

# On a posteriori error estimation

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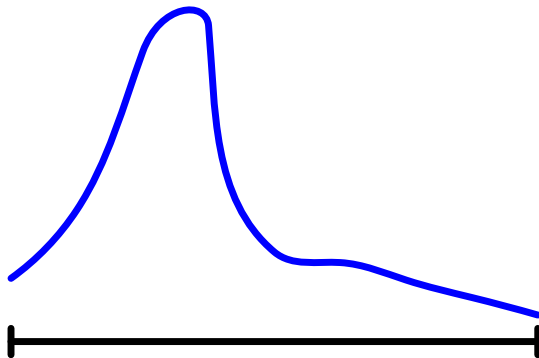


- ▶ Computations of approximate solutions of various mathematical problems
  - ▶ systems of algebraic equations (linear/nonlinear)
  - ▶ ODE
  - ▶ PDE
  - ▶ minimization, optimization
  - ▶ numerical quadrature
  - ▶ etc.
- ▶ Theoretical aspects
  - ▶ well posedness of discretized problems
  - ▶ convergence of numerical methods
  - ▶ qualitative properties of approximate solutions (DMP)
  - ▶ error estimates (a priori / a posteriori)
  - ▶ etc.
- ▶ Goal: Solve the problem
  - ▶ with given accuracy
  - ▶ efficiently

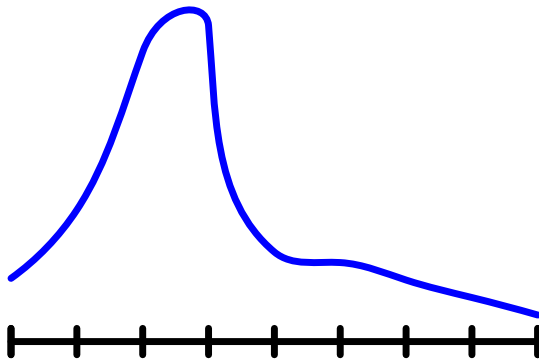


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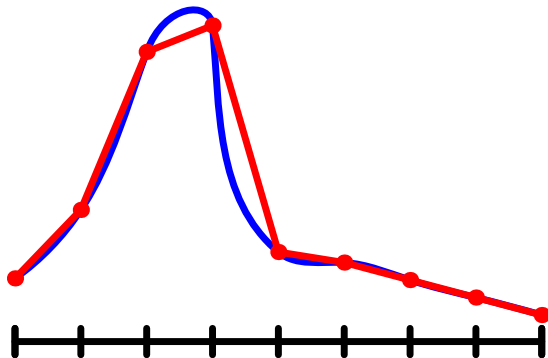
# Numerical PDE



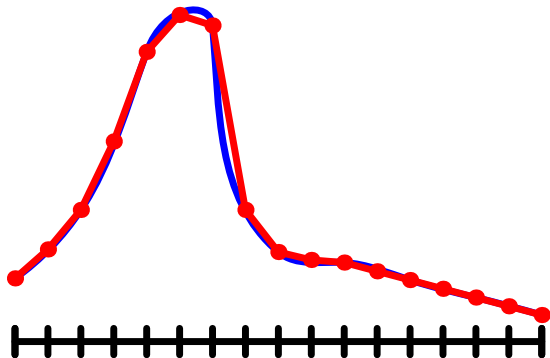
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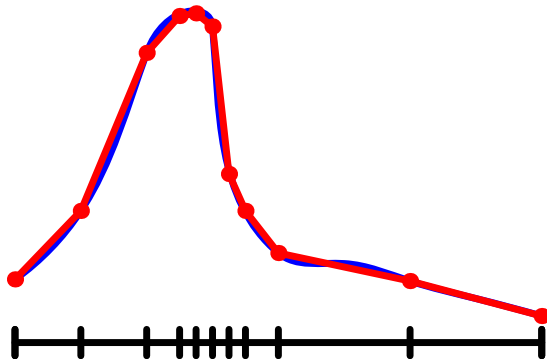
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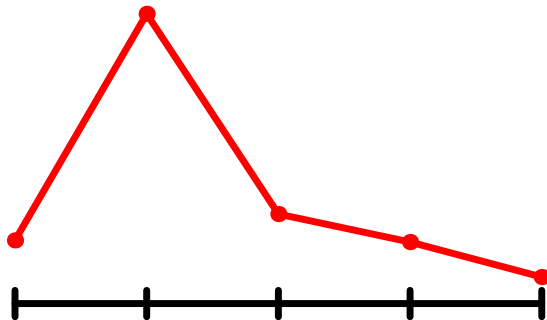
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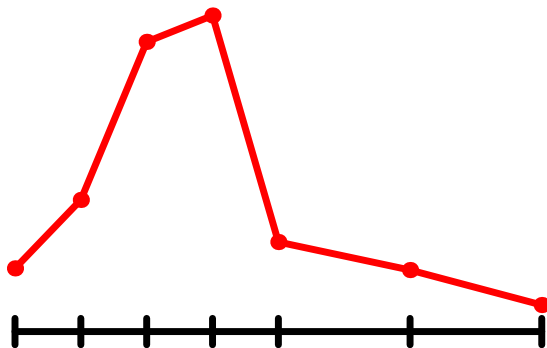
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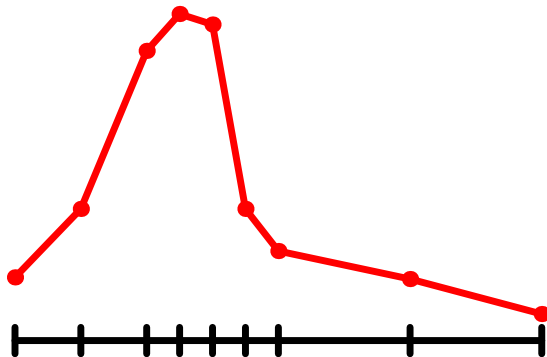
# Numerical PDE – adaptivity



