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The weighted cauchy problem for nonlinear singular differential equations with deviating arguments

Bedřich Půža, Zaza Sokhadze

Masaryk University, Institute of Mathematics of the Academy of Sciences of the Czech Republic, Brno, Czech Republic

Akaki Tsereteli State University, Kutaisi, Georgia

e-mail: puza@math.muni.cz, e-mail: z.soxadze@gmail.com

Let $-\infty < a < b < +\infty$, n be a natural number, $f\colon\ ]a,b[\,\times\mathbb{R}^n \to \mathbb{R}$ be a function, satisfying the local Carathéodory conditions, $\tau_i\colon\ ]a,b[\,\to\,]a,b]$ $(i = 1,\dots,n)$ be measurable functions, and $\rho\colon [a,b] \to [0,+\infty[$ be the $(n-1)$-times continuously differentiable function such that

$$\rho^{(i-1)}(a) = 0, \qquad \rho^{(i-1)}(t) > 0 \quad \text{for } a < t \le b \ (i = 1,\dots,n).$$

In the interval $]a,b[$ consider the differential equation

$$u^{(n)}(t) = f\Big(t, u(\tau_1(t)),\dots,u^{(n-1)}(\tau_n(t))\Big) \tag{1}$$

with the weighted initial conditions

$$\limsup_{t\to a}\left(\frac{|u^{(i-1)}(t)|}{\rho^{(i-1)}(t)}\right) < +\infty \quad (i = 1,\dots,n). \tag{2}$$

Theorem 1. *Let in the domain* $]a,b[\,\times\mathbb{R}^n$ *the condition*

$$\big|f(t,x_1,\dots,x_n)\big| \le \sum_{i=1}^n h_i(t)|x_i| + h_0(t)$$

hold, where $h_0 \in L([a,b])$ *and* $h_i \in L_{loc}(]a,b])$ $(i = 1,\dots,n)$ *are nonnegative functions. Let, moreover,*

$$\sup\left\{\left(\int_a^t h_0(s)\,ds\right)\Big/\rho^{(n-1)}(t):\ a < t \le b\right\} < +\infty \tag{3}$$

and there exist a number $\gamma \in\,]0,1[$ *such that*

$$\sum_{i=1}^n \int_a^t \rho^{(i-1)}(\tau_i(s))h_i(s)\,ds \le \gamma\rho^{(n-1)}(t) \quad \textit{for } a < t \le b. \tag{4}$$

Then the problem (1), (2) *has at least one solution.*

Theorem 2. *Let in the domain* $]a,b[\,\times\mathbb{R}^n$ *the condition*

$$\big|f(t,x_1,\dots,x_n)-f(t,y_1,\dots,y_n)\big|\le\sum_{i=1}^n h_i(t)|x_i-y_i|$$

be fulfilled, where $h_i\in L_{loc}(]a,b])$ $(i=1,\dots,n)$ *are nonnegative functions. Let, moreover, the inequalities* (3) *and* (4) *hold, where* $h_0(t)=|f(t,0,\dots,0)|$ *and* $\gamma\in\,]0,1[$. *Then the problem* (1), (2) *has one and only one solution.*

Theorem 3. *Let in the domain* $]a,b[\,\times\mathbb{R}^n$ *the inequality*

$$f(t,x_1,\dots,x_n)\ge\sum_{i=1}^n h_i(t)|x_i|+h_0(t)$$

hold, where $h_0\in L([a,b])$ *is a function satisfying the condition*

$$\inf\left\{\left(\int_a^t h_0(s)\,ds\right)\Big/\rho^{(n-1)}(t):\ a<t\le b\right\}>0$$

and $h_i\in L_{loc}(]a,b])$ $(i=1,\dots,n)$ *are nonnegative functions such that*

$$\sum_{i=1}^n\int_a^t \rho^{(i-1)}(\tau_i(s))h_i(s)\,ds\ge\rho^{(n-1)}(t)\quad\textit{for } a<t\le b.$$

Then the problem (1), (2) *has no solution.*

The above-formulated theorems cover the case in which the equation (1) is strongly singular, i.e., the case, where

$$\int_a^b (t-a)^{*(t,x)\,dt=+\infty}\qquad\text{for } \mu\ge 0,\ x>0,$$

where

$$f^*(t,x)=\max\left\{\big|f(t,x_1,\dots,x_n)\big|:\ \sum_{i=1}^n|x_i|\le x\right\}.$$

On the other hand, from Theorem 3 follows that the condition $\gamma\in\,]0,1[$ in Theorems 1 and 2 is unimprovable and it cannot be replaced by the condition $\gamma=1$.

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