Integral functionals that are continuous with respect to the weak topology on $W_0^{1,p}(0,1)$

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Abstract

For continuous (or, locally bounded Carathéodory) functions $g\colon [0,1]\times\mathbb{R}\to\mathbb{R}$ we prove that the functional $\Phi(u)=\int_0^1 g\big(x,u(x)\big)\,\mathrm{d}x$ is weakly continuous on $W_0^{1,p}(0,1)$, $1\le p<\infty$, if and only if g is linear in the second variable.

Key words: Weak continuity, non-linear integral functional, Sobolev spaces, linearity
1991 MSC: 49J45

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[1] Research was supported by the Research Project MSM 113200007 of the Czech Ministry of Education.
[2] Research was supported by the grant GAČR 201/02/D111.

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1 Introduction

Let $g\colon [0,1]\times\mathbb{R}\to\mathbb{R}$ be a continuous function. We prove that functional

$$\Phi(u) = \int_0^1 g\big(x, u(x)\big)\,\mathrm{d}x$$

is weakly continuous on the Sobolev space $W_0^{1,p}(0,1)$, $1\le p<\infty$, if and only if g is linear in the second variable (i.e., there are continuous functions k_1 and k_2 such that $g(x,u(x)) = k_1(x) + k_2(x)u(x)$). This essentially gives a negative answer to Problem 1 in [2].

Let us note that classical results in the calculus of variations (see e.g. [1,3]) usually deal with weakly *sequentially* continuous (or semicontinuous) functionals and therefore the results and techniques are different. In particular we briefly (without assumptions) recall some facts from which our result *does not* follow. First, every weakly continuous functional $u\mapsto \int_0^1 g(x,u(x),u'(x))\,\mathrm{d}x$ on $W^{1,p}$ is sequentially weakly continuous on $W^{1,p}$, hence sequentially w^*-continuous on $W^{1,\infty}$, which is known to be equivalent to g being linear in the third variable (i.e., *in the derivative* u'). Second, if $u\mapsto\int g(x,u)\,\mathrm{d}x$ is weakly continuous on L^p, then it is sequentially weakly continuous and this is equivalent to the linearity of g in u.

2 Preliminaries

We use the usual notation $W_0^{1,p}(0,1)$ for the Sobolev space, i.e., the set of all absolutely continuous functions on $[0,1]$ such that $f(0)=f(1)=0$ and $\|f\|_{W_0^{1,p}} = (\int_0^1 |f'|^p)^{1/p} < \infty$.

We will also use the fact that for every continuous linear functional Λ on $W_0^{1,p}(0,1)$, $1\le p<\infty$, there is a function $\phi\in L^{p'}(0,1)$ such that $\Lambda(f) = \int_0^1 f'(x)\phi(x)\,\mathrm{d}x$. Here p' denotes the conjugate Hölder index, i.e., $1/p+1/p'=1$.

Recall that a functional Φ is *weakly continuous* on $W_0^{1,p}(0,1)$ if for every $\varepsilon>0$ and $f_0\in W_0^{1,p}(0,1)$ there is a weak neighbourhood U of f_0 such that $|\Phi(f)-\Phi(f_0)|<\varepsilon$ for all $f\in U$. A set $U\subset W_0^{1,p}(0,1)$ is a *weak neighbourhood of* f_0 if we can find $k\in\mathbb{N}$ and continuous linear functionals $\Lambda_1,\dots,\Lambda_k\in(W_0^{1,p}(0,1))^*$ such that

$$\Big\{f\in W_0^{1,p}(0,1):\ |\Lambda_i(f-f_0)|<1 \text{ for every } i\in\{1,\dots,k\}\Big\}\subset U.$$

We denote $\operatorname{spt} f = \overline{\{x \in [0,1] : f(x) \neq 0\}}$. The integral average of a function is denoted by

$$⨍_a^b f = \frac{1}{b-a} \int_a^b f.$$

By $[x]$ we denote the integer part of $x > 0$. We use the notation $\#M$ for the number of elements of the set M. We write 2^M for the set of all subsets of M.

