On the Continuous Dependence on a Parameter of Solutions of IVP's for Linear GDE's

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Abstract. In the contribution the continuous dependence of solutions to linear generalized differential equations (*GDE's*) of the form

$$x(t) = x(0) + \int_0^t \mathrm{d}[A_k(s)]x(s), \quad t \in [0,1]$$

on a parameter $k \in \mathbf{N}$ is discussed.

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1 . Introduction

Throughout the paper $\mathbf{N}$ stands for the set of positive integers. Furthermore, $\mathbf{R}^{n\times m}$ denotes the space of real $n \times m$-matrices, $\mathbf{R}^n = \mathbf{R}^{n\times 1}$, $\mathbf{R}^1 = \mathbf{R}$. For a given $n \times m$-matrix $A \in \mathbf{R}^{n\times m}$, by $|A|$ we denote its norm,

$$|A| = \max_{i=1,\ldots,n} \sum_{j=1}^{m} |a_{i,j}|,$$

and det A is its determinant. The symbols I and 0 stand respectively for the identity and the zero matrix of the proper type.

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As usual, by $[0,1]$ and $(0,1)$ we denote the corresponding closed and open intervals, respectively. Furthermore, $[0,1)$ and $(0,1]$ are the corresponding half-open intervals.

The space of all functions $F:[0,1]\to\mathbf{R}^{n\times m}$ of bounded variation on $[0,1]$ is denoted by $\mathbf{BV}^{n\times m}$. It is well known that $\mathbf{BV}^{n\times m}$ equipped with the norm

$$F\in\mathbf{BV}^{n\times m}\to\|F\|_{\mathbf{BV}}=|F(0)|+\mathrm{var}_0^1F$$

is a Banach space. For a given $F\in\mathbf{BV}^{n\times m}$, we denote

$$\begin{gathered}F(t-)=\lim_{\tau\to t-}F(\tau)\ \text{ and }\ \Delta^-F(t)=F(t)-F(t-)\ \text{ for }\ t\in(0,1],\\ F(t+)=\lim_{\tau\to t+}F(\tau)\ \text{ and }\ \Delta^+F(t)=F(t+)-F(t)\ \text{ for }\ t\in[0,1),\\ F(0-)=F(0),\ \Delta^-F(0)=0,\ F(1+)=F(1),\ \Delta^+F(1)=0.\end{gathered}$$

As usual, the space of $n\times m$-matrix valued functions continuous on $[0,1]$ is denoted by $\mathbf{C}^{n\times m}$ and the space of $n\times m$-matrix valued functions Lebesgue integrable on $[0,1]$ is denoted by $\mathbf{L}_1^{n\times m}$. Instead of $\mathbf{BV}^{n\times 1}$ or $\mathbf{C}^{n\times 1}$ or $\mathbf{L}_1^{n\times 1}$ we write $\mathbf{BV}^n$ or $\mathbf{C}^n$ or $\mathbf{L}_1^n$, respectively. For given $F\in\mathbf{L}_1^{n\times m}$ and $G\in\mathbf{C}^{n\times m}$, we denote

$$\|F\|_{\mathbf{L}_1}=\int_0^1|F(t)|\mathrm{d}t\quad\text{and}\quad\|G\|=\sup_{t\in[0,1]}|G(t)|.$$

The integrals are considered in the *Perron-Stieltjes* sense. We work with the equivalent summation definition due to J. Kurzweil (cf. [5]) which is now usually called the *Kurzweil - Henstock integral* or the *gauge integral*.

Let $P_k\in\mathbf{L}_1^{n\times n}$ for $k\in\mathbf{N}\cup\{0\}$ and let $X_k\in\mathbf{AC}^{n\times n}$ be the corresponding *fundamental matrices*, i.e.

$$X_k(t)=\mathrm{I}+\int_0^t P_k(s)X_k(s)\mathrm{d}s\quad\text{on}\quad[0,1]\quad\text{for}\quad k\in\mathbf{N}\cup\{0\}.$$

The following two assertions are relatively representative examples of theorems on the continuous dependence of solutions of ordinary differential equations on a parameter.

Theorem 1.1. *If*

$$\lim_{k\to\infty}\int_0^1|P_k(s)-P_0(s)|\mathrm{d}s=0,$$

then

$$\lim_{k\to\infty}X_k(t)=X_0(t)\quad\textit{uniformly on}\quad[0,1].$$

Theorem 1.2. (Kurzweil & Vorel, [6]) *Let there exist* $m \in \mathbf{L}_1^1$ *such that*

$$|P_k(t)| \le m(t) \quad \text{a.e. on} \quad [0,1] \quad \text{for all} \quad k \in \mathbf{N} \tag{1.1}$$

and let

$$\lim_{k\to\infty} \int_0^t P_k(s)\mathrm{d}s = \int_0^t P_0(s)\mathrm{d}s \quad \text{uniformly on} \quad [0,1]. \tag{1.2}$$

Then

$$\lim_{k\to\infty} X_k(t) = X_0(t) \quad \text{uniformly on} \quad [0,1].$$

Remark 1.3. For $t \in [0,1]$ and $k \in \mathbf{N} \cup \{0\}$ denote

$$A_k(t) = \int_0^t P_k(s)\mathrm{d}s.$$

Then the assumptions of Theorem 1.2 may be reformulated for A_k as follows:

$$A_k \in \mathbf{AC}^{n\times n} \quad \text{for all} \quad k \in \mathbf{N} \cup \{0\}, \tag{1.3}$$

$$\sup_{k\in\mathbf{N}} \|A_k'\|_{\mathbf{L}_1} < \infty, \tag{1.4}$$

$$\lim_{k\to\infty} A_k(t) = A_0(t) \quad \text{uniformly on} \quad [0,1]. \tag{1.5}$$

Besides, the assumption (1.1) means that there exists a nondecreasing function $h_0 \in \mathbf{AC}$ such that

$$|A_k(t_2) - A_k(t_1)| \le |h_0(t_2) - h_0(t_1)| \quad \text{for all} \quad t_1, t_2 \in [0,1].$$

In fact, we may put

$$h_0(t) = \int_0^t m(s)\mathrm{d}s \quad \text{on} \quad [0,1].$$

2 . Linear GDE's - a survey of known results

The following basic existence result for linear generalized differential equations of the form

$$x(t) = \widetilde{x} + \int_0^t \mathrm{d}[A(s)]x(s), \quad t \in [0,1]$$

may be found e.g. in [9] (cf. Theorem III.1.4) or in [8] (cf. Theorem 6.13).

