Complementarity based a posteriori error estimates

Tomáš Vejchodský

Institute of Mathematics, Academy of Sciences
Žitná 25, 115 67 Praha 1
Czech Republic

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Introduction

2000- S. Repin (S. Korotov, J. Valdman, S. Sauter, M. Frolov, . . .)

M. Vohralík (R. Fučík, I. Cheddadi, M.I. Prieto, . . .)

1976- I. Hlaváček (M. Křížek, J. Vacek, J. Weisz, . . .)

1971 J.P. Aubin and H.G. Burchard

1957 J.L. Synge

Outline

- Primal problem:

$$-\Delta u + \kappa^2 u = f \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega$$

- Dual problem:

$$-\nabla \operatorname{div} \mathbf{y} + \kappa^2 \mathbf{y} = \nabla f \quad \text{in } \Omega, \qquad -\operatorname{div} \mathbf{y} = f \quad \text{on } \partial\Omega$$

- Error estimate:

$$|||u - u_h||| \leq \eta(u_h, \mathbf{y}_h) \quad \forall u_h \in V \quad \forall \mathbf{y}_h \in \mathbf{H}(\operatorname{div}, \Omega)$$

$$\eta^2(u_h, \mathbf{y}_h) = \left\| \kappa^{-1}(f - \kappa^2 u_h + \operatorname{div} \mathbf{y}_h) \right\|_0^2 + \|\mathbf{y}_h - \nabla u_h\|_0^2$$

- Case $\kappa = 0$
- Numerical examples

Primal Problem

Strong form.:
$$-\Delta u + \kappa^2 u = f \quad \text{in } \Omega \subset \mathbb{R}^2$$
$$u = 0 \quad \text{on } \partial\Omega$$

Weak form.: $u \in V: \quad \mathcal{B}(u,v) = \mathcal{F}(v) \quad \forall v \in V$

Lemma: $f \in L^2(\Omega) \quad \Rightarrow \quad \boldsymbol{\nabla} u \in \mathbf{H}(\mathrm{div},\Omega)$

Notation:

- $V = H_0^1(\Omega)$
- $\mathcal{B}(u,v) = \int_\Omega \boldsymbol{\nabla} u \cdot \boldsymbol{\nabla} v \,\mathrm{d}x + \int_\Omega \kappa^2 uv \,\mathrm{d}x$
- $\mathcal{F}(v) = \int_\Omega fv \,\mathrm{d}x$
- $\|v\|^2 = \mathcal{B}(v,v)$

Derivation of the estimate ($\kappa > 0$)

Divergence theorem: $v \in H^1(\Omega) \quad \mathbf{y} \in \mathbf{H}(\mathrm{div}, \Omega)$

$$\int_\Omega v \,\mathrm{div}\,\mathbf{y} \,\mathrm{d}x + \int_\Omega \mathbf{y} \cdot \nabla v \,\mathrm{d}x - \int_{\partial\Omega} v\mathbf{y} \cdot \mathbf{n} \,\mathrm{d}x = 0$$

$$\begin{aligned}
\mathcal{B}(u - u_h, v) &= \int_\Omega fv \,\mathrm{d}x - \int_\Omega \nabla u_h \cdot \nabla v \,\mathrm{d}x - \int_\Omega \kappa^2 u_h v \,\mathrm{d}x \qquad v \in V \\
&\quad + \int_\Omega v \,\mathrm{div}\,\mathbf{y} \,\mathrm{d}x + \int_\Omega \mathbf{y} \cdot \nabla v \,\mathrm{d}x \\
&= \int_\Omega \kappa^{-1}(f - \kappa^2 u_h + \mathrm{div}\,\mathbf{y})\kappa v \,\mathrm{d}x + \int_\Omega (\mathbf{y} - \nabla u_h) \cdot \nabla v \,\mathrm{d}x \\
&\le \left\| \kappa^{-1}(f - \kappa^2 u_h + \mathrm{div}\,\mathbf{y}) \right\|_0 \|\kappa v\|_0 + \|\mathbf{y} - \nabla u_h\|_0 \|\nabla v\|_0 \\
&\le \left(\left\| \kappa^{-1}(f - \kappa^2 u_h + \mathrm{div}\,\mathbf{y}) \right\|_0^2 + \|\mathbf{y} - \nabla u_h\|_0^2 \right)^{1/2} |||v|||
\end{aligned}$$

$$|||u - u_h||| \le \left(\left\| \kappa^{-1}(f - \kappa^2 u_h + \mathrm{div}\,\mathbf{y}) \right\|_0^2 + \|\mathbf{y} - \nabla u_h\|_0^2 \right)^{1/2}$$

The estimator ($\kappa > 0$)

Definition: $\eta^2(u_h, \mathbf{y}_h) = \left\|\kappa^{-1}(f - \kappa^2 u_h + \operatorname{div} \mathbf{y}_h)\right\|_0^2 + \|\mathbf{y}_h - \nabla u_h\|_0^2$