3 A combinatorial lemma

Lemma 3.1. *Let $l \in \mathbb{N}$ and $M = \{1, 2, \dots, 20l\}$. Then there is a system $\mathcal{A} \subset 2^M$ such that*

$$\text{(i)}\quad \#\mathcal{A} \geq 2^l, \qquad\qquad \text{(ii)}\quad A \in \mathcal{A} \;\Rightarrow\; \#A = 2l, \tag{1}$$

$$\text{(iii)}\quad A_1, A_2 \in \mathcal{A},\; A_1 \neq A_2 \;\Rightarrow\; \#(A_1 \cap A_2) \leq l. \tag{2}$$

PROOF. Let us denote by $\mathcal{A}_0$ the system of all subset of $\{1, \dots, 20l\}$ of cardinality $2l$. We will use induction to show that, for every $N = 1, \dots, 2^l$, there exists $\mathcal{A} \subset \mathcal{A}_0$ satisfying (ii), (iii) and $\#\mathcal{A} \geq N$.

We select $\{A_1\}$ as a solution of the task for $N = 1$. If $N = 2$, we use the elementary inequality

$$\binom{20l}{2l} \geq \left(\frac{18l}{2l}\right)^l \binom{18l}{l} = 9^l \binom{18l}{l}$$

to show that

$$\begin{aligned} \#\{A \in \mathcal{A}_0 : \#(A \cap A_1) > l\} &= \sum_{i=l+1}^{2l} \#\{A \in \mathcal{A}_0 : \#(A \cap A_1) = i\} \\ &= \sum_{i=l+1}^{2l} \binom{2l}{i} \binom{18l}{2l-i} \leq \binom{18l}{l} \sum_{i=0}^{2l} \binom{2l}{i} \\ &\leq 9^{-l} \binom{20l}{2l} (1+1)^{2l} = (4/9)^l \binom{20l}{2l}, \end{aligned}$$

so that there is enough space to choose A_2. Now, let $N \leq 2^l$ be arbitrary. By the induction hypothesis we find a system $\{A_1, \dots, A_{N-1}\}$ which solves the task for $N-1$. By the above estimate,

$$\begin{aligned} \#\Big\{A \in \mathcal{A}_0 : \#(A \cap A_i) > l \text{ for an } i = 1, \dots, N-1\Big\} \\ \leq (N-1) \left(\frac{4}{9}\right)^l \binom{20l}{2l} < \binom{20l}{2l} \end{aligned}$$

and thus there exits $A_N \in \mathcal{A}_0$ such that the system $\{A_1, \dots, A_N\}$ solves the task for N. $\square$

4 Construction of a suitable perturbation

Let $0 < \varepsilon \le 1/4$ and $n \in \mathbb{N}$. We will divide a given interval $[x_0, x_0 + \eta]$ into n subintervals $J_{n,j} = [x_0 + \eta\frac{j-1}{n}, x_0 + \eta\frac{j}{n}]$, $j \in \{1, \dots, n\}$. We denote by

$$J^1_{\varepsilon,n,j} = \left[x_0 + \eta\tfrac{j-1}{n}, x_0 + \eta\tfrac{j-1+\varepsilon}{n}\right] \quad \text{and} \quad J^2_{\varepsilon,n,j} = \left[x_0 + \eta\tfrac{j-\varepsilon}{n}, x_0 + \eta\tfrac{j}{n}\right]$$

the first and the last ε-part of these subintervals. Define a continuous piecewise linear function

$$f_{\varepsilon,n,j}(x) = \begin{cases} \frac{n}{\varepsilon\eta}\left(x - x_0 - \eta\frac{j-1}{n}\right) & \text{for } x \in J^1_{\varepsilon,n,j}\,, \\ 1 & \text{for } x \in J_{n,j} \setminus (J^1_{\varepsilon,n,j} \cup J^2_{\varepsilon,n,j})\,, \\ \frac{n}{\varepsilon\eta}\left(x_0 + \eta\frac{j}{n} - x\right) & \text{for } x \in J^2_{\varepsilon,n,j}\,, \\ 0 & \text{for } x \notin J_{n,j}\,. \end{cases} \tag{3}$$