Theorem 2.1. *Let $A \in \mathbf{BV}^{n\times n}$ be such that*

$$\det\left[\mathrm{I} - \Delta^- A(t)\right] \neq 0 \quad \textit{for all} \ \ t \in (0,1]. \tag{2.1}$$

Then there exists a unique $X \in \mathbf{BV}^{n\times n}$ such that

$$X(t) = \mathrm{I} + \int_0^t \mathrm{d}[A(s)]X(s) \quad \textit{on} \ \ [0,1]. \tag{2.2}$$

Definition 2.2. For a given $A \in \mathbf{BV}^{n\times n}$, the $n \times n$-matrix valued function $X \in \mathbf{BV}^{n\times n}$ such that (2.2) holds is called the *fundamental matrix corresponding to A.*

When restricted to the linear case, Theorem 8.8 from [8] modifies to

Theorem 2.3. *Let $A_0 \in \mathbf{BV}^{n\times n}$ satisfy (2.1) and let X_0 be the corresponding fundamental matrix. Let $A_k \in \mathbf{BV}^{n\times n}$, $k \in \mathbf{N}$, and scalar nondecreasing and left-continuous on $(0,1]$ functions h_k, $k \in \mathbf{N} \cup \{0\}$, be given such that h_0 is continuous on $[0,1]$ and*

$$\lim_{k\to\infty} A_k(t) = A_0(t) \quad \textit{on} \ \ [0,1], \tag{2.3}$$

$$|A_k(t_2) - A_k(t_1)| \leq |h_k(t_2) - h_k(t_1)| \qquad \textit{for all} \ \ t_1, t_2 \in [0,1] \ \textit{and} \ \ k \in \mathbf{N} \cup \{0\}, \tag{2.4}$$

$$\limsup_{k\to\infty} \left[h_k(t_2) - h_k(t_1)\right] \leq h_0(t_2) - h_0(t_1) \qquad \textit{whenever} \ \ 0 \leq t_1 \leq t_2 \leq 1. \tag{2.5}$$

Then for any $k \in \mathbf{N}$ sufficiently large there exists a fundamental matrix X_k corresponding to A_k and

$$\lim_{k\to\infty} X_k(t) = X_0(t) \quad \textit{uniformly on} \ \ [0,1].$$

Lemma 2.4. *Under the assumptions of Theorem 2.3 we have*

$$\sup_{k\in\mathbf{N}} \mathrm{var}_0^1 A_k < \infty \tag{2.6}$$

and

$$\lim_{k\to\infty} \left[A_k(t) - A_k(0)\right] = A_0(t) - A_0(0) \ \ \textit{uniformly on} \ \ [0,1]. \tag{2.7}$$

Proof. [1] i) By (2.5) there is $k_0 \in \mathbf{N}$ such that

$$h_k(1) - h_k(0) \le h_0(1) - h_0(0) + 1 \quad \text{for all} \quad k \ge k_0.$$

Hence for any $k \in \mathbf{N}$ we have

$$\operatorname{var}_0^1 A_k \le \alpha_0 = \max\Big(\big\{\operatorname{var}_0^1 A_k; k \le k_0\big\} \cup \big\{h_0(1) - h_0(0) + 1\big\}\Big) < \infty.$$

Thus we conclude that (2.6) is true.

ii) Suppose that

$$\lim_{k\to\infty} A_k(t) = A_0(t) \quad \text{uniformly on} \quad [0,1] \tag{2.8}$$

is not valid. Then there is $\widetilde{\varepsilon} > 0$ such that for any $\ell \in \mathbf{N}$ there exist $m_\ell \ge \ell$ and $t_\ell \in [0,1]$ such that

$$|A_{m_\ell}(t_\ell) - A_0(t_\ell)| \ge \widetilde{\varepsilon}. \tag{2.9}$$

We may assume that $m_{\ell+1} > m_\ell$ for any $\ell \in \mathbf{N}$ and

$$\lim_{\ell\to\infty} t_\ell = t_0 \in [0,1]. \tag{2.10}$$

Let $t_0 \in (0,1)$ and let an arbitrary $\varepsilon > 0$ be given. Since h_0 is continuous, we may choose $\eta > 0$ in such a way that $t_0 - \eta, t_0 + \eta \in [0,1]$ and

$$h_0(t_0 + \eta) - h_0(t_0 - \eta) < \varepsilon. \tag{2.11}$$

Furthermore, by (2.3) there is $\ell_1 \in \mathbf{N}$ such that

$$|A_{m_\ell}(t_0) - A_0(t_0)| < \varepsilon \quad \text{for all} \quad \ell \ge \ell_1 \tag{2.12}$$

and by (2.4), (2.5) and (2.11) there is $\ell_2 \in \mathbf{N}$, $\ell_2 \ge \ell_1$, such that

$$|A_{m_\ell}(\tau_2) - A_{m_\ell}(\tau_1)| \le h_0(t_0 + \eta) - h_0(t_0 - \eta) + \varepsilon < 2\varepsilon \quad \text{whenever} \quad \tau_1, \tau_2 \in (t_0 - \eta, t_0 + \eta) \text{ and } \ell \ge \ell_2. \tag{2.13}$$

The relations (2.3) and (2.13) imply immediately that

$$|A_0(\tau_2) - A_0(\tau_1)| = \lim_{\ell\to\infty} |A_{m_\ell}(\tau_2) - A_{m_\ell}(\tau_1)| \le 2\varepsilon \quad \text{whenever} \quad \tau_1, \tau_2 \in (t_0 - \eta, t_0 + \eta). \tag{2.14}$$

[1] The author is indebted to Ivo Vrkoč for his suggestions which led to a considerable simplification of this proof.

