A new reconstruction-enhanced discontinuous Galerkin method for time-dependent problems

Václav Kučera

Faculty of Mathematics and Physics
Charles University Prague

1. Finite volume method with reconstruction
   - Continuous Problem
   - Space semidiscretization

2. Discontinuous Galerkin method with reconstruction
   - Formulation
   - Theoretical results and numerical experiments

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with a Lipschitz boundary $\partial\Omega$.

Continuous Problem

Find $u : Q_T = \Omega \times (0, T) \to \mathbb{R}$ such that

$$\frac{\partial u}{\partial t} + \mathrm{div}\,\mathbf{f}(\mathrm{u}) = 0 \quad \text{in } \mathrm{Q_T},$$

$$u(x, 0) = u^0(x), \quad x \in \Omega,$$

where $\mathbf{f} = (f_1, \cdots, f_d)$ and f_s, $s = 1, \ldots, d$ are Lipschitz-continuous fluxes in the direction x_s, $s = 1, \ldots, d$.

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Let $\mathscr{T}_h$ be a partition of the closure $\overline{\Omega}$ into a finite number of closed triangles $K \in \mathscr{T}_h$.

- By $\mathscr{F}_h$ we denote the set of all edges.
- For each $\Gamma \in \mathscr{F}_h$ we define a unit normal vector $\mathbf{n}_\Gamma$. For each face $\Gamma \in \mathscr{F}_h^I$ there exist two neighbours $K_\Gamma^{(L)}, K_\Gamma^{(R)} \in \mathscr{T}_h$.
- Over $\mathscr{T}_h$ we define the *broken Sobolev space*

$$H^k(\Omega, \mathscr{T}_h) = \{v; v|_K \in H^k(K)\ \forall K \in \mathscr{T}_h\}$$

and for $v \in H^1(\Omega, \mathscr{T}_h)$ and $\Gamma \in \mathscr{F}_h^I$ we set

$$v|_\Gamma^{(L)} = \text{trace of } v|_{K_\Gamma^{(L)}} \text{ on } \Gamma, \quad v|_\Gamma^{(R)} = \text{trace of } v|_{K_\Gamma^{(R)}} \text{ on } \Gamma.$$

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Definition

We define the space of discontinuous piecewise polynomial functions

$$S_h^n = \{v; v|_K \in P_n(K)\ \forall K \in \mathscr{T}_h\},$$

where $P_n(K)$ is the set of all polynomials on K of degree $\leq n$.

- S_h^0 - finite volume space,
- S_h^n - discontinuous Galerkin space,
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We integrate over $K \in \mathscr{T}_h$ and apply Green's theorem

$$\frac{d}{dt}\int_K u(t)\,dx + \int_{\partial K} \mathbf{f}(u)\cdot\mathbf{n}\,dS = 0.$$

We define

$$\bar{u}_K(t) := \frac{1}{|K|}\int_K u(t)\,dx$$

and obtain

$$\frac{d}{dt}\bar{u}_K(t) + \frac{1}{|K|}\int_{\partial K} \mathbf{f}(u)\cdot\mathbf{n}\,dS = 0.$$

We assume, that there exists a piecewise polynomial function $U_h^N(t) \in S_h^N$ such that

$$U_h^N(x,t) = u(x,t) + O(h^{N+1}), \quad \forall x \in \Omega,\ \forall t \in (0,T).$$

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The boundary convective terms will be treated with the aid of a numerical flux $H(u, v, \mathbf{n})$:

$$\int_\Gamma \mathbf{f}(u) \cdot \mathbf{n}\, dS \approx \int_\Gamma H(U_h^{N,(L)}, U_h^{N,(R)}, \mathbf{n})\, dS.$$

Lemma

The averages of the exact solution u satisfy

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Definition (FV reconstruction problem)

Let $v:\Omega\to\mathbb{R}$ be sufficiently regular. Given $\bar{v}_K$ for all $K\in\mathscr{T}_h$, find $v_h^N\in S_h^N$ such that $v-v_h^N=O(h^{N+1})$ in Ω.
We define the corresponding reconstruction operator $R:S_h^0\to S_h^N$ by $R\,\bar{v}:=v_h^N$.

