

On Izobov's problem for a nonlinear third-order differential equation

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Abstract

Sufficient and necessary conditions for existence of Kneser solutions vanishing at infinity of a nonlinear third-order differential equation with singular nonlinearity will be discussed.

Introduction

Consider the differential equation

$$y^{(n)} = p(x)|y|^{\lambda-1}y, \quad 0 < \lambda < 1, \quad x \geq 0 \quad (1)$$

with

$$(-1)^{(n)}p(x) \geq 0, \quad x \geq 0. \quad (2)$$

Definition 1 ([1], [4]). A solution $y(x)$ of (1) is called a *Kneser solution vanishing at infinity* if

$$(-1)^{(i)}y^{(i)}(x) > 0, \quad (3)$$

$$|y^{(i-1)}(x)| \downarrow 0, \quad x \rightarrow \infty, \quad i = 1, 2, \dots, n. \quad (4)$$

In [2] a sufficient condition was obtained for existence of solutions $y(x)$ satisfying (3), (4):

Theorem 1. *If a continuous function $p(x)$ satisfies the condition*

$$\int_0^{+\infty} \tau^{n-1}|p(\tau)|d\tau < \infty \quad (5)$$

then (1) has solutions $y(x)$ such that (3), (4) hold.

Later in [3] N. Izobov proved that (5) is not necessary if $n = 2$:

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Theorem 2 (N. Izobov). Let $n = 2$. For any $\mu \geq \frac{1}{(n-1)\lambda+1}$ and any function $\varphi(x) > 0$ there exists a piecewise continuous non-negative function $p(x)$ satisfying the condition

$$\int_0^{+\infty} p^\mu(\tau)\varphi(\tau)d\tau = \infty \quad (6)$$

such that equation (1) has a Kneser solution $y(x)$ vanishing at infinity.

Corollary 1 (N. Izobov). Let $n = 2$. There exists a piecewise continuous non-negative function $p(x)$ satisfying the condition

$$\int_0^{+\infty} \tau^{n-1}|p(\tau)|d\tau = \infty \quad (7)$$

such that equation (1) has a solution $y(x)$ with (3), (4).

Problem (N. Izobov): Is it possible to prove for $n > 2$ the analogue of Theorem 2?

A partial answer is given here for $n = 3$.

Main result

Theorem 3. Suppose $n = 3$ and $0 < \lambda < 1$. Then for any $\mu > \frac{1}{2\lambda+1}$ and any continuous positive function $\varphi(x)$ $x \geq 0$, there exists a smooth negative function $p(x)$ such that the condition

$$\int_0^{+\infty} |p(\tau)|^\mu \varphi(\tau)d\tau = \infty \quad (8)$$

holds and equation (1) has a solution satisfying conditions (3), (4).

In fact the following result is proved:

Theorem 4. Suppose $n = 3, 0 < \lambda < 1$, $\varphi(x)$ is a continuous positive function for $x \geq 0$, $\mu > \frac{1}{2\lambda+1}$. Then there exists a C^∞ function $y(x)$, $x \geq 0$, such that

$$|y^{(i)}(x)| \Downarrow 0, \quad x \rightarrow \infty, \quad i = 0, 1, 2, \quad (9)$$

and

$$\int_0^{+\infty} |y'''(\xi)|^\mu |y(\xi)|^{-\mu\lambda} d\xi = \infty. \quad (10)$$

Remark 1. To generalize Theorem 2 the inequality $\mu \geq \frac{1}{2\lambda+1}$ is needed. We prove Theorem 3 for $\mu > \frac{1}{2\lambda+1}$ only, so Izobov's problem is partially solved. However we prove existence of a smooth negative function $p(x)$, which is a piecewise continuous non-positive function in Theorem 2.

Remark 2. Asymptotic behavior of solutions of the third-order equation (1) is described in [5, 6].

References

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