On a posteriori error estimation

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Outline

- Computation with prescribed accuracy
- Adaptive algorithm for PDEs
- A posteriori error estimates
  - General
  - Complementary estimates
- Summary

Numerical computations

Goal: Solve the problem (a) with prescribed accuracy,
(b) efficiently.

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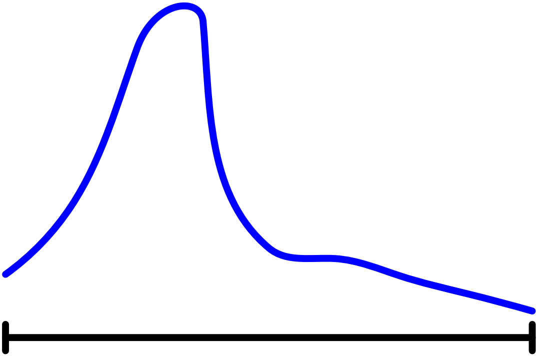
Example 2:

- $I = \int_{-100}^{200} \mathrm{e}^{-|x|} \, \mathrm{d}x$
- Matlab: $I_h \doteq 7.3 \times 10^{-6}$ (absolute error tolerance 10^{-6})

Numerical PDE



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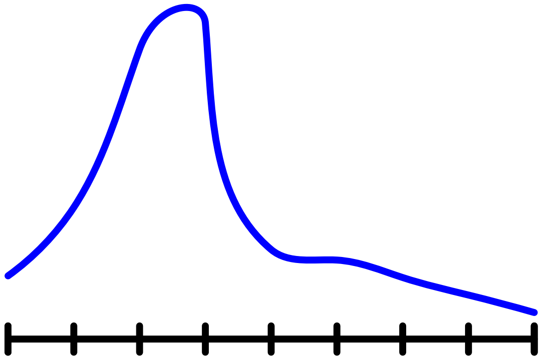

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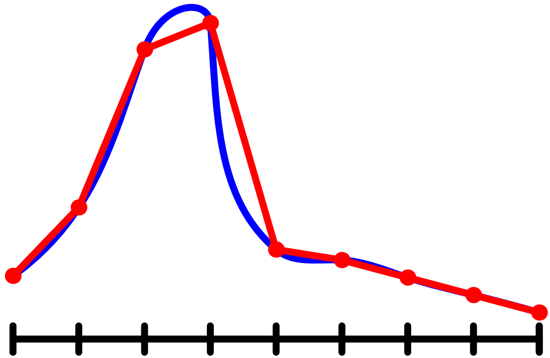
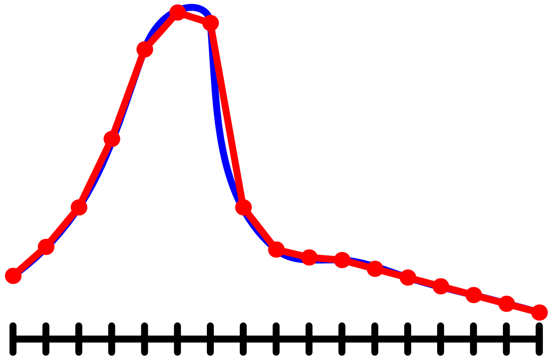


Numerical PDE



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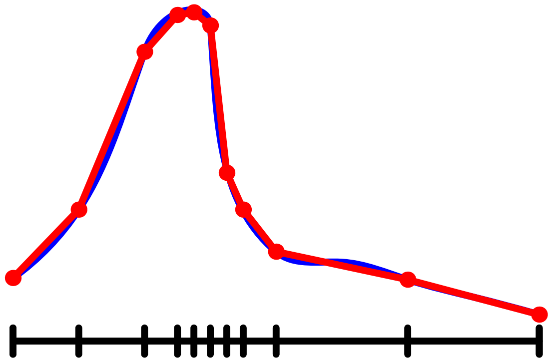
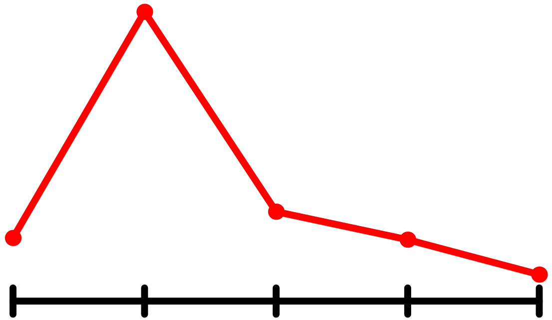


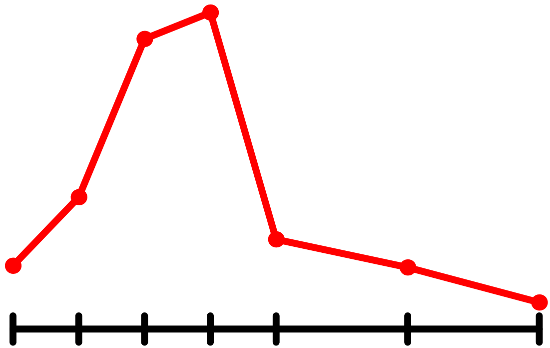
Numerical PDE

?