Definition (Reconstructed FV scheme)

We seek $u_h(t)\in S_h^0$ such that

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$$\|u(t) - R\,\bar{u}_h(t)\| = O(h^N),$$

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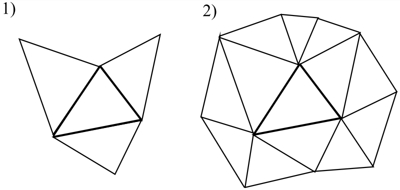
'Standard' FV reconstruction operator



Reconstruction stencil

For each $K \in \mathscr{T}_h$ we choose the *reconstruction stencil* $S_K \subset \mathscr{T}_h$, usually some neighborhood of K.

For each $K \in \mathscr{T}_h$, we seek a polynomial $p_{S_K} \in P^N(S_K)$, s.t.

$$\frac{1}{|K'|}\int_{K'} p_{S_K}\,dx = u_h\big|_{K'} \quad \forall K' \in S_K.$$

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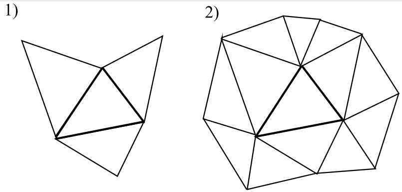

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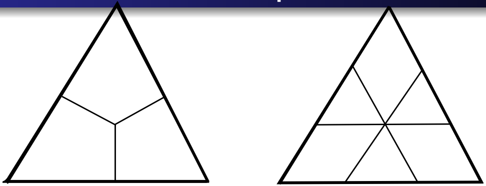

Spectral and *control* volumes

Let $\mathscr{T}_h^S$ be a partition of $\overline{\Omega}$ into simplices $S \in \mathscr{T}_h^S$, called *spectral volumes*. The FV triangulation $\mathscr{T}_h$ is formed by subdividing each $S \in \mathscr{T}_h^S$ into so-called *control volumes* $K \subset S$.

For each spectral volume $S \in \mathscr{T}_h^S$ we seek $p_S \in P^N(S)$, s.t.

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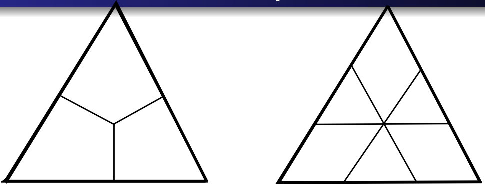

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'Standard' FV

- R must be constructed (and stored) for each $K \in \mathscr{T}_h$ independently (on unstructured meshes).
- Stencil size impractical for $N > 2$.
- Construction of stencils near $\partial\Omega$.
- Explicit construction in 1D.

Spectral FV

- All spectral volumes are affine equivalent $\Rightarrow$ R is constructed and stored only on a reference configuration.
- The construction of partitions of spectral volumes into control volumes is not straightforward for higher N and 3D.
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Let $v \in L^2(\Omega)$. Define by $\Pi_h^n v$ the $L^2(\Omega)$-projection of v on S_h^n:

$$\Pi_h^n v \in S_h^n, \quad (\Pi_h^n v - v, \varphi_h^n) = 0, \qquad \forall \varphi_h^n \in S_h^n.$$

The basis of the FV schemes consisted of the identity

$$\frac{d}{dt}\bar{u}_K(t) + \frac{1}{|K|}\int_{\partial K} H((R\bar{u})^{(L)}, (R\bar{u})^{(R)}, \mathbf{n})\, dS = O(h^N)$$

Since $\bar{u}(t) = \Pi_h^0 u(t)$, we may view this as an identity for $\Pi_h^0 u(t)$.

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We multiply our problem by an arbitrary $\varphi_h^n \in S_h^n$, integrate over an element $K \in \mathscr{T}_h$ and apply Green's theorem

$$\frac{d}{dt}\int_K u(t)\,\varphi_h^n\,dx + \int_{\partial K} \mathbf{f}(u)\cdot\mathbf{n}\,\varphi_h^n\big|_K\,dS - \int_K \mathbf{f}(u)\cdot\nabla\varphi_h^n\,dx = 0.$$

By summing over all $K \in \mathscr{T}_h$ and rearranging, we get

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$$\int_\Gamma \mathbf{f}(u)\cdot\mathbf{n}\,[\varphi_h^n]\,dS \approx \int_\Gamma H(u^{(L)}, u^{(R)}, \mathbf{n})[\varphi_h^n]\,dS.$$

Definition

$$b_h(u,\varphi) = \int_{\mathscr{F}_h} H(u^{(L)}, u^{(R)}, \mathbf{n})[\varphi]\,dS - \sum_{K\in\mathscr{T}_h}\int_K \mathbf{f}(u)\cdot\nabla\varphi\,dx.$$

