A note on Grzegorczyk's logic

Emil Jeřábek
Mathematical Institute, Czech Academy of Sciences
jerabek@math.cas.cz

December 1, 2003

Abstract

Grzegorczyk's modal logic (*Grz*) corresponds to the class of upwards well-founded partially ordered Kripke frames, however all known proofs of this fact utilize some form of the Axiom of Choice; G. Boolos asked in [1], whether it is provable in plain *ZF*. We answer his question negatively: *Grz* corresponds (in *ZF*) to a class of frames, which does *not* provably coincide with upwards well-founded posets in *ZF* alone.

Definition 1 *Grzegorczyk's logic* (*Grz*) [2] is a normal modal logic axiomatized by the schema

$$\Box(\Box(\varphi \to \Box\varphi) \to \varphi) \to \varphi.$$

We denote by $\mathcal{K}_1$ the class of upwards well-founded posets, $\mathcal{K}_3$ the class of posets without any strictly increasing infinite chain, and $\mathcal{K}_2$ the class of posets $\langle W, \leq \rangle$ satisfying

$$\forall X \subseteq W\, (X \neq \emptyset \to \exists x \in X\, \forall y \geq x\, \forall z \geq y\, (z \in X \to y \in X)). \tag{1}$$

Recall that the *Principle of Dependent Choices* (*DC*) is the following weak version of the Axiom of Choice: let R be a binary relation on a nonempty set A such that $\forall x \in A\, \exists y \in A\, \langle x, y \rangle \in R$, then there is an infinite sequence $\{a_n;\, n \in \omega\} \in A^\omega$ such that $\langle a_n, a_{n+1} \rangle \in R$ for every $n \in \omega$.

Lemma 2 *ZF proves* $\mathcal{K}_1 \subseteq \mathcal{K}_2 \subseteq \mathcal{K}_3$.

Proof: If $\langle W, \leq \rangle \in \mathcal{K}_1$, and $X \subseteq W$ nonempty, then any $<$-maximal element $x \in X$ witnesses that (1) holds, hence $\langle W, \leq \rangle \in \mathcal{K}_2$. Assume that there is $\langle W, \leq \rangle \in \mathcal{K}_2 \setminus \mathcal{K}_3$. Fix an infinite increasing chain $x_0 < x_1 < x_2 < \cdots$ in W, and put $X = \{x_n;\, n \text{ odd}\}$. Then for any $x \in X$ there are $z \geq y \geq x$ such that $z \in X$ and $y \notin X$, contradicting (1). $\Box$

Proposition 3 (*ZF* $\vdash$:) *A frame* $\mathbf{W} = \langle W, \leq \rangle$ *is a model of Grz under all valuations if and only if* $\mathbf{W} \in \mathcal{K}_2$.

Proof: ("if") Let $\Vdash$ be a valuation in $\mathbf{W}$, $w \in W$, and $w \nVdash \varphi$. Define $X = \{v;\, w \leq v \,\&\, v \nVdash \varphi\}$, and let $x \in X$ be as in (1). If $y \geq x$, and $y \Vdash \varphi$, then $y \Vdash \Box\varphi$ by (1), hence $x \Vdash \Box(\varphi \to \Box\varphi)$, and $w \nVdash \Box(\Box(\varphi \to \Box\varphi) \to \varphi)$.

("only if") It is well-known that *Grz* contains $S4$, hence all *Grz*-frames are reflexive and transitive (i.e., preorderings). Assume that $X \subseteq W$ is a counterexample to (1), and put $w \Vdash p$ iff $w \notin X$, where p is an atom. Let $x \in W$, and $x \Vdash \Box(p \to \Box p)$. This means $\forall y \geq x \, \forall z \geq y \, (y \notin X \to z \notin X)$, hence $x \notin X$ (by our assumption on X), thus $x \Vdash p$. In other words, $\Box(\Box(p \to \Box p) \to p)$ is valid in all nodes of W, however p is not, because X is nonempty. This contradicts $\mathbf{W} \Vdash$ *Grz*.

