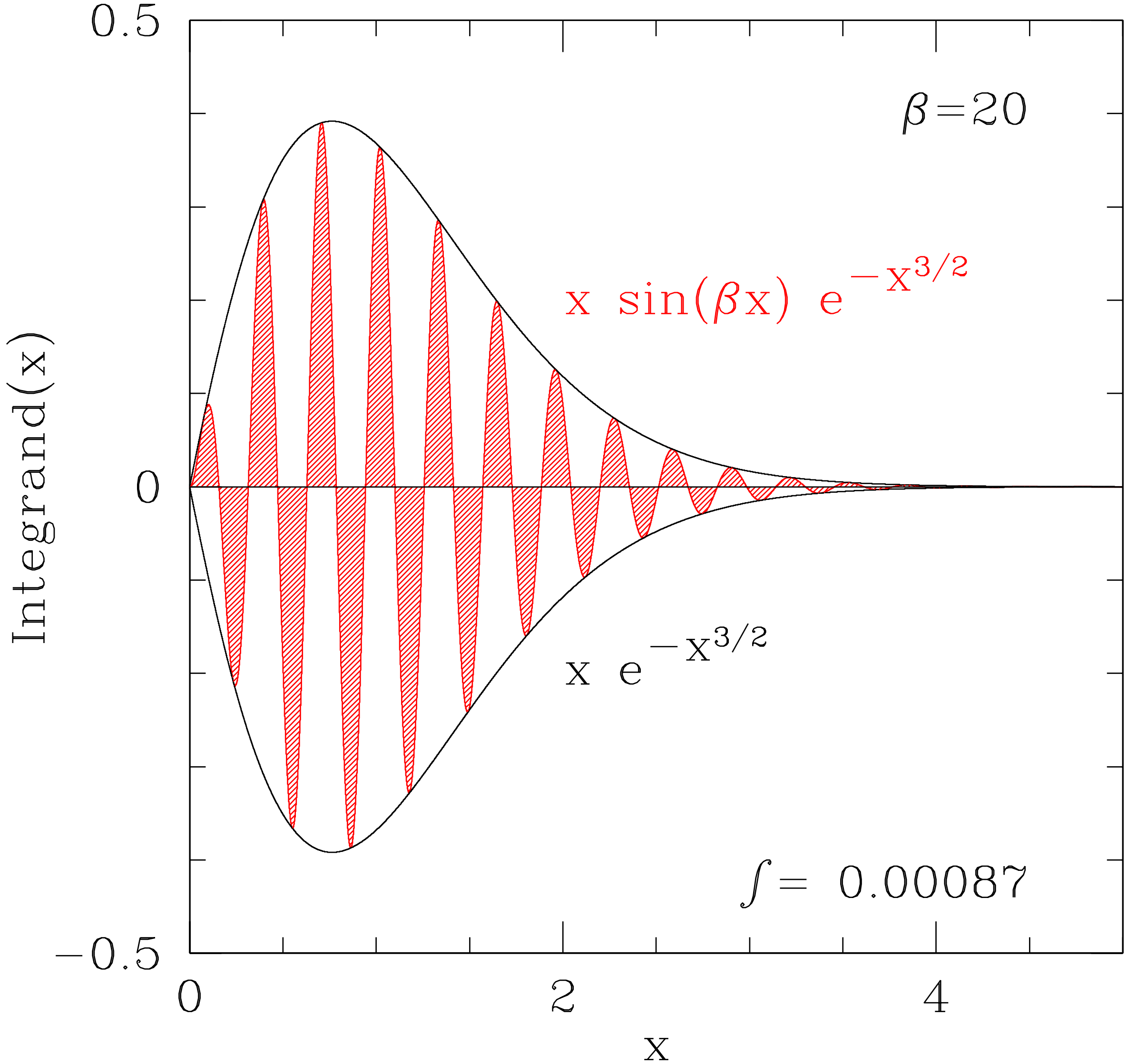
$$H(\beta)d\beta = W(F)dF$$

and since $\beta = F/F_0$:

$$H(\beta) = F_0 W(F)$$

$$H(\beta) = \frac{2}{\pi}\beta \int_0^\infty e^{-x^{3/2}} \sin{(x\beta)} x \, dx$$

How does one integrate this? Numerically! For $\beta >>$:



A general solution for $\beta >>$ can be obtained by expanding the integrand in a series and obtain:

$$H(\beta) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n!}\Gamma(\frac{3n+4}{2})\sin(\frac{3n\pi}{4})\beta^{-(3n+2)/2}$$