Finally, let $\ell_3 \in \mathbf{N}$ be such that $\ell_3 \geq \ell_2$ and

$$|t_\ell - t_0| < \eta \quad \text{for all} \quad \ell \geq \ell_3, \tag{2.15}$$

then in virtue of the relations (2.10)–(2.15) we have

$$\begin{aligned}
&|A_{m_\ell}(t_\ell) - A_0(t_\ell)| \\
&\qquad \leq |A_{m_\ell}(t_\ell) - A_{m_\ell}(t_0)| + |A_{m_\ell}(t_0) - A_0(t_0)| + |A_0(t_0) - A_0(t_\ell)| \\
&\qquad \leq 5\varepsilon.
\end{aligned}$$

Hence, choosing $\varepsilon < \frac{1}{5}\widetilde{\varepsilon}$, we obtain by (2.9) that

$$\widetilde{\varepsilon} > |A_{m_\ell}(t_\ell) - A_0(t_\ell)| \geq \widetilde{\varepsilon}.$$

This being impossible, the relation (2.8) has to be true. The modification of the proof in the cases $t_0 = 0$ or $t_0 = 1$ and the extension of (2.8) to (2.7) is obvious. □

Thus, Theorem 2.3 is a special case of the following result due to M. Ashordia (cf.[1]).

Theorem 2.5. *Let $A_0 \in \mathbf{BV}^{n\times n}$ satisfy (2.1), let X_0 be the corresponding fundamental matrix and let $\{A_k\}_{k=1}^\infty \subset \mathbf{BV}^{n\times n}$ be such that (2.6) and (2.7) hold. Then for any $k \in \mathbf{N}$ sufficiently large there exists a fundamental matrix X_k corresponding to A_k and*

$$\lim_{k\to\infty} X_k(t) = X_0(t) \quad \textit{uniformly on} \quad [0,1].$$

Remark 2.6. Under the assumptions of Theorem 2.5 we obviously have

$$\lim_{k\to\infty} A_k(t-) = A_0(t-) \text{ and } \lim_{k\to\infty} A_k(s+) = A_0(s+)$$

for all $t \in (0,1]$ and all $s \in [0,1)$, respectively. Thus Theorem 2.5 cannot cover the case that there is a $t_0 \in (0,1]$ such that

$$A_k(t_0-) = A_k(t_0) \quad \text{for all} \quad k \in \mathbf{N}, \quad \text{while} \quad A_0(t_0-) \neq A_0(t_0).$$

In particular, Theorem 2.5 does not apply to the following simple example.

Example 2.7. Consider the sequence of initial value problems

$$x_k' = a_k'(t)x_k \quad \text{on} \quad [-1,1], \quad x(-1) = \widetilde{x},$$

where

$$a_k(t) = \begin{cases} 0 & \text{if } t \le \alpha_k, \\ \frac{t-\alpha_k}{\beta_k-\alpha_k} & \text{if } t \in (\alpha_k, \beta_k), \\ 1 & \text{if } t \ge \beta_k; \end{cases}$$

$\{\alpha_k\}_{k=1}^{\infty}$ is an arbitrary increasing sequence in $[-1, 0)$ such that

$$\lim_{k\to\infty} \alpha_k = 0;$$

$\{\beta_k\}_{k=1}^{\infty}$ is an arbitrary decreasing sequence in $(0, 1]$ such that

$$\lim_{k\to\infty} \beta_k = 0$$

and

$$\lim_{k\to\infty} \frac{\alpha_k}{\alpha_k - \beta_k} = \varkappa \in [0, 1).$$

For the corresponding solutions we have

$$x_k(t) = \begin{cases} \widetilde{x} & \text{if } \ t \le \alpha_k, \\ \mathrm{e}^{\frac{t-\alpha_k}{\beta_k-\alpha_k}}\, \widetilde{x} & \text{if } \ t \in (\alpha_k, \beta_k), \\ \mathrm{e}\, \widetilde{x} & \text{if } \ t \ge \beta_k \end{cases}$$

$$x_0(t) = \lim_{k\to\infty} x_k(t) = \begin{cases} \widetilde{x} & \text{if } \ t < 0, \\ \mathrm{e}^{\varkappa}\, \widetilde{x} & \text{if } \ t = 0, \\ \mathrm{e}\, \widetilde{x} & \text{if } \ t > 0, \end{cases}$$

while the unique solution $x(t)$ of the "limit" equation

$$x(t) = \widetilde{x} + \int_{-1}^{t} \mathrm{d}[a(s)]x(s), \quad t \in [-1, 1],$$

where

$$a(t) = \lim_{k\to\infty} a_k(t) = \begin{cases} 0 & \text{if } \ t < 0, \\ \varkappa & \text{if } \ t = 0, \\ 1 & \text{if } \ t > 0, \end{cases}$$

is given by

$$x(t) = \left\{ \begin{array}{cl} \widetilde{x} & \text{if } \ t < 0 \\ \frac{1}{1-\varkappa}\, \widetilde{x} & \text{if } \ t = 0 \\ \frac{2-\varkappa}{1-\varkappa}\, \widetilde{x} & \text{if } \ t > 0 \end{array} \right\} \ne x_0(t).$$

On the other hand, x_0 is a solution to

$$x_0(t) = \widetilde{x} + \int_{-1}^{t} \mathrm{d}[a_0(t)]x_0(s) \quad \text{on} \quad [-1, 1],$$

where

$$a_0(t) = \begin{cases} 0 & \text{if } t < 0, \\ 1 - \mathrm{e}^{-\varkappa} & \text{if } t = 0, \\ (\mathrm{e} - 1)\,\mathrm{e}^{-\varkappa} & \text{if } t > 0 \end{cases}$$

and a_k tends to a_0 in the following sense:

(a) *given arbitrary* $\alpha \in (-1,0)$ *and* $\beta \in (0,1)$, $\lim_{k\to\infty} a_k(t) = a_0(t)$ *uniformly on* $[-1,\alpha]$ *and* $\lim_{k\to\infty}[a_k(t) - a_k(\beta)] = a_0(t) - a_0(\beta)$ *uniformly on* $[\beta,1]$;

(b) $\lim_{k\to\infty} a_k(t) = a_0(t) + \widetilde{a_0}(t)$, *where*

$$\widetilde{a_0}(t) = \begin{cases} 0 & \textit{if } t < 0, \\ \varkappa + \mathrm{e}^{-\varkappa} - 1 & \textit{if } t = 0, \\ 1 - \mathrm{e}^{1-\varkappa} + \mathrm{e}^{-\varkappa} & \textit{if } t > 0; \end{cases}$$