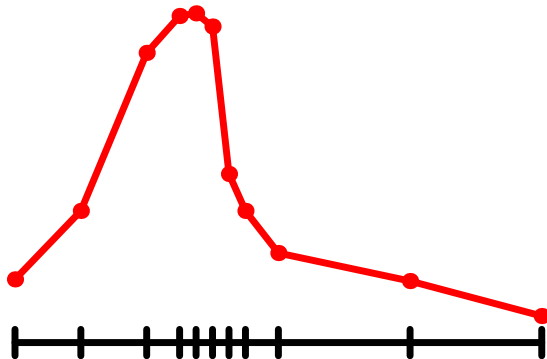
# Numerical PDE – adaptivity



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# Numerical PDE – adaptivity





1. **Initialize:** Construct the initial mesh  $\mathcal{T}_h$ .
2. **Solve:** Find  $u_h$  on  $\mathcal{T}_h$ .
3. **Error indicators:** Compute  $\eta_K$  for all  $K \in \mathcal{T}_h$ .
4. **Error estimator:**  $\eta^2 = \sum_{K \in \mathcal{T}_h} \eta_K^2$ .
5. **Stopping criterion:** If  $\eta \leq \text{TOL} \Rightarrow \text{STOP}$ .
6. **Mark:** If  $\eta_K \geq \Theta \max_{K \in \mathcal{T}_h} \eta_K \Rightarrow \text{mark } K$ .  $0 < \Theta < 1$
7. **Refine:** Refine marked elements and build the new mesh  $\mathcal{T}_h$ .
8. GO TO 2.



- ▶ Automatic hp-adaptivity in L-shape domain  
mesherror.avi
  - ▶ error reduction
  - ▶ singularity in the re-entrant corner
  
- ▶ Automatic hp-adaptivity in a waveguide  
waveguide\_sol.avi
  - ▶ square waveguide
  - ▶ circular load (different permittivity)
  - ▶ sinusoidal current in the left edge
  - ▶ time-harmonic Maxwell's equations
  - ▶ 32 steps of the adaptive process



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**Remark:** Guaranteed upper bound:  $\|u - u_h\| \leq \eta \leq \text{TOL}$

**Remark:** Nonlocal error





# Model Problem

Strong form.: 
$$\begin{aligned} -\Delta u &= f & \text{in } \Omega \subset \mathbb{R}^d \\ u &= 0 & \text{on } \partial\Omega \end{aligned}$$

Weak form.:  $u \in H_0^1(\Omega) : (\nabla u, \nabla v) = (f, v) \quad \forall v \in H_0^1(\Omega)$

Notation:

- ▶  $(\mathbf{p}, \mathbf{q}) = \int_{\Omega} \mathbf{p} \cdot \mathbf{q} \, dx$
- ▶  $\|v\|^2 = (\nabla v, \nabla v)$

Error:  $e = u - u_h$



# A posteriori error estimates

## Definition

- ▶  $\eta \approx \|e\|$  (or  $\|e\| \leq \eta$ , or  $\eta \leq \|e\|$ )
- ▶  $\eta = \eta(u_h, f, \Omega, \mathcal{T}_h, \dots)$