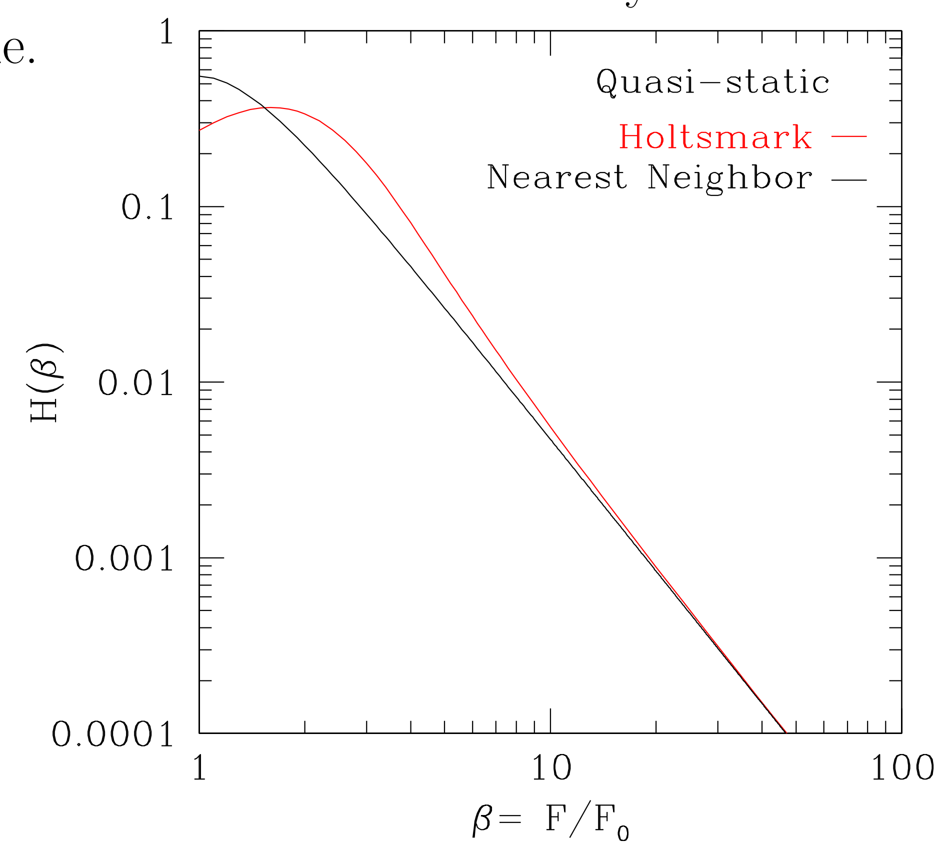
and retaining only the $n = 1$ term of the series:

$$H(\beta) \approx 1.496\beta^{-5/2} \propto \Delta\omega^{-5/2}$$

in agreement with the **nearest neighbor**:

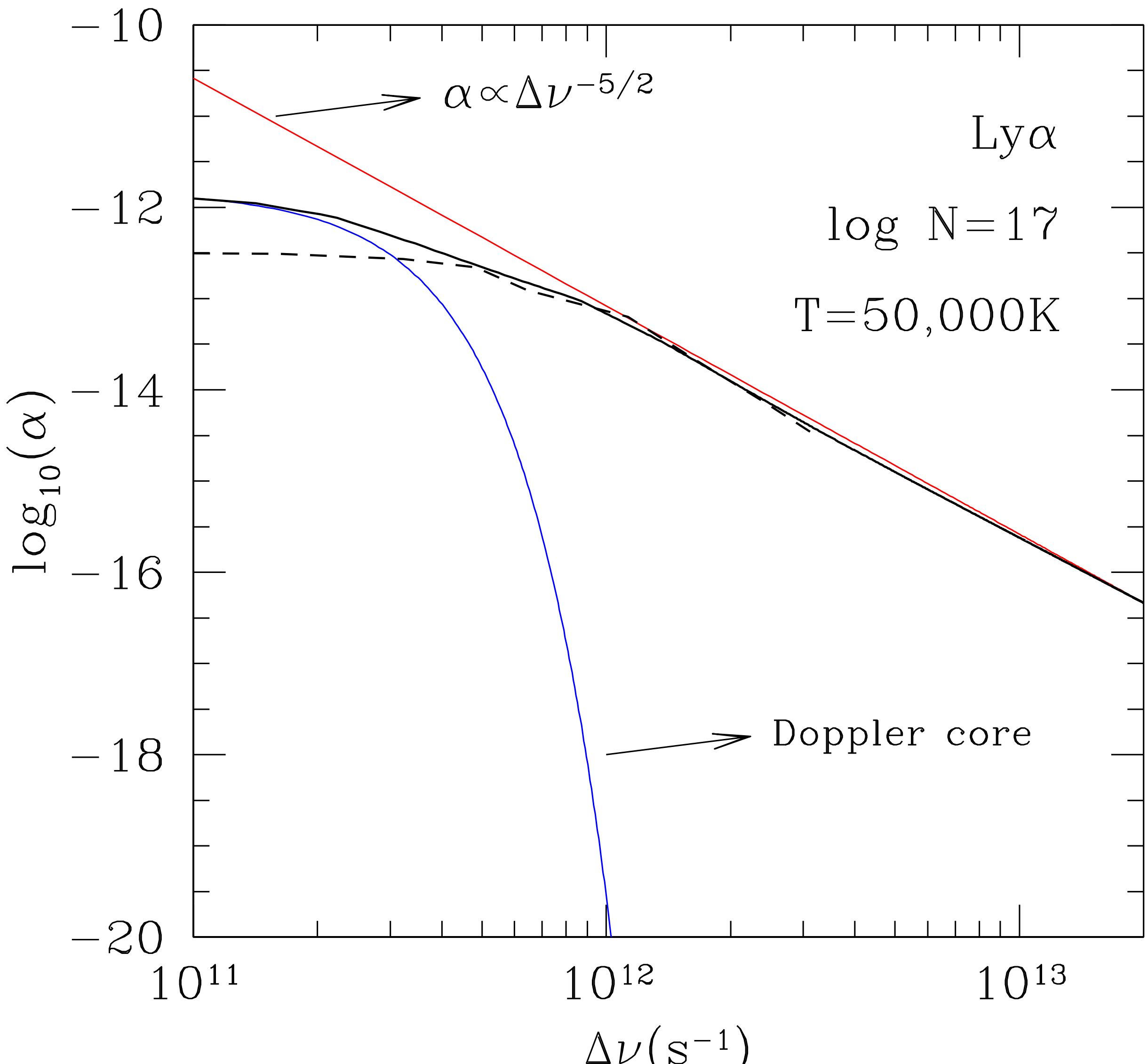
$$H(\beta) \approx 1.5\beta^{-5/2} \propto \Delta\omega^{-5/2}$$

But we did not exclude weak fields, so we expect the distribution to be correct everywhere in the line profile.



4.6 How does **Holtsmark theory** compare to recent calculations ? Line opacities $\alpha_\nu$ are given by:

$$\alpha(\Delta\nu)d(\Delta\nu) = H(\beta)d\beta$$




In red, the line wing behavior, in blue, the Doppler profile, with adashed line, the **Holtsmark theory**, and with a full line, the full impact/Holtsmark theory of Vidal, Cooper, and Smith (1973).

5.1 White dwarfs. Computed using Vennes & Kawka codes (employs Lemke and Vidal, Cooper & Smith line profiles based on **Holtsmark theory** in the line wings, and impact broadening theory in the center).