Numerical PDE – adaptivity



Numerical PDE – adaptivity



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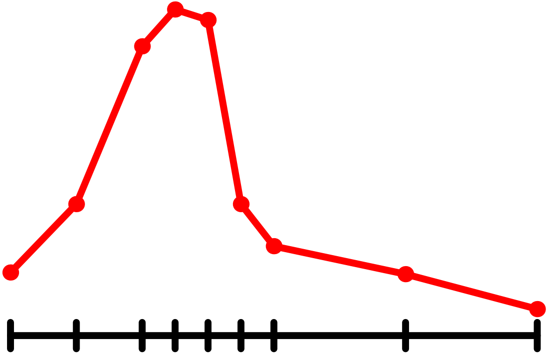
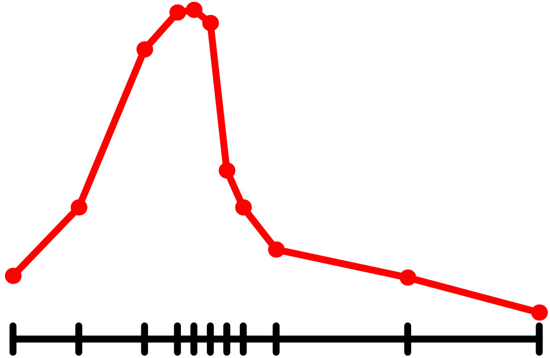


Numerical PDE – adaptivity



Adaptive algorithm

1. Initialize: Construct the initial mesh $\mathcal{T}_h$.
2. Solve: Find u_h on $\mathcal{T}_h$.
3. Error indicators: Compute η_K for all $K \in \mathcal{T}_h$.
4. Error estimator: $\eta^2 = \sum_{K \in \mathcal{T}_h} \eta_K^2$.
5. Stopping criterion: If $\eta \leq \mathrm{TOL} \quad \Rightarrow \quad$ STOP.
6. Mark: If $\eta_K \geq \Theta \max_{K \in \mathcal{T}_h} \eta_K \quad \Rightarrow \quad$ mark K. $\qquad 0 < \Theta < 1$
7. Refine: Refine marked elements and build the new mesh $\mathcal{T}_h$.
8. GO TO 2.

Video – Hermes2D (P. Šolín and col.)

- Automatic hp-adaptivity in L-shape domain
  mesherror.avi
  - error reduction
  - singularity in the re-entrant corner
- Automatic hp-adaptivity in a waveguide
  waveguide_sol.avi
  - square waveguide
  - circular load (different permittivity)
  - sinusoidal current in the left edge
  - time-harmonic Maxwell's equations
  - 32 steps of the adaptive process

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Remark: Guaranteed upper bound: $\|u - u_h\| \leq \eta \leq \mathrm{TOL}$

Remark: Nonlocal error

Model Problem

Strong form.:

$$-\Delta u = f \quad \text{in } \Omega \subset \mathbb{R}^d$$
$$u = 0 \quad \text{on } \partial\Omega$$

Weak form.: $u \in H_0^1(\Omega): \quad (\nabla u, \nabla v) = (f, v) \quad \forall v \in H_0^1(\Omega)$

Notation:

- $(\mathbf{p}, \mathbf{q}) = \int_\Omega \mathbf{p} \cdot \mathbf{q} \, \mathrm{d}x$
- $\|v\|^2 = (\nabla v, \nabla v)$

Error: $e = u - u_h$

A posteriori error estimates

Definition

- $\eta \approx \|e\| \quad (\text{or } \|e\| \leq \eta, \quad \text{or } \eta \leq \|e\|)$
- $\eta = \eta(u_h, f, \Omega, \mathcal{T}_h, \dots)$

Properties

- Efficiency and reliability: $C_1\eta \leq \|e\| \leq C_2\eta$
- Guaranteed upper/lower bound: $\|e\| \leq \eta \quad / \quad \eta \leq \|e\|$
- Asymptotic exactness: $\lim_{h\to 0} I_{\text{eff}} = 1, \quad I_{\text{eff}} = \frac{\eta}{\|e\|}$
- Robustness:
  C_1 and C_2 are independent from quantities like coefficients in the equation, mesh aspect ratio etc.
- Locality: $\eta^2 = \sum_{K\in\mathcal{T}_h} \eta_K^2$