Lemma

The projections $\Pi_h^n u(t)$ of the exact solution satisfy

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Let $v : \Omega \to \mathbb{R}$ be sufficiently regular. Given $\Pi_h^n v \in S_h^n$, find $v_h^N \in S_h^N$ such that $v - v_h^N = O(h^{N+1})$ in Ω.
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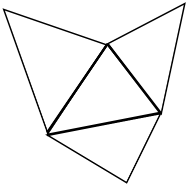
Analogy of 'standard' FV reconstruction operator



Reconstruction stencil

For each $K \in \mathscr{T}_h$ we choose the *reconstruction stencil* $S_K \subset \mathscr{T}_h$, usually some neighborhood of K.

For each $K \in \mathscr{T}_h$, we seek a polynomial $p_{S_K} \in P^N(S_K)$, s.t.

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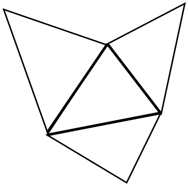

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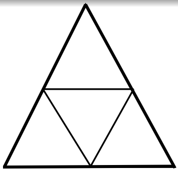

Spectral and _control_ volumes

Let $\mathscr{T}_h^S$ be a partition of $\overline{\Omega}$ into simplices $S \in \mathscr{T}_h^S$, called *spectral volumes*. The DG triangulation $\mathscr{T}_h$ is formed by subdividing each $S \in \mathscr{T}_h^S$ into so-called *control volumes* $K \subset S$.

For each spectral volume $S \in \mathscr{T}_h^S$ we seek $p_S \in P^N(S)$, s.t.

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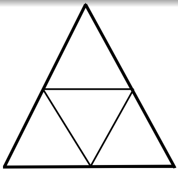

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- Stencil size need not be increased! To obtain higher orders we simply increase n.
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Reconstructed DG vs *Standard DG*

- Test functions only of order n as opposed to N.
- Fewer quadrature points, flux evaluations.
- CFL condition permits larger time steps. Mass matrices of order $n \times n$ instead of $N \times N$.
- The reconstruction procedure is problem-independent.

The von Neumann neighborhood allows us to reconstruct:

- 1D: S_h^{3n+2} from S_h^n.
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Therefore, error estimates for the reconstructed DG scheme may be derived from estimates for the auxiliary problem

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This is similar to the standard DG scheme

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Let $v \in H^{N+1}(\Omega), v_h \in S_h^N$. Then

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Numerical experiments

N	$\|\|e_h\|\|_{L^\infty(\Omega)}$	α	$\|\|e_h\|\|_{L^2(\Omega)}$	α	$\|e_h\|_{H^1(\Omega,\mathscr{T}_h)}$	α
4	9.30E-01	–	6.23E-01	–	4.05E+00	–
8	2.22E-01	2.07	1.55E-01	2.00	1.29E+00	1.65
16	3.25E-02	2.77	2.21E-02	2.81	2.47E-01	2.38
32	4.09E-03	2.99	2.82E-03	2.97	4.63E-02	2.41
64	5.07E-04	3.01	3.53E-04	3.00	9.46E-03	2.29
128	6.31E-05	3.01	4.41E-05	3.00	2.10E-03	2.17
256	7.86E-06	3.00	5.50E-06	3.00	4.91E-04	2.10

Table: 1D advection of sine wave, P^0 elements with P^2 reconstruction.

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8	7.53E-05	6.27	4.43E-05	6,30	1.06E-03	5,11
16	9.07E-07	6.38	5.95E-07	6,22	3.58E-05	4,89
32	1.82E-08	5.64	8.70E-09	6,10	1.16E-06	4,95
64	3.41E-10	5.74	1.33E-10	6,03	3.67E-08	4,98

Table: 1D advection of sine wave, P^1 elements with P^5 reconstruction.

Numerical experiments

N	$\|\|e_h\|\|_{L^\infty(\Omega)}$	α	$\|\|e_h\|\|_{L^2(\Omega)}$	α	$\|e_h\|_{H^1(\Omega,\mathscr{T}_h)}$	α
4	2.90E-03	–	1.85E-03	–	1.63E-02	–
8	7.75E-06	8.55	3.56E-06	9.02	1.03E-04	7.30
16	2.10E-08	8.53	6.64E-09	9.07	4.34E-07	7.89
32	7.21E-11	8.18	4.02E-11	7.37	1.76E-09	7.94

Table: 1D advection of sine wave, P^2 elements with P^8 reconstruction.

Thank you for your attention