Lemma 4.1. *Let $r > 0$, $0 < \varepsilon \le 1/4$ and $k \in \mathbb{N}$. Suppose that $x_0 \in \mathbb{R}$, $\eta > 0$ and $\phi_i \in L^1(0,1)$ for $i \in \{1, \dots, k\}$. Then there is a continuous piecewise linear function $f_1\colon [0,1] \to [-r, r]$ such that* $\operatorname{spt} f_1 \subset [x_0, x_0 + \eta]$,

$$\left|\int_0^1 f_1'\phi_i\right| < 1 \quad \textit{for every} \;\; i \in \{1, \dots, k\}, \tag{4}$$

$$\operatorname{meas} f_1^{-1}(\{-r, r\}) \ge \frac{\eta}{40} \quad \textit{and} \quad \operatorname{meas} f_1^{-1}(\mathbb{R} \setminus \{-r, 0, r\}) \le 2\varepsilon\eta. \tag{5}$$

(*In fact, there are $n \in \mathbb{N}$ and numbers $s_j \in \{-1, 0, 1\}$ for $j \in \{1, \dots, n\}$ such that*

$$\#\Big\{j \in \{1, \dots, n\} : \;\; s_j \ne 0\Big\} \ge \frac{n}{20} \tag{6}$$

and the function

$$f_1(x) = r\sum_{j=1}^{n} s_j f_{\varepsilon,n,j}(x) \tag{7}$$

satisfies the above properties.)

PROOF. Choose $l \in \mathbb{N}$ such that

$$([16rlK] + 1)^k < 2^l \quad \text{where} \quad K = \frac{3}{\eta\varepsilon}\sum_{i=1}^{k} \|\phi_i\|_{L^1}. \tag{8}$$

Set $n = 30l$ and

$$B = \Big\{ j \in \{1, \dots, n\} : \ \Big| \fint_{J^s_{\varepsilon,n,j}} \phi_i \Big| > K \text{ for some } s \in \{1,2\} \text{ and } i \in \{1, \dots, k\} \Big\}. \tag{9}$$

Since the intervals $J^s_{\varepsilon,n,j}$ are disjoint, we have $\sum_{i=1}^k \|\phi_i\|_{L^1} \geq \#B\, K\eta\varepsilon/n$ and hence $\#B \leq n/3 = 10l$.

We fix a set $M \subset \{1, \dots, n\} \setminus B$ such that $\#M = 20l$. In view of Lemma 3.1 we can choose a system $\mathcal{A}$ of subsets of M such that (1) and (2) are valid. Consider the following set of functions

$$\mathcal{H} = \Big\{ h_A : \ A \in \mathcal{A} \Big\} \qquad \text{where} \qquad h_A(x) = r \sum_{j \in A} f_{\varepsilon,n,j}(x).$$

For every $\phi \in L^1(0,1)$ we have

$$\int_0^1 r f'_{\varepsilon,n,j}\, \phi = r \Big(\fint_{J^1_{\varepsilon,n,j}} \phi - \fint_{J^2_{\varepsilon,n,j}} \phi \Big).$$

Hence for every $h = h_A \in \mathcal{H}$ and $i \in \{1, \dots, k\}$ we obtain from (1) (ii), $M \cap B = \emptyset$ and (9) that

$$\Big| \int_0^1 h'\, \phi_i \Big| \leq r \sum_{j \in A} \Big(\fint_{J^1_{\varepsilon,n,j}} |\phi_i| + \fint_{J^2_{\varepsilon,n,j}} |\phi_i| \Big) \leq 4lrK.$$