## Properties

- ▶ Efficiency and reliability:  $C_1 \eta \leq \|e\| \leq C_2 \eta$
- ▶ Guaranteed upper/lower bound:  $\|e\| \leq \eta$  /  $\eta \leq \|e\|$
- ▶ Asymptotic exactness:  $\lim_{h \rightarrow 0} l_{\text{eff}} = 1$ ,  $l_{\text{eff}} = \frac{\eta}{\|e\|}$
- ▶ Robustness:  
     $C_1$  and  $C_2$  are independent from quantities like  
    coefficients in the equation, mesh aspect ratio etc.
- ▶ Locality:  $\eta^2 = \sum_{K \in \mathcal{T}_h} \eta_K^2$



# A posteriori error estimates – types

- ▶ Explicit residual
- ▶ Implicit residual – Dirichlet type
- ▶ Implicit residual – Neumann type
- ▶ Hierarchical
- ▶ Based on postprocessing
- ▶ Complementarity
- ▶ Quantity of interest,  $\eta \approx |J(u) - J(u_h)|$

# Estimates based on complementarity

Divergence theorem:  $v \in H^1(\Omega)$   $\mathbf{y} \in \mathbf{H}(\text{div}, \Omega)$

$$\int_{\Omega} v \operatorname{div} \mathbf{y} \, dx + \int_{\Omega} \mathbf{y} \cdot \nabla v \, dx - \int_{\partial\Omega} v \mathbf{y} \cdot \mathbf{n} \, dx = 0$$

Friedrichs inequality:

$$\|v\|_0 \leq C_F \|\nabla v\|_0 \quad \forall v \in H_0^1(\Omega)$$

Derivation:  $v \in H_0^1(\Omega)$ ,  $u_h \in H_0^1(\Omega)$ ,  $\mathbf{y} \in \mathbf{H}(\text{div}, \Omega)$

$$\begin{aligned} (\nabla u - \nabla u_h, \nabla v) &= (f, v) - (\nabla u_h, \nabla v) + (v, \operatorname{div} \mathbf{y}) + (\mathbf{y}, \nabla v) \\ &= (f + \operatorname{div} \mathbf{y}, v) + (\mathbf{y} - \nabla u_h, \nabla v) \\ &\leq \|f + \operatorname{div} \mathbf{y}\|_0 \|v\|_0 + \|\mathbf{y} - \nabla u_h\|_0 \|\nabla v\|_0 \\ &\leq (C_F \|f + \operatorname{div} \mathbf{y}\|_0 + \|\mathbf{y} - \nabla u_h\|_0) \|\nabla v\|_0 \end{aligned}$$

$$\|u - u_h\| \leq C_F \|f + \operatorname{div} \mathbf{y}\|_0 + \|\mathbf{y} - \nabla u_h\|_0$$



# The estimator

**Definition:**  $\eta(u_h, \mathbf{y}_h) = C_F \|f + \operatorname{div} \mathbf{y}_h\|_0 + \|\mathbf{y}_h - \nabla u_h\|_0$

**Theorem:**  $\|u - u_h\| \leq \eta(u_h, \mathbf{y}_h) \quad \forall u_h \in V \quad \forall \mathbf{y}_h \in \mathbf{H}(\operatorname{div}, \Omega)$



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**Complementary problem:**

Find  $\mathbf{y} \in \mathbf{H}(\operatorname{div}, \Omega) : \eta(u_h, \mathbf{y}) \leq \eta(u_h, \mathbf{w}) \quad \forall \mathbf{w} \in \mathbf{H}(\operatorname{div}, \Omega)$

- ▶  $\exists!$  solution
- ▶  $\mathbf{y} = \nabla u$  and  $\|u - u_h\| = \eta(u_h, \mathbf{y})$
- ▶  $C_F \leq \frac{1}{\pi} \left( \frac{1}{|a_1|} + \dots + \frac{1}{|a_d|} \right)^{-1/2}, \quad \Omega \subset a_1 \times \dots \times a_d$



# The estimator

**Definition:**  $\eta(u_h, \mathbf{y}_h) = C_F \|f + \operatorname{div} \mathbf{y}_h\|_0 + \|\mathbf{y}_h - \nabla u_h\|_0$

**Theorem:**  $\|u - u_h\| \leq \eta(u_h, \mathbf{y}_h) \quad \forall u_h \in V \quad \forall \mathbf{y}_h \in \mathbf{H}(\operatorname{div}, \Omega)$

**Special case:**

▶  $\|u - u_h\| \leq \tilde{\eta}(u_h, \mathbf{y}_h) = \|\mathbf{y}_h - \nabla u_h\|_0 \quad \forall \mathbf{y}_h \in \mathbf{Q}(f)$

▶  $\mathbf{Q}(f) = \{\mathbf{y} \in \mathbf{H}(\operatorname{div}, \Omega) : (\mathbf{y}, \nabla v) = (f, v) \quad \forall v \in H_0^1(\Omega)\}$