(c) *for any* $z \in \mathbf{R}$ *and* $\varepsilon > 0$, *there is* $\delta > 0$ *such that for any* $\delta' \in (0,\delta)$ *there is* $k_0 \in \mathbf{N}$ *such that for any* $k \geq k_0$ *we have* $\alpha_k \geq -\delta'$, $\beta_k \leq \delta'$ *and the relations*

$$\left| y_k(0) - y_k(-\delta') - \frac{\Delta^- a_0(0) z}{1 - \Delta^- a_0(0)} \right| < \varepsilon$$

and

$$|z_k(\delta') - z_k(0) - \Delta^+ a_0(0) z| < \varepsilon$$

are satisfied for any solution y_k *on* $[-\delta',0]$ *of*

$$y_k' = a_k'(t) y_k \quad \textit{with} \;\; y_k(-\delta') \in (z-\delta, z+\delta)$$

and any solution z_k *on* $[0,\delta']$ *of*

$$z_k' = a_k'(t) z_k \quad \textit{with} \;\; z_k(0) \in (z-\delta, z+\delta).$$

In fact, for given $z \in \mathbf{R}$, $\delta' > 0$ and $k \in \mathbf{N}$ such that $\alpha_k \geq -\delta'$ we have

$$y_k(t) = \mathrm{e}^{\frac{t-\alpha_k}{\beta_k-\alpha_k}}\, y_k(-\delta') \quad \text{on} \;\; [\alpha_k, 0]$$

and thus

$$\begin{aligned} &\left| y_k(0) - y_k(-\delta') - \frac{\Delta^- a_0(0) z}{1 - \Delta^- a_0(0)} \right| \\ &\quad = \left| \left(\mathrm{e}^{\frac{-\alpha_k}{\beta_k-\alpha_k}} - 1\right) y_k(-\delta') - \left(\mathrm{e}^{\varkappa} - 1\right) z \right| \\ &\quad \leq \left| \mathrm{e}^{\frac{-\alpha_k}{\beta_k-\alpha_k}} - \mathrm{e}^{\varkappa} \right| |z| + \left| \mathrm{e}^{\frac{-\alpha_k}{\beta_k-\alpha_k}} - 1 \right| \left| z - y_k(-\delta') \right|, \end{aligned}$$

where

$$\lim_{k\to\infty}\left|\mathrm{e}^{\frac{-\alpha_k}{\beta_k-\alpha_k}}-\mathrm{e}^{\varkappa}\right|=0,\quad \left|\mathrm{e}^{\frac{-\alpha_k}{\beta_k-\alpha_k}}-1\right|\le 2$$

and

$$\left|z-y_k(-\delta')\right|\le\delta.$$

Analogously, if $k\in\mathbf{N}$ is such that $\beta_k\le\delta'$, we have

$$z_k(t)=\mathrm{e}^{\frac{\beta_k}{\beta_k-\alpha_k}}z_k(0)\quad\text{on}\quad[0,\delta']$$

and thus

$$\begin{aligned}&\left|z_k(\delta')-z_k(0)-\Delta^+a_0(0)z\right|\\&\quad=\left|\left(\mathrm{e}^{\frac{\beta_k}{\beta_k-\alpha_k}}-1\right)z_k(-\delta')-\left(\mathrm{e}^{1-\varkappa}-1\right)z\right|\\&\quad\le\left|\mathrm{e}^{\frac{\beta_k}{\beta_k-\alpha_k}}-\mathrm{e}^{1-\varkappa}\right||z|+\left|\mathrm{e}^{\frac{\beta_k}{\beta_k-\alpha_k}}-1\right|\left|z-z_k(0)\right|,\end{aligned}$$

where

$$\lim_{k\to\infty}\left|\mathrm{e}^{\frac{\beta_k}{\beta_k-\alpha_k}}-\mathrm{e}^{1-\varkappa}\right|=0,\quad\left|\mathrm{e}^{\frac{\beta_k}{\beta_k-\alpha_k}}-1\right|\le 2$$

and

$$\left|z-z_k(0)\right|\le\delta.$$

Notice that if

$$x_0(t)=\widetilde{x}+\int_{-1}^{t}\mathrm{d}[a_0(t)]x_0(s)\quad\text{on}\quad[-1,1],$$

then

$$\Delta^-x_0(0)=\Big(\frac{1}{1-\Delta^-a_0(0)}-1\Big)x_0(0-)=\frac{\Delta^-a_0(0)}{1-\Delta^-a_0(0)}x_0(0-).$$

The convergence described in Example 2.7 is closely related to the notion of the *emphatic convergence* introduced by J. Kurzweil (cf. [5]).

Definition 2.8. A sequence $\{A_k\}_{k=1}^{\infty}\subset\mathbf{BV}^{n\times n}$ *converges emphatically to* $A_0\in\mathbf{BV}^{n\times n}$ on $[0,1]$ if

(i) there exist nondecreasing functions $h_k:[0,1]\to\mathbf{R}$, $k\in\mathbf{N}\cup\{0\}$, which are left-continuous on $(0,1]$ and such that

$$|A_k(t_2)-A_k(t_1)|\le|h_k(t_2)-h_k(t_1)|$$

for all $k\in\mathbf{N}\cup\{0\}$ and $t_1,t_2\in[0,1]$;

(ii) $\limsup_{k\to\infty} \big[h_k(t_2)-h_k(t_1)\big] \le \big[h_0(t_2)-h_0(t_1)\big]$ whenever $0 \le t_1 \le t_2 \le 1$ and h_0 is continuous at t_1 and t_2;

(iii) there is $\widetilde{A}_0 \in \mathbf{BV}^{n\times n}$ such that $\lim_{k\to\infty} A_k(t) = A_0(t)+\widetilde{A}_0(t)$ whenever $h_0(t) = h_0(t+)$ and $|\widetilde{A}_0(t_2) - \widetilde{A}_0(t_1)| \le |\widetilde{h}_0(t_2) - \widetilde{h}_0(t_1)|$ for all $t_1, t_2 \in [0,1]$, where $\widetilde{h}_0$ stands for the break part of h_0;

(iv) if $h_0(t_0+) > h_0(t_0)$, then for any $z \in \mathbf{R}^n$ and any $\varepsilon > 0$ there exists $\delta > 0$ such that for any $\delta' \in (0,\delta)$ there is $k_0 \in \mathbf{N}$ such that

$$|y_k(t_0+\delta') - y_k(t_0-\delta') - \Delta^+ A_0(t_0)z| \le \varepsilon$$

holds for any $k \ge k_0$, any $\widetilde{y}_k \in \mathbf{R}^n$ such that $|z - \widetilde{y}_k| \le \delta$ and any solution y_k of the equation

$$y_k(t) = \widetilde{y}_k + \int_{t_0-\delta'}^{t} \mathrm{d}[A_k(s)]y_k(s) \quad \text{on} \quad [t_0-\delta', t_0+\delta'].$$

The following assertion is a restriction of Theorem 4.1 from [5] to the linear case.