We can divide the interval $[-4lrK, 4lrK]$ into $[16rlK] + 1$ subintervals of length at most $1/2$ and therefore the cube $[-4lrK, 4lrK]^k$ can be covered by $N := ([16rlK] + 1)^k$ translates of cube $[0, 1/2]^k$. By (8), $N < 2^l$. From (1) we know that $\#\mathcal{H} \geq 2^l$ and therefore by (8) there are two different functions $h_1, h_2 \in \mathcal{H}$ such that the vectors $(\int_0^1 h_1' \phi_i)_{i=1}^k$, $(\int_0^1 h_2' \phi_i)_{i=1}^k$ lie in the same translate of $[0, 1/2]^k$, that is, for each $i \in \{1, \dots, k\}$ we have

$$\Big| \int_0^1 (h_1 - h_2)'\, \phi_i \Big| = \Big| \int_0^1 h_1'\, \phi_i - \int_0^1 h_2'\, \phi_i \Big| \leq \frac{1}{2}. \tag{10}$$

Set $f_1 = h_1 - h_2$; this function is clearly of the form (7) with $s_j \in \{-1, 0, 1\}$. From (10) we obtain (4). It is not difficult to see from (1) (ii) and (2) that among $2l$ intervals $J_{n,j}$ where h_1 is non-zero there are at least l intervals where h_2 is zero; we have $s_j = 1$ on these intervals. Analogously we obtain at least l intervals where $s_j = -1$ and (6) and (5) follow. $\square$

5 Main theorem

Theorem 5.1. *Let* $1 \le p < \infty$ *and let* $g\colon [0,1]\times\mathbb{R}\to\mathbb{R}$ *be a continuous function. The functional*

$$\Phi(u) = \int_0^1 g\big(x, u(x)\big)\,\mathrm{d}x$$

is weakly continuous on $W_0^{1,p}(0,1)$ *if and only if* $g(x,u(x)) = k_1(x)+k_2(x)u(x)$*, where* $k_1(x)$ *and* $k_2(x)$ *are continuous functions.*

PROOF. Let $g(x,u(x)) = k_1(x) + k_2(x)u(x)$, where $k_1(x)$ and $k_2(x)$ are continuous functions. Then $u \mapsto \Phi(u) - \Phi(0) = \int_0^1 k_2(x)u(x)\,\mathrm{d}x$ is obviously continuous linear functional on $L^p(0,1)$ and hence on $W_0^{1,p}(0,1)$. Therefore it is weakly continuous.

We will prove the reverse implication by contradiction. Suppose that Φ is weakly continuous and that g is not linear in the second variable. Then we can find $x_0 \in (0,1)$, $a \in \mathbb{R}$ and $r > 0$ such that

$$2g(x_0,a) \ne g(x_0,a-r) + g(x_0,a+r).$$

Replacing g by $\tilde{g}(x,y) = \pm g(x,y) + cy$ will not change the weak continuity of Φ and therefore we can assume without loss of generality that

$$g(x_0,a-r) > g(x_0,a) \quad \text{and} \quad g(x_0,a+r) > g(x_0,a).$$

Since g is continuous there are $\eta > 0$ and $A > 0$ such that $[x_0, x_0+\eta] \subset (0,1)$ and for $x \in [x_0, x_0+\eta]$ we have

$$g(x,a-r) > g(x,a) + A \quad \text{and} \quad g(x,a+r) > g(x,a) + A. \tag{11}$$

Let f_0 be a smooth function on $[0,1]$ with $f_0(0) = f_0(1) = 0$ and $f_0(x) = a$ for every $x \in [x_0, x_0+\eta]$. By the continuity of Φ we can find a weak neighbourhood U of the function f_0 such that