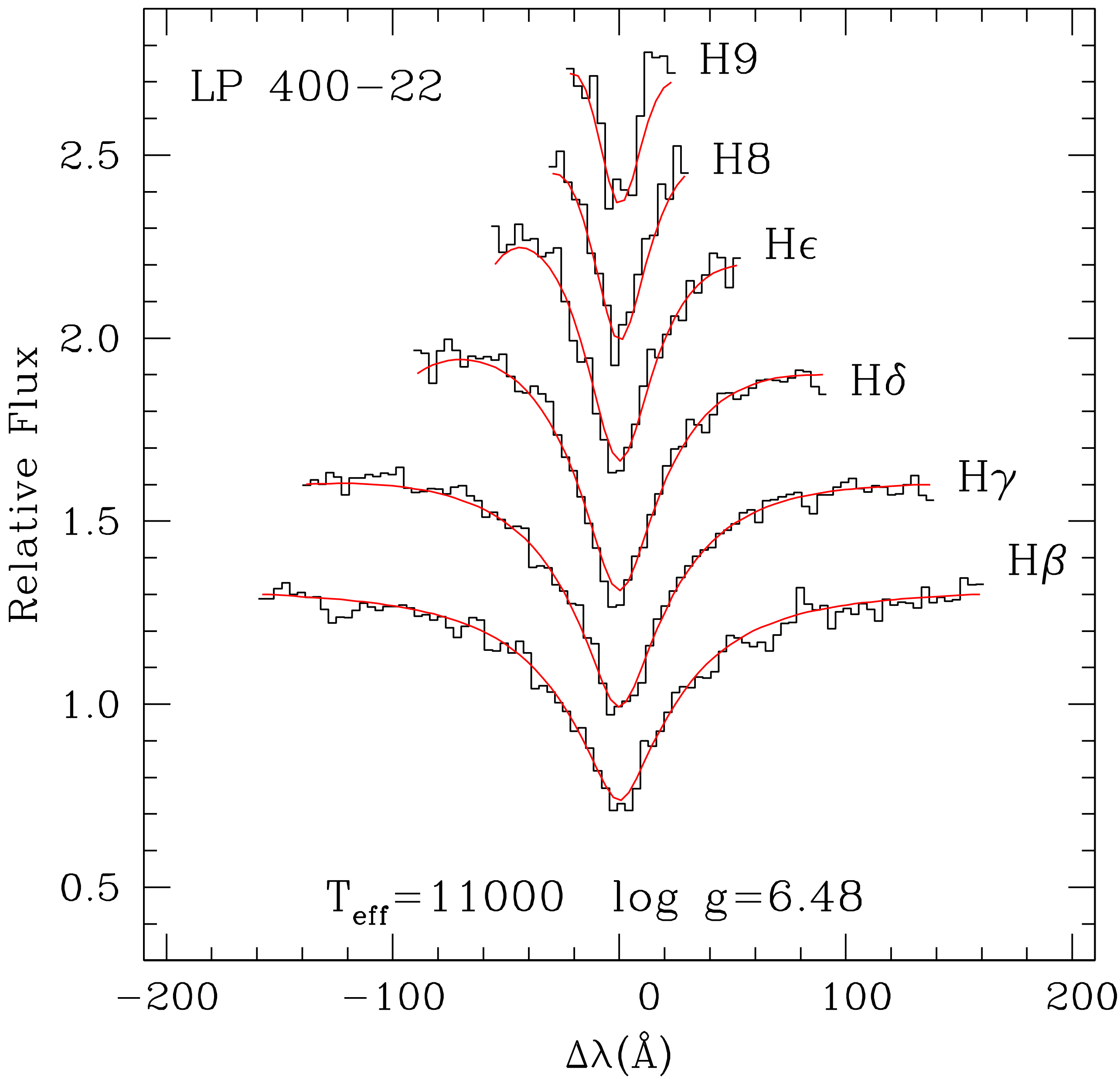



Figure 3: Temperature and gravity diagnostics for the low-mass white dwarf LP400-22 ($M_V = 9.1$).