A posteriori error estimates – types

- Explicit residual
- Implicit residual – Dirichlet type
- Implicit residual – Neumann type
- Hierarchical
- Based on postprocessing
- Complementarity
- Quantity of interest, $\eta \approx |J(u) - J(u_h)|$

Estimates based on complementarity

Divergence theorem: $v \in H^1(\Omega) \quad \mathbf{y} \in \mathbf{H}(\mathrm{div}, \Omega)$

$$\int_\Omega v \,\mathrm{div}\, \mathbf{y} \,\mathrm{d}x + \int_\Omega \mathbf{y} \cdot \nabla v \,\mathrm{d}x - \int_{\partial\Omega} v\mathbf{y} \cdot \mathbf{n} \,\mathrm{d}x = 0$$

Friedrichs inequality:

$$\|v\|_0 \le C_F \|\nabla v\|_0 \quad \forall v \in H_0^1(\Omega)$$

Derivation: $v \in H_0^1(\Omega), \quad u_h \in H_0^1(\Omega), \quad \mathbf{y} \in \mathbf{H}(\mathrm{div}, \Omega)$

$$\begin{aligned}
(\nabla u - \nabla u_h, \nabla v) &= (f, v) - (\nabla u_h, \nabla v) + (v, \mathrm{div}\, \mathbf{y}) + (\mathbf{y}, \nabla v) \\
&= (f + \mathrm{div}\, \mathbf{y}, v) + (\mathbf{y} - \nabla u_h, \nabla v) \\
&\le \|f + \mathrm{div}\, \mathbf{y}\|_0 \|v\|_0 + \|\mathbf{y} - \nabla u_h\|_0 \|\nabla v\|_0 \\
&\le \left(C_F \|f + \mathrm{div}\, \mathbf{y}\|_0 + \|\mathbf{y} - \nabla u_h\|_0 \right) \|\nabla v\|_0
\end{aligned}$$

$$|\!|\!| u - u_h |\!|\!| \le C_F \|f + \mathrm{div}\, \mathbf{y}\|_0 + \|\mathbf{y} - \nabla u_h\|_0$$

The estimator

Definition: $\eta(u_h, \mathbf{y}_h) = C_F \, \|f + \operatorname{div} \mathbf{y}_h\|_0 + \|\mathbf{y}_h - \boldsymbol{\nabla} u_h\|_0$

Theorem: $\| u - u_h \| \leq \eta(u_h, \mathbf{y}_h) \qquad \forall u_h \in H_0^1(\Omega) \quad \forall \mathbf{y}_h \in \mathbf{H}(\operatorname{div}, \Omega)$

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Complementary problem:

Find $\mathbf{y} \in \mathbf{H}(\operatorname{div}, \Omega) : \eta(u_h, \mathbf{y}) \leq \eta(u_h, \mathbf{w}) \quad \forall \mathbf{w} \in \mathbf{H}(\operatorname{div}, \Omega)$

- $\exists!$ solution
- $\mathbf{y} = \boldsymbol{\nabla} u$ and $|\!|\!| u - u_h |\!|\!| = \eta(u_h, \mathbf{y})$
- $C_F \leq \frac{1}{\pi} \left(\frac{1}{|a_1|} + \cdots + \frac{1}{|a_d|} \right)^{-1/2}, \quad \Omega \subset a_1 \times \cdots \times a_d$

The estimator

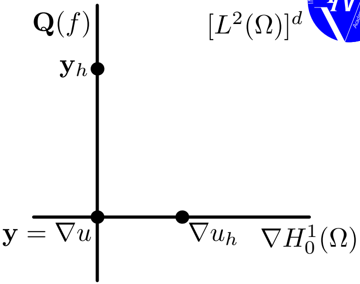
Definition: $\eta(u_h, \mathbf{y}_h) = C_F \|f + \operatorname{div} \mathbf{y}_h\|_0 + \|\mathbf{y}_h - \nabla u_h\|_0$

Theorem: $|\!|\!| u - u_h |\!|\!| \leq \eta(u_h, \mathbf{y}_h) \qquad \forall u_h \in H_0^1(\Omega) \quad \forall \mathbf{y}_h \in \mathbf{H}(\operatorname{div}, \Omega)$

Special case:

- $|\!|\!| u - u_h |\!|\!| \leq \widetilde{\eta}(u_h, \mathbf{y}_h) = \|\mathbf{y}_h - \nabla u_h\|_0 \qquad \forall \mathbf{y}_h \in \mathbf{Q}(f)$
- $\mathbf{Q}(f) = \left\{ \mathbf{y} \in \mathbf{H}(\operatorname{div}, \Omega) : (\mathbf{y}, \nabla v) = (f, v) \quad \forall v \in H_0^1(\Omega) \right\}$
- $\widetilde{\eta}^2(u, \mathbf{y}_h) + \widetilde{\eta}^2(u_h, \mathbf{y}) = \widetilde{\eta}^2(u_h, \mathbf{y}_h)$
  $\|\mathbf{y}_h - \mathbf{y}\|_0^2 + \|\nabla u - \nabla u_h\|_0^2 = \|\mathbf{y}_h - \nabla u_h\|_0^2$

The estimator

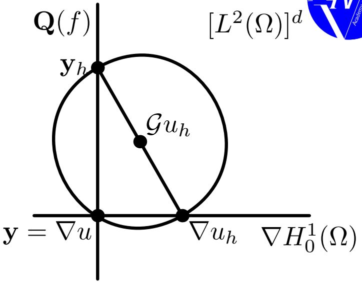


Special case:

- $\widetilde{\eta}^2(u, \mathbf{y}_h) + \widetilde{\eta}^2(u_h, \mathbf{y}) = \widetilde{\eta}^2(u_h, \mathbf{y}_h)$

  $\|\mathbf{y}_h - \mathbf{y}\|_0^2 + \|\boldsymbol{\nabla} u - \boldsymbol{\nabla} u_h\|_0^2 = \|\mathbf{y}_h - \boldsymbol{\nabla} u_h\|_0^2$

Method of hypercircle



Theorem: If

- $u \in H_0^1(\Omega)$ is primal solution
- $u_h \in H_0^1(\Omega)$, $\mathbf{y}_h \in \mathbf{Q}(f)$ arbitrary
- $\mathcal{G}u_h = (\mathbf{y}_h + \nabla u_h)/2$

Then

$$\|\nabla u - \mathcal{G}u_h\|_0 = \frac{1}{2}\tilde{\eta}(u_h, \mathbf{y}_h).$$

Proof:

$$\begin{aligned} 4\,\|\nabla u - \mathcal{G}u_h\|_0^2 &= \|\nabla u - \mathbf{y}_h + \nabla u - \nabla u_h\|_0^2 \\ &= \|\nabla u - \mathbf{y}_h\|_0^2 + \|\nabla u - \nabla u_h\|_0^2 = \|\nabla u_h - \mathbf{y}_h\|_0^2 \end{aligned}$$

Recent result

M. Ainsworth and T. Vejchodský: *Fully computable robust a posteriori error bounds for singularly perturbed reaction–diffusion problems*, accepted by Numer. Math., 2010.

$$
\begin{aligned}
-\Delta u + \kappa^2 u = f \quad & \text{in } \Omega \subset \mathbb{R}^2 \\
u = 0 \quad & \text{on } \partial\Omega
\end{aligned}
$$

Definition

$$
\eta_K(\mathbf{y}_K)^2 = \|\mathbf{y}_K - \nabla u_h\|_{0,K}^2 + \kappa^{-2} \left\| \Pi_K f - \kappa^2 u_h + \operatorname{div} \mathbf{y}_K \right\|_{0,K}^2
$$

$\mathbf{y}_K^*$ given by an explicit formula

Theorem

$$
\|\!|e|\!\|^2 \leq \sum_{K \in \mathcal{T}_h} \left[\eta_K(\mathbf{y}_K^*) + \min(h_K/\pi, \kappa^{-1}) \|f - \Pi_K f\|_{0,K} \right]^2
$$

$$
\eta_K(\mathbf{y}_K^*) \leq C \left[\|\!|e|\!\|_{\widetilde{K}} + \min(h_K, \kappa^{-1}) \|f - \Pi f\|_{0,\widetilde{K}} \right]
$$

Summary

- Adaptive algorithm guaranteed error bounds $\Rightarrow$ efficiency guaranteed accuracy
- Complementary technique $\Rightarrow$ guaranteed error bounds
- Guaranteed error bounds are often complicated and expensive
- But, it is possible to find an error bound that is
  - efficient
  - guaranteed
  - robust
  - local (fast)
  - fully computable

References

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- M. Ainsworth, T. Vejchodský: Fully computable robust a posteriori error bounds for singularly perturbed reaction-diffusion problems, submitted to Numer. Math., 2010. (Preprint 208.)
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Recommended books

- I. Babuška, J.R. Whiteman, T. Strouboulis, Finite elements: an introduction to the method and error estimation, Oxford University Press, Oxford, 2011.
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Thank you for your attention

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