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On the Cauchy-Nicoletti multipoint boundary value problem for systems of linear generalized differential equations with singularities

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For the system of linear generalized ordinary differential equations with singularities the two point boundary value problem

$$dx(t) = dA(t) \cdot x(t) + df(t), \tag{1}$$
$$x_i(t_i-) = 0, \quad x_i(t_i+) = 0 \quad (i = 1, ..., n), \tag{2}$$

is considered, where $x_1, \dots, x_n$ are the components of the desired solution x, $-\infty < a < t_i \leq t_{i+1} < b < +\infty$, $f = (f_l)_{l=1}^n \colon [a, b] \to \mathbb{R}^n$ is a vector-function the components of which have bounded variations, $A = (a_{il})_{i,l=1}^n \colon [a, b] \to \mathbb{R}^{n\times n}$ is a matrix-function such that the functions a_{il} $(i \neq l;\ i, l = 1, ..., n)$ have bounded variations on $[a, b]$, and the function a_{ii} have bounded variation on every closed interval from $[a, b]$ which do not include the point t_i for every $i \in \{1, \dots, n\}$.

The sufficient conditions are established for this problem to be uniquely solvable in the case when system (1) is singular, i. e., the components of the matrix-function A maybe to have unbounded variation on $[a, b]$.

Generalized ordinary differential equations have been introduced by J. Kurzweil in connection with the investigation of the question of the correctness of the Cauchy problem for ordinary differential equations [J. Kurzweil, *Generalized ordinary differential equations and continuous dependence on a parameter.* Czechoslovak Math. J. **7** (1957), No. 3, 418–449].

The theory of generalized ordinary differential equations enables one to investigate ordinary differential, impulsive and difference equations from a unified point of view.

$\mathrm{BV}_{loc}(a, b, t_i; \mathbb{R}^n)$ is the set of all functions $\varphi \colon [a, b] \setminus \{t_i\} \to \mathbb{R}$ having bounded variation on every closed interval $[s, d] \subset [a, b] \setminus \{t_i\}$ $(i = 1, \dots, n)$.

Under a solution of problem (1), (2) we mean a vector-function $x = (x_i)_i^n$ with $x_i \in \mathrm{BV}_{loc}(a, b, t_i; \mathbb{R})$ $(i = 1, \dots, n)$, satisfying condition (2) and system (1), i. e., such that

$$x_i(t) = x_i(s) + \sum_{l=1}^n \int_s^t x_l(\tau)\,da_{il}(\tau) + f_i(t) - f_i(s) \quad \text{for every } [s, t] \subset [a, b],\ t_i \notin [s, t]\ (i = 1, ..., n),$$

where the integral is considered in the Lebesgue-Stieltjes sense.

Let $\det(I_n + (-1)^j d_j A(t)) \neq 0$ for $t \in [a, b]$ $(j = 1, 2)$, where I_n is the identity $n \times n$-matrix.

By $\gamma_\alpha(\cdot,s)$ we denote the unique solution of the Cauchy problem

$$d\gamma(t)=\gamma(t)\,d\alpha(t),\quad \gamma(s)=1.$$

Definition 1. We say that a matrix-function $C=(c_{il})_{i,l=1}^n\in \mathrm{BV}([a,b],\mathbb{R}^{n\times n})$ belongs to the set $\mathcal{U}(a,b,t_1,\dots,t_n)$ if the functions c_{il} $(i\neq l;\ i,l=1,\dots,n)$ are nondecreasing on $[a,b]$ and the system

$$\operatorname{sgn}(t-t_i)\cdot dx_i(t)\leq \sum_{l=1}^n x_l(t)\,dc_{il}(t)\quad \text{for } t\in[a,b]\ (i=1,\dots,n)$$

has no nontrivial, nonnegative solution satisfying condition (2).