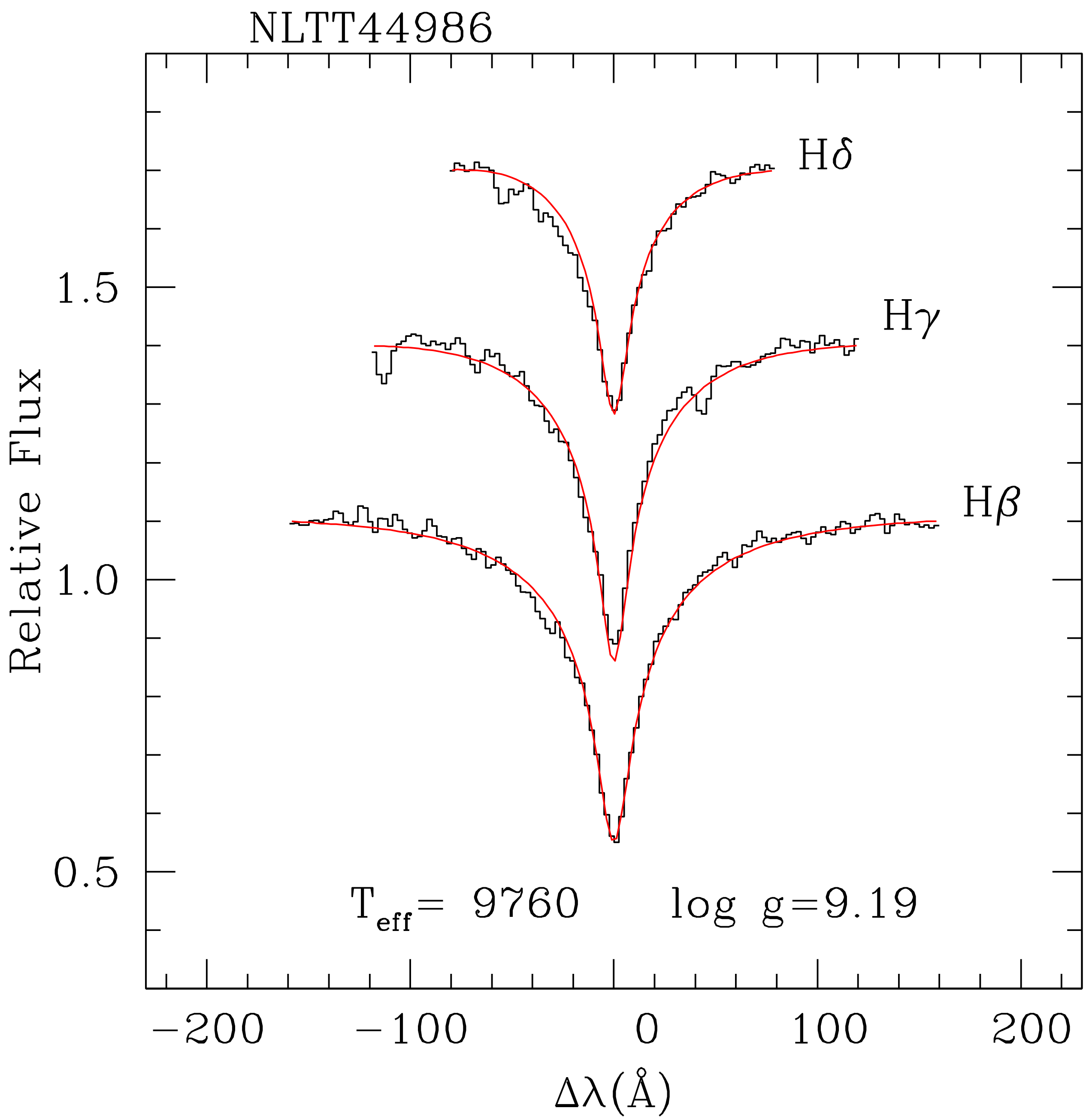



Figure 4: Temperature and gravity diagnostics for the high-mass white dwarf NLTT 44986 ($M_V = 14.5$).