$$\left|\int_0^1 g\big(x, f(x)\big) - g(x, f_0(x))\,\mathrm{d}x\right| < \frac{\eta}{200}A \tag{12}$$

for every $f \in U$. From the properties of the weak topology (see Preliminaries) we can find $k \in \mathbb{N}$ and functions $\phi_i \in L^{p'}(0,1)$ for $i \in \{1,\ldots,k\}$ such that

$$\left\{f \in W_0^{1,p}(0,1) : \left|\int_0^1 (f-f_0)'\phi_i\right| < 1 \text{ for } i \in \{1,\ldots,k\}\right\} \subset U. \tag{13}$$

We set

$$K = \max_{\substack{x\in[0,1]\\ t\in[a-r,a+r]}} |g(x,t)|, \qquad \varepsilon = \min\left\{\frac{A}{320K}, \frac{1}{4}\right\} \tag{14}$$

and find a function f_1 as in Lemma 4.1. From (4) and (13) we obtain $f_1 + f_0 \in U$, thus (12) implies

$$\left|\int_0^1 g(x, f_1(x) + f_0(x)) - g(x, f_0(x))\,\mathrm{d}x\right| < \frac{\eta}{200}A. \tag{15}$$

Denote $Z = f_1^{-1}(\mathbb{R} \setminus \{-r, 0, r\})$. By (5) we have $\operatorname{meas} Z \leq 2\varepsilon\eta$. In view of (5), (11) and (14) we get

$$\left|\int_0^1 g(x, f_0(x) + f_1(x)) - g(x, f_0(x))\right| = \left|\int_{\operatorname{spt} f_1} g(x, a + f_1(x)) - g(x, a)\right| \geq$$

$$\geq \int_{\{f_1=r\}} (g(x, a+r) - g(x,a)) + \int_{\{f_1=-r\}} (g(x, a-r) - g(x,a)) -$$

$$- \int_Z |g(x, a + f_1(x)) - g(x,a)| \geq \frac{\eta}{40}A - 2K \operatorname{meas} Z \geq \frac{\eta}{80}A.$$

This contradicts (15). ☐

Remark 5.2. In Theorem 5.1, the hypotheses on the weak continuity of Φ on the whole space can be replaced by its weak continuity at the zero function. (The proof is similar, we only have to choose f_0 in the corresponding weak neighbourhood of the zero function. This can be obtained by an additional use of Lemma 4.1 with $r := |a|$; the interval $[x_0, x_0 + \eta]$ must be changed afterwards accordingly.)

Remark 5.3. The continuity assumption on g can be replaced by the following one: g is a Carathéodory function, bounded on bounded sets. The conclusion of the theorem is that $g(x, \cdot)$ is linear for almost every $x \in [0,1]$.
The proof is to be modified as follows. If, for a fixed x, the function $g(x, \cdot)$ is not linear, then we get, by its continuity, $s \in \{-1, 1\}$ and rational numbers $A > 0$, a, r such that

$$\Big(g(x, a-r) + g(x, a+r) - 2g(x,a)\Big)s > 2A. \tag{16}$$

Hence there are A, a, r and s such that $G := \{x \in [0,1] : (16) \text{ is true}\}$ has positive measure. We let $x_0 \in (0,1)$ be a point of density of G and η so small that $\operatorname{meas}([x_0, x_0+\eta] \setminus G) < \frac{\eta A}{800(2K+A)}$. For the rest of the proof we again replace g with $\tilde{g}(x,y) = (g(x,y) - c(x)y)s$, where $c(x) = (g(x, a+r) - g(x, a-r))/2r$, which does not change the weak continuity of Φ and makes (16) equivalent to (11). By the choice of η, the inequalities at the end of the proof are not disturbed too much and still give a contradiction with (15).

Acknowledgement. The authors would like to thank Jan Malý for drawing their attention to the problem and for many stimulating conversations. We also thank the referee for the simplified proof of Lemma 3.1.

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