Finally, notice that any preordering satisfying (1) is a partial ordering: taking $X = \{x\}$, (1) yields $x \geq y \geq x \to x = y$. □

Lemma 4 *The following are equivalent over ZF:*

(*i*) *DC*,

(*ii*) $\mathcal{K}_1 = \mathcal{K}_3$,

(*iii*) $\mathcal{K}_2 = \mathcal{K}_3$.

Proof: The implication $DC \to \mathcal{K}_1 = \mathcal{K}_3$ follows directly from the definition, and $\mathcal{K}_1 = \mathcal{K}_3$ implies $\mathcal{K}_2 = \mathcal{K}_3$ by Lemma 2, it remains to show $\mathcal{K}_2 = \mathcal{K}_3 \to DC$. Assume $\mathcal{K}_3 \subseteq \mathcal{K}_2$, let $R \subseteq A^2$ be a relation without a maximal element, and let $a_0 \in A$. Define U as the set of all finite sequences $\langle a_0, \dots, a_n \rangle \in A^{<\omega}$ such that $\langle a_i, a_{i+1} \rangle \in R$ for all $i < n$, ordered by inclusion (i.e., $s \leq t$ iff t extends s). By taking $X = \{s \in U;\, lh(s) \text{ odd}\}$ we see that $U \notin \mathcal{K}_2$, hence (by assumption) $U \notin \mathcal{K}_3$. Consequently U contains an infinite strictly increasing chain, and the union of such a chain is clearly an infinite sequence $\{a_n;\, n < \omega\} \in A^\omega$ such that $\langle a_i, a_{i+1} \rangle \in R$ for all $i \in \omega$. □

Proposition 5 *There is a model of ZF, in which* $\mathcal{K}_1 \neq \mathcal{K}_2 \neq \mathcal{K}_3$ (*unless ZF is inconsistent*).

Proof: By Lemma 4, it suffices to find a model of $\mathcal{K}_1 \neq \mathcal{K}_2$.

The following property holds in the Ordered Mostowski Model [7]: there is a dense linear ordering $\mathbf{W} = \langle W, \leq \rangle$ such that any subset of W is a finite union of intervals. (Mostowski's permutation model is a model of *ZFA*, the set theory with atoms, but it is possible to transfer this result into *ZF*, using e.g. the Jech-Sochor Embedding Theorem [5], [6].) Clearly $\mathbf{W} \notin \mathcal{K}_1$, we claim that $\mathbf{W} \in \mathcal{K}_2$: let X be a nonempty subset of W, we may write X as a disjoint union $X = I_1 \cup \dots \cup I_n$ of nonempty intervals (possibly degenerate) such that $I_1 < \dots < I_n$. Then any $x \in I_n$ witnesses (1).

Note: Halpern [3] has shown that the Boolean Prime Ideal Theorem (*BPI*) holds in Ordered Mostowski's Model (cf. also [4]), hence even *ZF* + *BPI* doesn't prove $\mathcal{K}_1 = \mathcal{K}_2 \vee \mathcal{K}_2 = \mathcal{K}_3$. □

Corollary 6 *It is relatively consistent with ZF that there is a Grz-frame which is not upwards well-founded.* □

References

[1] G. Boolos, *The Logic of Provability,* Cambridge University Press 1993, p. 163

[2] A. Grzegorczyk, Some relational systems and the associated topological spaces, *Fundamenta Mathematicae* 60 (1967), pp. 223–231

[3] J. D. Halpern, The independence of the axiom of choice from the Boolean prime ideal theorem, *Fundamenta Mathematicae* 55 (1964), pp. 57–66

[4] T. Jech, *The Axiom of Choice,* North-Holland, Amsterdam 1973

[5] T. Jech, A. Sochor, On θ-model of set theory, *Bulletin de l'Académie Polonaise des Sciences* 14 (1966), pp. 297–303

[6] T. Jech, A. Sochor, Applications of the Θ-model, *Bulletin de l'Académie Polonaise des Sciences* 14 (1966), pp. 351–355

[7] A. Mostowski, Über die Unabhängigkeit des Wohlordnungssatzes vom Ordnungsprinzip, *Fundamenta Mathematicae* 32 (1939), pp. 201–252