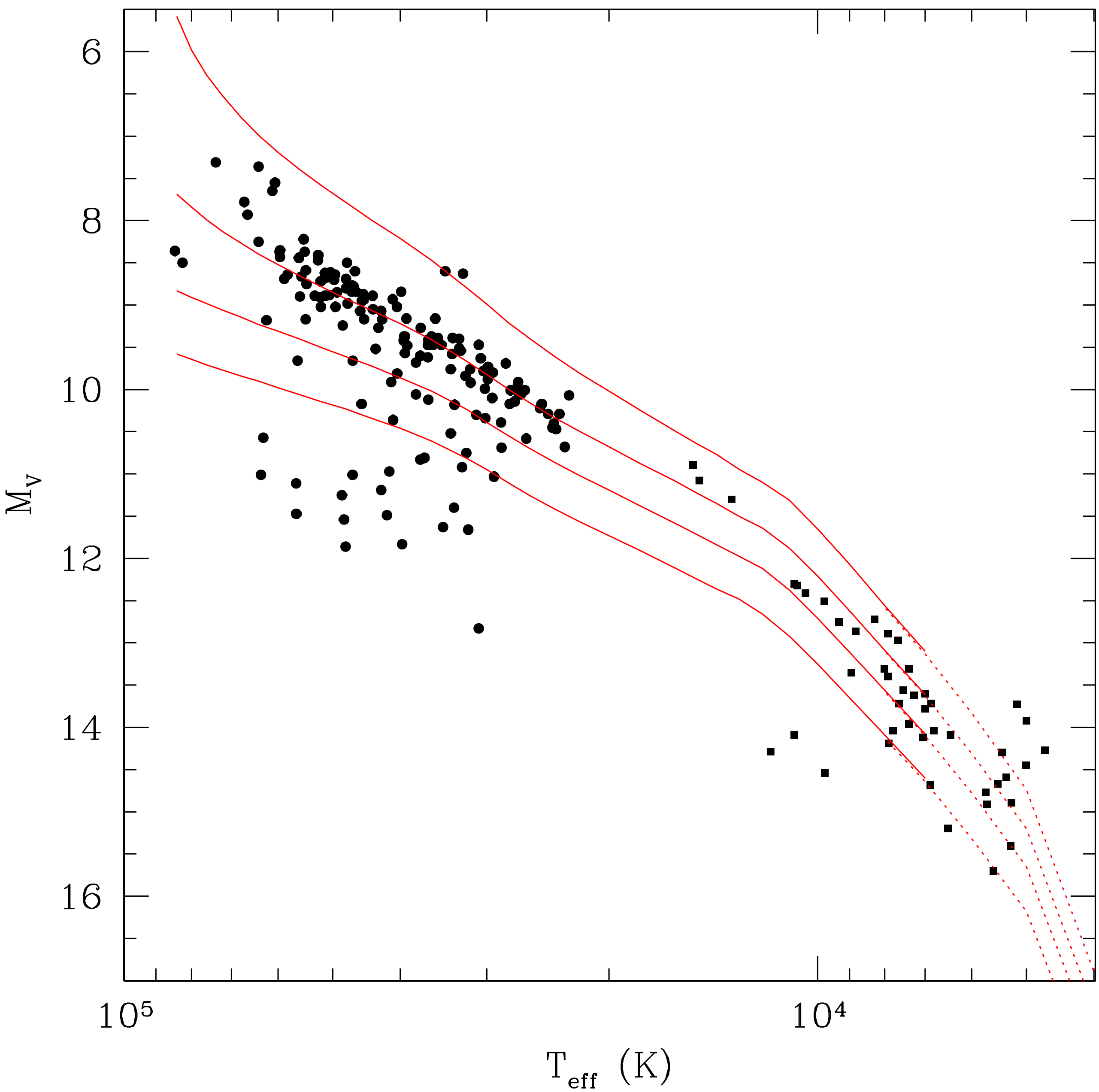



Figure 5: H-R diagram for white dwarfs built using temperature and gravity diagnostics based on Balmer line profiles.