Theorem 2.9. *Let A_k converge emphatically on $[0,1]$ to A_0. Let the sequence $\{X_k\}_{k=1}^{\infty} \subset \mathbf{BV}^{n\times n}$ of the fundamental matrices corresponding respectively to A_k, $k \in \mathbf{N}$, be uniformly bounded on $[0,1]$ and such that*

$$\lim_{k\to\infty} X_k(t) = Z_0(t) \quad \textit{on} \quad [0,1] \quad \textit{whenever} \quad h_0(t+) = h_0(t).$$

Then $Z_0 \in \mathbf{BV}^{n\times n}$ and the function X_0 defined by

$$X_0(t) = \begin{cases} Z_0(t) & \textit{if} \quad h_0(t+) = h_0(t), \\ Z_0(t-) & \textit{otherwise} \end{cases}$$

is the fundamental matrix corresponding to A_0.

Remark 2.10. Let us notice that necessary and sufficient conditions assuring the uniform convergence of fundamental matrices X_k corresponding to A_k, $k \in \mathbf{N}$, to the fundamental matrix X_0 corresponding to A_0 may be found in the paper [2] by M. Ashordia.

Results related to Theorem 2.9 obtained by the method of "*prolongation*" of functions of bounded variation to continuous functions along monotone functions and using the concept of *convergence under substitution* instead of the emphatic convergence were obtained by D. Fraňková in [3] (cf. also [4]), as well.

3 . Linear GDE's - new results

Notation 3.1. For a given function $F \in \mathbf{BV}^{n\times n}$, the symbol $\mathbf{S}(F)$ stands for the set of the points of discontinuity of F in $[0,1]$, while

$$\mathbf{S}^+(F) = \{t \in [0,1); \Delta^+ F(t) \neq 0\} \text{ and } \mathbf{S}^-(F) = \{t \in [0,1); \Delta^- F(t) \neq 0\}.$$

If F is such that $\mathbf{S}(F)$ possesses at most a finite number of points, then for an arbitrary compact set M such that

$$M = \bigcup_{j=1}^{m} [\alpha_j, \beta_j] \subset [0,1] \setminus \mathbf{S}(F)$$

with $[\alpha_j, \beta_j] \cap [\alpha_k, \beta_k] = \emptyset$ for $j \neq k$, we define

$$F^M(t) = F(t) - F(\alpha_j) \quad \text{if} \ \ t \in [\alpha_j, \beta_j].$$

Provided the set $\mathbf{S}(A_0)$ contains at most a finite number of elements, we can extend Theorem 2.9 to the case that the functions A_k, $k \in \mathbf{N} \cup \{0\}$, need not be left-continuous on $(0,1]$ in the following way.

Theorem 3.2. *Let* $A_0 \in \mathbf{BV}^{n\times n}$, $\mathbf{S}(A_0) = \{\tau_j\}_{j=1}^m$,

$$\det\big[\mathrm{I} - \Delta^- A_0(t)\big] \neq 0 \quad \textit{on} \quad [0,1]$$

and let X_0 *be the fundamental matrix solution corresponding to* A_0. *Let the sequence* $\{A_k\}_{k=1}^\infty \subset \mathbf{BV}^{n\times n}$ *be such that*

(i) $\sup_k \operatorname{var}_0^1 A_k < \infty$ *and* $\det\big[\mathrm{I} - \Delta^- A_k(t)\big] \neq 0$ *on* $(0,1]$ *for all* $k \in \mathbf{N}$*;*

(ii) $\lim_{k\to\infty} A_k^M(s) = A_0^M(s)$ *uniformly on* M *for any* $M \subset [0,1] \setminus \mathbf{S}(A_0)$ *such that* $M = \bigcup_{j=1}^m [\alpha_j, \beta_j]$, *where* $[\alpha_j, \beta_j] \cap [\alpha_k, \beta_k] = \emptyset$ *for* $j \neq k$*;*

(iii) *if* $\tau \in \mathbf{S}(A_0)$ *then for any* $z \in \mathbf{R}^n$ *and any* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that for any* $\delta' \in (0,\delta)$ *there is* $k_0 \in \mathbf{N}$ *such that the relations*

$$\Big| y_k(\tau) - y_k(\tau - \delta') - \Delta^- A_0(\tau)\big[\mathrm{I} - \Delta^- A_0(\tau)\big]^{-1} z \Big| \le \varepsilon$$

and

$$\big| z_k(\tau + \delta') - z_k(\tau) - \Delta^+ A_0(\tau) z \big| \le \varepsilon$$

are satisfied for any $k \ge k_0$ *and* y_k *and* z_k *such that*

$$\begin{aligned} y_k(t) &= y_k(\tau-\delta') &+ \textstyle\int_{\tau-\delta'}^t \mathrm{d}[A_k(s)]y_k(s) \quad &\textit{on} \ \ [\tau-\delta',\tau], \\ z_k(t) &= z_k(\tau) &+ \textstyle\int_{\tau}^t \mathrm{d}[A_k(s)]z_k(s) \quad &\textit{on} \ \ [\tau,\tau+\delta'] \end{aligned}$$

and

$$|z-y_k(\tau-\delta')|\le\delta \quad \textit{and} \quad |z-z_k(\tau)|\le\delta.$$

Then for any $k\in\mathbf{N}$ sufficiently large the fundamental matrix X_k corresponding to A_k is defined on $[0,1]$ and

$$\lim_{k\to\infty} X_k(t)=X_0(t) \quad \textit{on} \ \ [0,1].$$

Proof. Let us restrict ourselves to the case that $m=1$, i.e. let $\mathbf{S}(A_0)=\{\tau\}$, where $\tau\in(0,1)$.