Theorem 1. *Let the vector-function f have bounded variation and let the matrix-function $A=(a_{il})_{i,l=1}^n$ be such that the inequalities*

$$\begin{gathered}\big(s_0(a_{ii})(t)-s_0(a_{ii})(s)\big)\operatorname{sgn}(t-t_i)\leq s_0(c_{ii}-\alpha_i)(t)-s_0(c_{ii}-\alpha_i)(s),\\ (-1)^j\big(\big|1+(-1)^jd_ja_{ii}(t)\big|-1\big)\operatorname{sgn}(t-t_i)\leq d_j\big(c_{ii}(t)-\alpha_i(t)\big)\ (j=1,2),\\ \big|s_0(a_{il})(t)-s_0(a_{il})(s)\big|\leq s_0(c_{il})(t)-s_0(c_{il})(s),\\ |d_ja_{il}(t)|\leq d_jc_{il}(t)\ (j=1,2)\end{gathered}$$

hold for $a\leq s<t<t_i$ or $t_i<s<t\leq b$ $(i\neq l;\ i,l=1,\dots,n)$, where $C=(c_{il})_{i,l=1}^n\in\mathcal{U}(a,b,t_1,\dots,t_n)$, and $\alpha_i\colon[a,b]\to\mathbb{R}$ $(i=1,\dots,n)$ are nondecreasing on intervals $[a,t_i[$ and $]t_i,b]$ functions such that

$$\lim_{t\to t_i+}d_1\alpha_i(t)<1\quad(i=1,\dots,n),\quad \lim_{t\to t_i-}d_2\alpha_i(t)<1\quad(i=1,\dots,n), \tag{3}$$

$$\begin{gathered}\lim_{t\to t_i+}\sup\big\{\gamma_{\alpha_i}(t,t_i+1/k):k=1,2,\dots\big\}=0\quad(i=1,\dots,n),\\ \lim_{t\to t_i-}\sup\big\{\gamma_{\alpha_i}(t,t_i-1/k):k=1,2,\dots\big\}=0\quad(i=1,\dots,n).\end{gathered} \tag{4}$$

Then problem (1), (2) *has one and only one solution.*

Corollary 1. *Let the vector-function f have bounded variation and let the elements of the matrix-function $A=(a_{il})_{i,l=1}^n$ satisfy the conditions given in Theorem 1, where $c_{il}(t)\equiv h_{il}\beta(t)$ $(i,l=1,\dots,n)$, $\alpha_i(t)\equiv\alpha(t)$ $(i=1,\dots,n)$, $\alpha\colon[a,b]\to\mathbb{R}$ is a nondecreasing function satisfying conditions* (3) *and* (4)*, β is a nondecreasing on $[a,b]$ function having not more than a finite number of discontinuity points, $h_{ii}\in\mathbb{R}$, and $h_{il}\in\mathbb{R}_+$ $(i\neq l,\ i,l=1,\dots,n)$. Let, moreover, $\rho\,r(\mathcal{H})<1$, where $\mathcal{H}=(h_{ik})_{i,k=1}^n$, $\rho=\max\big\{\lambda_{m0}+\lambda_{m1}+\lambda_{m2}:m=0,1,2\big\}$, $\lambda_{00}=2\pi^{-1}\big(s_0(\beta)(b)-s_0(\beta)(a)\big)$, and*

$$\begin{gathered}\lambda_{0j}=\lambda_{j0}=\big(s_0(\beta)(b)-s_0(\alpha)(a)\big)^{\frac12}\big(s_j(\beta)(b)-s_j(\beta)(a)\big)^{\frac12}\quad(j=1,2),\\ \lambda_{mj}=\frac12\left(\mu_{\alpha_m}\nu_{\alpha_m alpha_j}\right)^{\frac12}\sin^{-1}\frac{\pi}{4n_{\alpha_m+2}+2}\quad(m,j=1,2).\end{gathered}$$

Then problem (1), (2) *has one and only one solution.*

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