## 5.2 $\delta$ Scuti stars. Computed using Kurucz's ATLAS codes (employs **Holtsmark theory**).

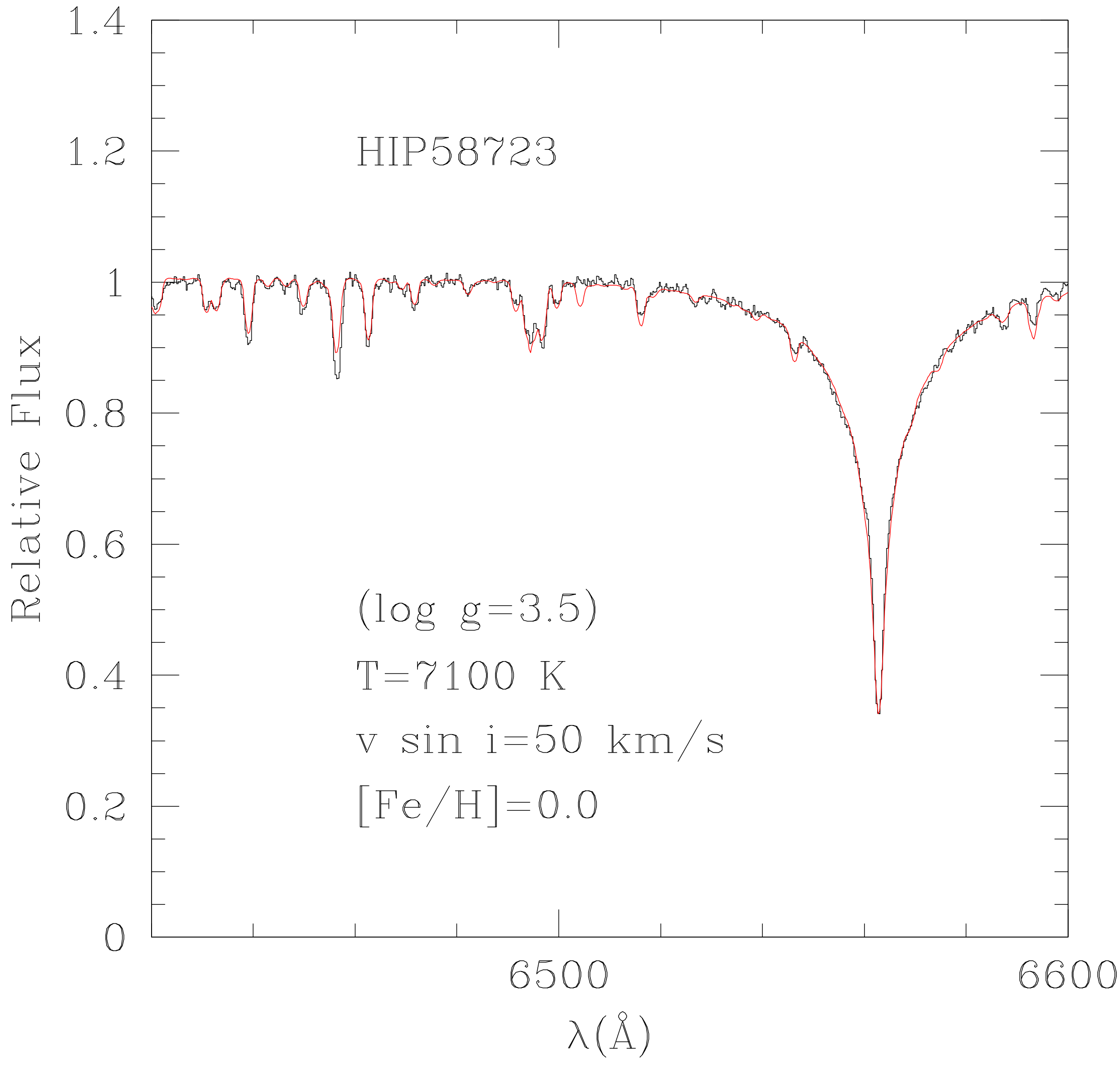



Figure 6: Atmospheric diagnostics for a $\delta$ Scuti star.

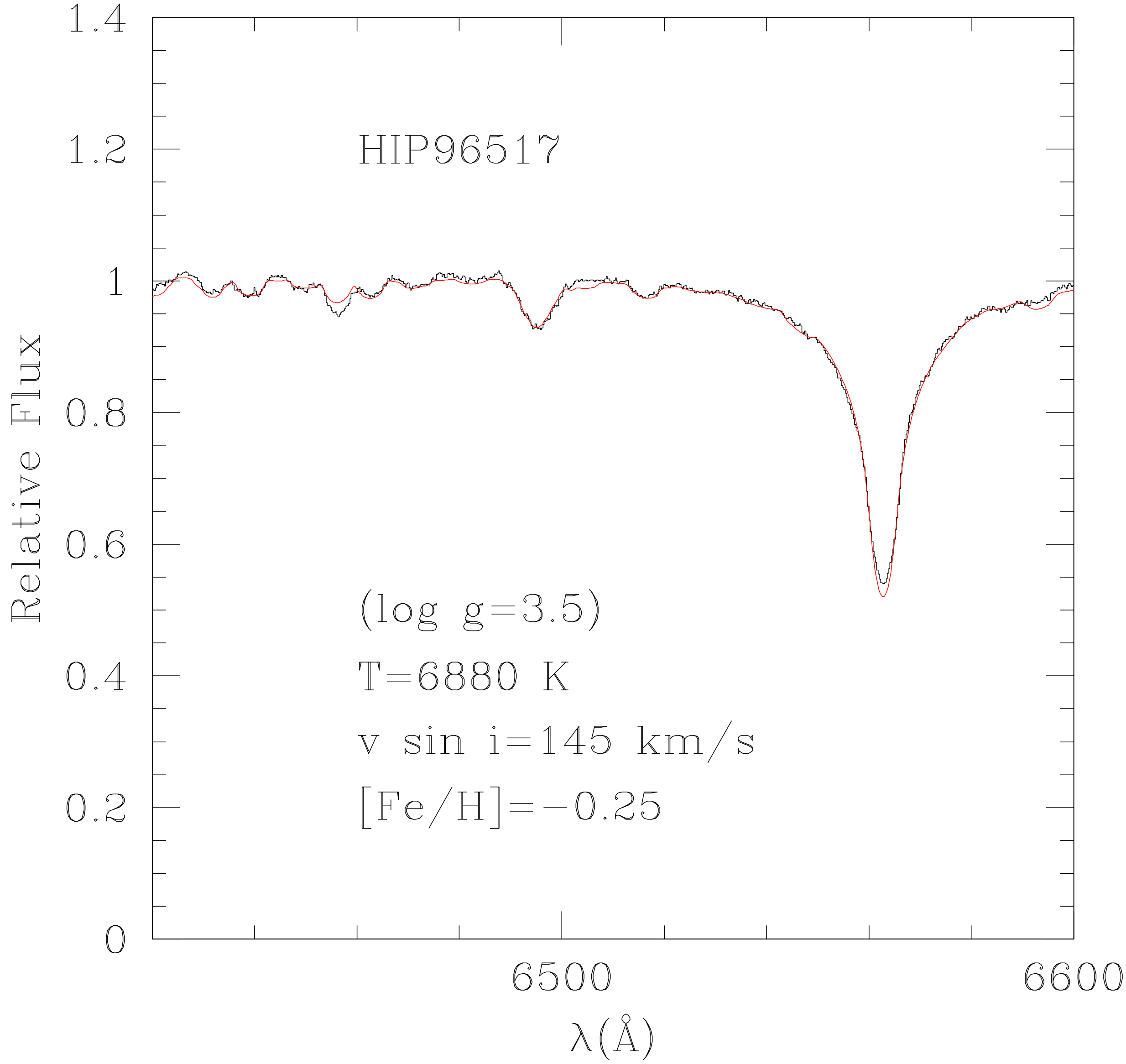



Figure 7: Atmospheric diagnostics for a $\delta$ Scuti star. Note the rotationally broadened line profiles.