Let an arbitrary $\widetilde{x}\in\mathbf{R}^n$ be given and let x_k for any $k\in\mathbf{N}\cup\{0\}$ denote the solution to the equation

$$x_k(t)=\widetilde{x}+\int_0^t \mathrm{d}[A_k(s)]x_k(s) \quad \text{on} \ \ [0,1].$$

Our assumptions (i) and (ii) by Theorem 2.5 imply that for any $\alpha\in(0,\tau)$ we have

$$\lim_{k\to\infty} x_k(t)=x_0(t) \quad \text{uniformly on} \ \ [0,\alpha]. \tag{3.1}$$

Consequently,

$$\lim_{k\to\infty} x_k(t)=x_0(t) \quad \text{for all} \ \ t\in[0,\tau). \tag{3.2}$$

Furthermore, for any $\delta'\in(0,\tau)$ and $k\in\mathbf{N}$ we have

$$\begin{aligned} &|x_0(\tau)-x_k(\tau)| \\ &\le \left|x_0(\tau)-x_0(\tau-\delta')-\Delta^-A_0(\tau)\left[\mathrm{I}-\Delta^-A_0(\tau)\right]^{-1}x_0(\tau-)\right| \\ &\quad+\left|\Delta^-A_0(\tau)\left[\mathrm{I}-\Delta^-A_0(\tau)\right]^{-1}x_0(\tau-)-\left(x_k(\tau)-x_k(\tau-\delta')\right)\right| \\ &\quad+|x_0(\tau-\delta')-x_k(\tau-\delta')|. \end{aligned} \tag{3.3}$$

Let an arbitrary $\varepsilon>0$ be given. By the assumption (iii) there exists $\delta\in(0,\varepsilon)$ such that for all $\delta'\in(0,\delta)$ there exists $k_1=k_1(\delta')\in\mathbf{N}$ such that for any $k\ge k_1$ and for any solution y_k of the equation

$$y_k(t)=y_k(\tau-\delta')+\int_{\tau-\delta'}^t \mathrm{d}[A_k(s)]y_k(s) \quad \text{on} \ \ [\tau-\delta',\tau]$$

such that $|y_k(\tau-\delta')-x_0(\tau-)|<\delta$ we have

$$\Big|y_k(\tau)-y_k(\tau-\delta')-\Delta^-A_0(\tau)\big[\mathrm{I}-\Delta^-A_0(\tau)\big]^{-1}x_0(\tau-)\Big|<\varepsilon. \tag{3.4}$$

Let us choose $\delta'\in(0,\delta)$ in such a way that

$$|x_0(\tau-)-x(\tau-\delta')|<\frac{\delta}{2} \tag{3.5}$$

is true. Furthermore, according to (3.2) there is $k_0\in\mathbf{N}$ such that $k_0\ge k_1$ and

$$|x_0(\tau-\delta')-x_k(\tau-\delta')|<\frac{\delta}{2}\quad\text{for all}\quad k\ge k_0. \tag{3.6}$$

In particular, for $k\ge k_0$ we have

$$|x_0(\tau-)-x_k(\tau-\delta')|<\delta. \tag{3.7}$$

Thus, if we put $y_k(t)=x_k(t)$ on $[\tau-\delta',\tau]$, then the relation (3.4) will be satisfied for any $k\ge k_0$, i.e. we have

$$\Big|x_k(\tau)-x_k(\tau-\delta')-\Delta^-A_0(\tau)\big[\mathrm{I}-\Delta^-A_0(\tau)\big]^{-1}x_0(\tau-)\Big|<\varepsilon \tag{3.8}$$

for all $k\ge k_0$. Now, inserting (3.6)-(3.8) into (3.3), we obtain that

$$|x_k(\tau)-x_0(\tau)|<\frac{\delta}{2}+\frac{\delta}{2}+\varepsilon<2\varepsilon$$

is satisfied for any $k\ge k_0$, i.e.

$$\lim_{k\to\infty}x_k(\tau)=x_0(\tau). \tag{3.9}$$

Further, we will prove that there is $\eta>0$ such that

$$\lim_{k\to\infty}x_k(t)=x_0(t)$$

is true on $(\tau,\tau+\eta)$ as well. To this aim, let $\varepsilon>0$ be given and let $\eta_0\in(0,\varepsilon)$ be such that

$$|x_0(s)-x_0(\tau+)|<\varepsilon\quad\text{for all}\quad s\in(\tau,\tau+\eta_0). \tag{3.10}$$

By the assumption (iii) there exists $\eta\in(0,\eta_0)$ such that for any $\eta'\in(0,\eta)$ there is $\ell_1=\ell_1(\eta')\in\mathbf{N}$ such that for any $k\ge\ell_1$ and for any solution z_k of the equation

$$z_k(t)=z_k(\tau)+\int_\tau^t \mathrm{d}[A_k(s)]z_k(s)\ \text{on}\ \ [\tau,\tau+\eta']$$

such that $|z_k(\tau) - x_0(\tau)| < \eta$ we have

$$\big|z_k(\tau+\eta') - z_k(\tau) - \Delta^+ A_0(\tau) x_0(\tau)\big| < \varepsilon. \tag{3.11}$$

Let us choose $\eta' \in (0,\eta)$ arbitrarily. By (3.10), we have

$$|x_0(\tau-\eta') - x_0(\tau+)| < \varepsilon. \tag{3.12}$$

Furthermore, by (3.9) there is $\ell_0 \in \mathbf{N}$ such that $\ell_0 \geq \ell_1$ and

$$|x_k(\tau) - x_0(\tau)| < \eta \quad \text{for all} \quad k \geq \ell_0. \tag{3.13}$$

Thus, by (3.11), for any $k \geq \ell_0$ we have

$$\big|x_k(\tau+\eta') - x_k(\tau) - \Delta^+ A_0(\tau) x_0(\tau)\big| < \varepsilon. \tag{3.14}$$

Making use of (3.12)-(3.14) we finally get for any $k \geq k_0$

$$\begin{aligned} &|x_k(\tau+\eta') - x_0(\tau+\eta')| \\ &\quad \leq \big|x_k(\tau+\eta') - x_k(\tau) - x_0(\tau+) + x_0(\tau)\big| \\ &\qquad + \big|x_0(\tau+\eta') - x_0(\tau+)\big| + \big|x_k(\tau) - x_0(\tau)\big| \\ &\quad = \big|x_k(\tau+\eta') - x_k(\tau) - \Delta^+ A_0(\tau) x_0(\tau)\big| \\ &\qquad + \big|x_0(\tau+) - x_0(\tau+\eta')\big| + \big|x_k(\tau) - x_0(\tau)\big| < 3\varepsilon, \end{aligned}$$

i.e.

$$\lim_{k\to\infty} x_k(t) = x_0(t) \quad \text{for all} \quad t \in (\tau, \tau+\eta).$$

The proof of the theorem can be completed by making use of Theorem 2.5 and taking into account that $\widetilde{x} \in \mathbf{R}^n$ was chosen arbitrarily. The extension to a general case $m \in \mathbf{N}$ is obvious. □

Remark 3.3. Obviously, if we did not restrict ourselves to the case of only a finite number of discontinuities of A_0, we should replace the assumptions (i)-(ii) in Theorem 3.2 by assumptions of the form (i)-(ii) from Definition 2.8.