▶  $\tilde{\eta}^2(u, \mathbf{y}_h) + \tilde{\eta}^2(u_h, \mathbf{y}) = \tilde{\eta}^2(u_h, \mathbf{y}_h)$

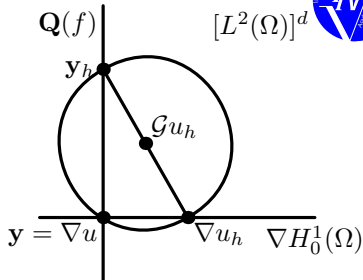
$$\|\mathbf{y}_h - \mathbf{y}\|_0^2 + \|\nabla u - \nabla u_h\|_0^2 = \|\mathbf{y}_h - \nabla u_h\|_0^2$$

# Method of hypercircle



Theorem: If

- ▶  $u \in H_0^1(\Omega)$  is primal solution
- ▶  $u_h \in H_0^1(\Omega)$ ,  $\mathbf{y}_h \in \mathbf{Q}(f)$  arbitrary
- ▶  $\mathcal{G}u_h = (\mathbf{y}_h + \nabla u_h)/2$



Then

$$\|\nabla u - \mathcal{G}u_h\|_0 = \frac{1}{2} \tilde{\eta}(u_h, \mathbf{y}_h).$$

Proof:

$$\begin{aligned} 4 \|\nabla u - \mathcal{G}u_h\|_0^2 &= \|\nabla u - \mathbf{y}_h + \nabla u - \nabla u_h\|_0^2 \\ &= \|\nabla u - \mathbf{y}_h\|_0^2 + \|\nabla u - \nabla u_h\|_0^2 = \|\nabla u_h - \mathbf{y}_h\|_0^2 \end{aligned}$$





## Recent result

M. Ainsworth and T. Vejchodský: *Fully computable robust a posteriori error bounds for singularly perturbed reaction–diffusion problems*, submitted to Numer. Math., 2010.

$$\begin{aligned} -\Delta u + \kappa^2 u &= f \quad \text{in } \Omega \subset \mathbb{R}^2 \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned}$$





### Definition

$$\eta_K(\mathbf{y}_K)^2 = \|\mathbf{y}_K - \nabla u_h\|_{0,K}^2 + \kappa^{-2} \|\Pi_K f - \kappa^2 u_h + \operatorname{div} \mathbf{y}_K\|_{0,K}^2$$

$\mathbf{y}_K^*$  given by an explicit formula







### Theorem

$$\begin{aligned} \|e\|^2 &\leq \sum_{K \in \mathcal{T}_h} \left[ \eta_K(\mathbf{y}_K^*) + \min(h_K/\pi, \kappa^{-1}) \|f - \Pi_K f\|_{0,K} \right]^2 \\ \eta_K(\mathbf{y}_K^*) &\leq C \left[ \|e\|_{\tilde{K}} + \min(h_K, \kappa^{-1}) \|f - \Pi f\|_{0,\tilde{K}} \right] \end{aligned}$$

-  T. Vejchodský: Complementary error bounds for elliptic systems and applications, submitted to Appl. Math. Comput., 2010. (Preprint 232.)
-  T. Vejchodský: Complementarity - the way towards guaranteed error estimates, in: Programs and Algorithms of Numerical Mathematics 15, Institute of Mathematics, Prague, 2010, pp. 205–220. (Preprint 231.)
-  M. Ainsworth, T. Vejchodský: Fully computable robust a posteriori error bounds for singularly perturbed reaction-diffusion problems, submitted to Numer. Math., 2010. (Preprint 208.)
-  T. Vejchodský: Complementarity based a posteriori error estimates and their properties, submitted to Math. Comput. Simulation, 2009. (Preprint 190.)

# Recommended books



-  I. Babuška, J.R. Whiteman, T. Strouboulis, Finite elements: an introduction to the method and error estimation, Oxford University Press, Oxford, 2011.
-  M. Ainsworth, J.T. Oden, A posteriori error estimation in finite element analysis, Wiley, New York, 2000.
-  I. Babuška, T. Strouboulis, The finite element method and its reliability, Clarendon Press, Oxford University Press, New York, 2001.
-  P. Neittaanmäki, S. Repin, Reliable methods for computer simulation, error control and a posteriori estimates, Elsevier, Amsterdam, 2004.
-  S. Repin, A posteriori estimates for partial differential equations, de Gruyter, Berlin, 2008.
-  R. Verfürth, A review of a posteriori error estimation and adaptive mesh-refinement techniques., Wiley-Teubner, Chichester/Stuttgart, 1996.

Thank you for your attention

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