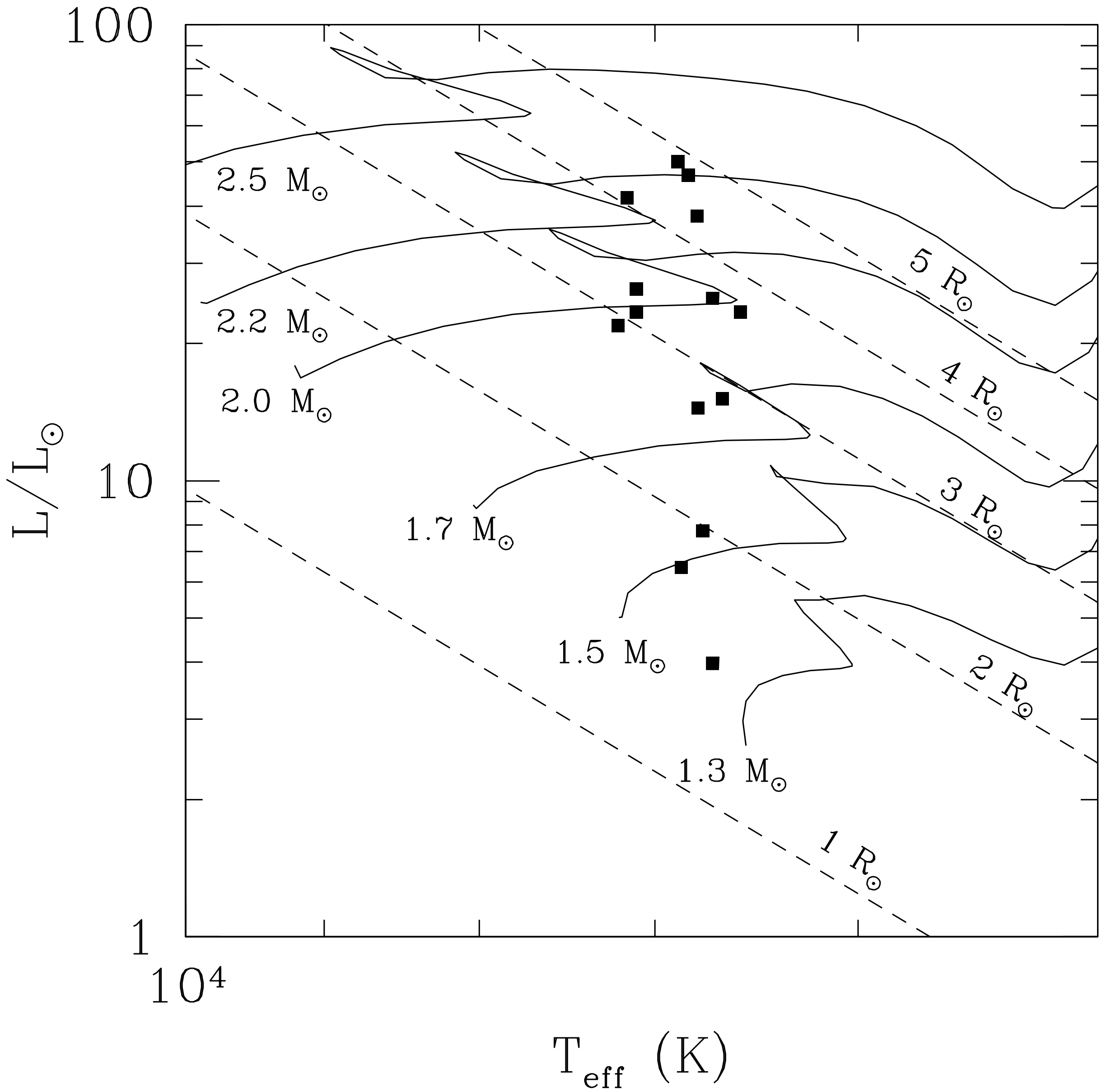



Figure 8: H-R diagram for a sample of variable giants and sub-giants built using line profile diagnostics.