Remark 3.4. The following concept due to M. Pelant (cf. [7]) leads to another interesting convergence effect which most probably cannot be explained by Theorem 3.2.

Let $A \in \mathbf{BV}^{n\times n}$ and let the divisions $\mathcal{P}_k = \{0 = t_0^k < \cdots < t_{p_k}^k = 1\}$, $k \in \mathbf{N}$, of $[0,1]$ be such that

$$\begin{aligned} \mathcal{P}_k \supset \mathcal{D}_k \quad = \quad & \{t \in [0,1]; t = \frac{i}{2^k}, i = 0,1,...2^k\} \\ & \cup \{t \in (0,1]; |\Delta^- A(t)| \geq \frac{1}{k}\} \\ & \cup \{t \in [0.1); |\Delta^+ A(t)| \geq \frac{1}{k}\}. \end{aligned}$$

For a given $k \in \mathbf{N}$, let us put

$$A_k(t) = \begin{cases} A(t) & \text{if } \ t \in \mathcal{P}_k, \\ A(t_{i-1}^k) + \frac{A(t_i^k)-A(t_{i-1}^k)}{t_i^k - t_{i-1}^k}\,(t - t_{i-1}^k) & \\ & \text{if } \ t \in (t_{i-1}^k, t_i^k). \end{cases}$$

Then we say that the sequence $\{A_k, \mathcal{P}_k\}_{k=1}^{\infty}$ *piecewise linearly approximates* A.

Furthermore, for a given $A \in \mathbf{BV}^{n\times n}$, let us define A_0 on $[0,1]$ by

$$\begin{aligned} A_0(t) = A(t) &- \sum_{s\in \mathbf{S}^-(A)} \Delta^- A(s)\chi_{[s,1]}(t) \\ &- \sum_{s\in \mathbf{S}^+(A)} \Delta^+ A(s)\chi_{(s,1]}(t) \\ &+ \sum_{s\in \mathbf{S}^-(A)} \Big(\mathrm{I} - \big[\exp\big(\Delta^- A(s)\big)\big]^{-1}\Big)\chi_{[s,1]}(t) \\ &+ \sum_{s\in \mathbf{S}^+(A)} \Big(\exp\big(\Delta^+ A(s)\big) - \mathrm{I}\Big)\chi_{(s,1]}(t). \end{aligned} \tag{3.15}$$

Then, obviously

$$\det\big[\mathrm{I} - \Delta^- A_0(t)\big] \neq 0 \quad \text{on} \quad [0,1]$$

holds and the following assertion may be proved (cf. [7]).

Let $A \in \mathbf{BV}^{n\times n}$, *let* A_0 *be given by* (3.15), *let* $\{A_k, \mathcal{P}_k\}_{k=1}^{\infty}$ *piecewise linearly approximate* A *and let for a given* $k \in \mathbf{N}$, X_k *denote the fundamental matrix corresponding to* A_k. *Then*

$$\lim_{k\to\infty} X_k(t) = X_0(t) \quad \textit{for all} \ \ t \in [0,1].$$

Furthermore, if $A \in \mathbf{BV}^{n\times n}$ is such that the relations

$$\det\big[\mathrm{I} - \Delta^- A(t)\big] \neq 0 \quad \text{and} \quad \det\big[\mathrm{I} + \Delta^+ A(t)\big] \neq 0 \quad \text{on} \quad [0,1] \tag{3.16}$$

are true, then for $t \in [0,1]$ we can define

$$A_0^*(t) = A(t) - \sum_{s\in\mathbf{S}^-(A)} \Delta^- A(s)\chi_{[s,1]}(t) - \sum_{s\in\mathbf{S}^+(A)} \Delta^+ A(s)\chi_{(s,1]}(t) + \sum_{s\in\mathbf{S}^-(A)} \ln\left[\mathrm{I} - \Delta^- A(s)\right]^{-1}\chi_{[s,1]}(t) + \sum_{s\in\mathbf{S}^+(A)} \ln\left[\mathrm{I} + \Delta^+ A(s)\right]\chi_{(s,1]}(t) \tag{3.17}$$

and the following assertion is an immediate corollary of the above mentioned result of M. Pelant.

Theorem 3.5. *Let $A \in \mathbf{BV}^{n\times n}$ be such that (3.16) holds and let X be the fundamental matrix corresponding to A. Let A_0^* be given by (3.17), let $\{A_k, \mathcal{P}_k\}_{k=1}^\infty$ piecewise linearly approximate A_0^** and let for a given $k \in \mathbf{N}$, X_k denote the fundamental matrix corresponding to A_k. Then

$$\lim_{k\to\infty} X_k(t) = X(t) \quad \textit{for all} \quad t \in [0,1].$$

4 . Appendix (2010)

When restricted to the linear case, Theorem 8.2 from [8] modifies to

Theorem 4.1. *Let $A_k \in \mathbf{BV}^{n\times n}$, $k \in \mathbf{N}\cup\{0\}$, and a nondecreasing function $h : [0,1] \to \mathbf{R}$ be given such that*

$$\lim_{k\to\infty} A_k(t) = A_0(t) \quad \textit{on} \quad [0,1], \tag{4.1}$$

$$\left.\begin{aligned} &|A_k(t_2) - A_k(t_1)| \le |h(t_2) - h(t_1)| \\ &\qquad \textit{for } t_1, t_2 \in [0,1] \textit{ and } k \in \mathbf{N}\cup\{0\}. \end{aligned}\right\} \tag{4.2}$$

Let X_k be the fundamental matrix solutions corresponding to A_k for $k \in \mathbf{N}$ and let

$$\lim_{k\to\infty} X_k(t) = X_0(t) \quad \textit{for } t \in [0,1].$$

Then $X_0 \in \mathbf{BV}^{n\times n}$ and X_0 is the fundamental matrix solution corresponding to A_0.

Proposition 4.2. *Under the assumptions of Theorem 4.1 we have*

$$\sup_{k\in\mathbf{N}} \operatorname{var}_0^1 A_k < \infty \tag{4.3}$$

and

$$\lim_{k\to\infty} A_k(t) = A_0(t) \ \textit{uniformly on } [0,1]. \tag{4.4}$$

Proof. i) The relation (4.3) follows immediately from (4.2).

ii) Notice that (4.1) and (4.2) imply that

$$|A_k(t-) - A_k(s)| \le |h(t-) - h(s)| \quad \text{for} \ \ t \in (0,1], s \in [0,1], k \in \mathbf{N}\cup\{0\} \tag{4.5}$$

and

$$|A_k(t+) - A_k(s)| \le |h(t+) - h(s)| \quad \text{for} \ \ t \in [0,1), s \in [0,1], k \in \mathbf{N}\cup\{0\}. \tag{4.6}$$

iii) Let $\varepsilon > 0$ and $t \in (0,1]$ be given and let us choose $s_0 \in (0,t)$ and $k_0 \in \mathbf{N}$ so that

$$|h(t-) - h(s_0)| < \frac{\varepsilon}{3} \quad \text{and} \quad |A_k(s_0) - A_0(s_0)| < \frac{\varepsilon}{3} \quad \text{for } k \ge k_0. \tag{4.7}$$

Then, by (4.5) and (4.7),

$$\begin{aligned}|A_k(t-) - A_0(t-)| &\le |A_k(t-) - A_k(s_0)| + |A_k(s_0) - A_0(s_0)| + |A_0(s_0) - A_k(t-)| \\ &< |h(t-) - h(s_0)| + \frac{\varepsilon}{3} + |h(t-) - h(s_0)| < \varepsilon.\end{aligned}$$

This means that

$$\lim_{k\to\infty} A_k(t-) = A_0(t-) \quad \text{holds for} \ \ t \in (0,1]. \tag{4.8}$$

Similarly, using (4.6) and (4.7), we get

$$\lim_{k\to\infty} A_k(t+) = A_0(t+) \quad \text{holds for} \ \ t \in [0,1). \tag{4.9}$$

iii) Now, suppose that (4.4) is not valid. Then there is $\widetilde{\varepsilon} > 0$ such that for any $\ell \in \mathbf{N}$ there exist $m_\ell \ge \ell$ and $t_\ell \in [0,1]$ such that

$$|A_{m_\ell}(t_\ell) - A_0(t_\ell)| \ge \widetilde{\varepsilon}. \tag{4.10}$$

We may assume that $m_{\ell+1} > m_\ell$ for any $\ell \in \mathbf{N}$ and

$$\lim_{\ell\to\infty} t_\ell = t_0 \in [0,1]. \tag{4.11}$$

Let $t_0 \in (0,1]$ and assume that the set of those $\ell \in \mathbf{N}$ for which $t_\ell \in (0,t_0)$ has infinitely many elements, i.e. there is a sequence $\{\ell_k\}_{k\in\mathbf{N}} \subset \mathbf{N}$ such that $t_{\ell_k} \in (0,t_0)$ for all $k \in \mathbf{N}$ and $\lim_{k\to\infty} t_{\ell_k} = t_0$. Denote $s_k = t_{\ell_k}$ and $B_k = A_{m_{\ell_k}}$ for $k \in \mathbf{N}$. Then, in view of (4.10) we have

$$s_k \in (0,t_0) \quad \text{for } k \in \mathbf{N}, \quad \lim_{k\to\infty} s_k = t_0 \tag{4.12}$$

and

$$|B_k(s_k) - A_0(s_k)| \ge \widetilde{\varepsilon} \quad \text{for } k \in \mathbf{N}. \tag{4.13}$$

By (4.5), we have

$$|A_0(t_0-) - A_0(s_k)| \le h(t_0-) - h(s_k)$$

and

$$|B_k(t_0-) - A_0(s_k)| \le h(t_0-) - h(s_k).$$

Therefore, by (4.8) and since $\lim\limits_{k\to\infty}(h(t_0-) - h(s_k)) = 0$ due to (4.12), we can choose $k_0 \in \mathbf{N}$ so that

$$|A_k(t_0-) - A_0(t_0-)| < \frac{\widetilde{\varepsilon}}{3}$$

$$|A_0(t_0-) - A_0(s_{k_0})| \le h(t_0-) - h(s_{k_0}) < \frac{\widetilde{\varepsilon}}{3}$$

and

$$|B_{k_0}(t_0-) - A_0(s_{k_0})| < \frac{\widetilde{\varepsilon}}{3}.$$

As a consequence, we get finally by (4.13)

$$\begin{aligned}\widetilde{\varepsilon} &\le |B_{k_0}(s_{k_0}) - A_0(s_{k_0})| \\ &\le |B_{k_0}(s_{k_0}) - A_k(t_0-)| + |A_k(t_0-) - A_0(t_0-)| + |A_0(t_0-) - A_0(s_{k_0})| < \widetilde{\varepsilon},\end{aligned}$$

a contradiction.

If $t_0 \in [0,1)$ and the set of those $\ell \in \mathbf{N}$ for which $t_\ell \in (0,t_0)$ has only finitely many elements, then there is a sequence $\{\ell_k\}_{k\in\mathbf{N}} \subset \mathbf{N}$ such that $t_{\ell_k} \in (t_0,1]$ for all $k \in \mathbf{N}$ and $\lim_{k\to\infty} t_{\ell_k} = t_0$. As before, let $s_k = t_{\ell_k}$ and $B_k = A_{m_{\ell_k}}$ for $k \in \mathbf{N}$ and notice that

$$s_k \in (t_0,1) \quad \text{for } k \in \mathbf{N}, \quad \lim_{k\to\infty} s_k = t_0$$

and (4.13) are true. Arguing similarly as before we get that there is $k_0 \in \mathbf{N}$ such that

$$\begin{aligned}\widetilde{\varepsilon} &\le |B_{k_0}(s_{k_0}) - A_0(s_{k_0})| \\ &\le |B_{k_0}(s_{k_0}) - A_k(t_0+)| + |A_k(t_0+) - A_0(t_0+)| + |A_0(t_0+) - A_0(s_{k_0})| < \widetilde{\varepsilon},\end{aligned}$$

a contradiction. □

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