Theorem: $\|u - u_h\| \le \eta(u_h, \mathbf{y}_h) \qquad \forall u_h \in V \quad \forall \mathbf{y}_h \in \mathbf{H}(\operatorname{div}, \Omega)$

Dual problem:

(A) Find $\mathbf{y} \in \mathbf{W} : \eta(u_h, \mathbf{y}) \le \eta(u_h, \mathbf{w}) \quad \forall \mathbf{w} \in \mathbf{W}$

(B) Find $\mathbf{y} \in \mathbf{W} : \mathcal{B}^*(\mathbf{y}, \mathbf{w}) = \mathcal{F}^*(\mathbf{w}) \quad \forall \mathbf{w} \in \mathbf{W}$

- $\mathbf{W} = \mathbf{H}(\operatorname{div}, \Omega)$
- $\mathcal{B}^*(\mathbf{y}, \mathbf{w}) = \int_\Omega \kappa^{-2} \operatorname{div} \mathbf{y} \operatorname{div} \mathbf{w} \, \mathrm{d}x + \int_\Omega \mathbf{y} \cdot \mathbf{w} \, \mathrm{d}x$
- $\mathcal{F}^*(\mathbf{w}) = -\int_\Omega \kappa^{-2} f \operatorname{div} \mathbf{w} \, \mathrm{d}x$
- $\|\mathbf{w}\|_*^2 = \mathcal{B}^*(\mathbf{w}, \mathbf{w})$

Properties

- (A) $\Leftrightarrow$ (B)
- $\exists ! \mathbf{y} \in \mathbf{W}$
- $\mathbf{y} = \boldsymbol{\nabla} u$
- $\eta(u_h, \mathbf{y}) = \|u - u_h\|$
- $\eta(u, \mathbf{y}_h) = \|\mathbf{y} - \mathbf{y}_h\|_*$
- $\eta^2(u_h, \mathbf{y}) + \eta^2(u, \mathbf{y}_h) = \eta^2(u_h, \mathbf{y}_h)$
- $\|u - u_h\|^2 + \|\mathbf{y} - \mathbf{y}_h\|_*^2 = \eta^2(u_h, \mathbf{y}_h)$

u is primal solution; $\mathbf{y}$ is dual solution; $u_h \in V$; $\mathbf{y}_h \in \mathbf{W}$

Method of hypercircle ($\kappa > 0$)

Theorem: If

- $u \in V$ is primal solution
- $u_h \in V$, $\mathbf{y}_h \in \mathbf{W}$ arbitrary
- $\bar{u}_h = [\kappa^{-2}(f + \operatorname{div} \mathbf{y}_h) + u_h]/2$
- $\mathcal{G}\bar{u}_h = (\mathbf{y}_h + \nabla u_h)/2$

Then

$$\|\nabla u - \mathcal{G}\bar{u}_h\|_0^2 + \|\kappa(u - \bar{u}_h)\|_0^2 = \frac{1}{4}\eta^2(u_h, \mathbf{y}_h).$$

Poisson problem ($\kappa \geq 0$)

I. Error majorants (Friedrichs' inequality)

II. Complementary approach (dual finite elements)

I. Error majorants ($\kappa \geq 0$)

Friedrichs' inequality: $\|v\|_0 \leq C_\Omega \|\nabla v\|_0 \quad \forall v \in V$

Remark: $C_\Omega \leq \frac{1}{\pi}\left(\frac{1}{|a_1|} + \cdots + \frac{1}{|a_d|}\right)^{-1/2}, \quad \Omega \subset a_1 \times \cdots \times a_d$

$$\begin{aligned}\mathcal{B}(u - u_h, v) &= \int_\Omega (f - \kappa^2 u_h + \operatorname{div} \widehat{\mathbf{y}}) v \, \mathrm{d}x + \int_\Omega (\widehat{\mathbf{y}} - \nabla u_h) \cdot \nabla v \, \mathrm{d}x \\ &\leq \left(C_\Omega \left\| f - \kappa^2 u_h + \operatorname{div} \widehat{\mathbf{y}} \right\|_0 + \|\widehat{\mathbf{y}} - \nabla u_h\|_0 \right) \|\!| v |\!\|\end{aligned}$$

$$\|\!| u - u_h |\!\| \leq \widehat{\eta}(u_h, \widehat{\mathbf{y}}) \quad \forall \widehat{\mathbf{y}} \in \mathbf{W}$$

$$\widehat{\eta}(u_h, \widehat{\mathbf{y}}) = C_\Omega \left\| f - \kappa^2 u_h + \operatorname{div} \widehat{\mathbf{y}} \right\|_0 + \|\widehat{\mathbf{y}} - \nabla u_h\|_0$$

$$\widehat{\eta}^2(u_h, \widehat{\mathbf{y}}) \leq 2 C_\Omega^2 \left\| f - \kappa^2 u_h + \operatorname{div} \widehat{\mathbf{y}} \right\|_0^2 + 2 \|\widehat{\mathbf{y}} - \nabla u_h\|_0^2$$

II. Complementarity approach ($\kappa \geq 0$)

Definition:

$$\mathbf{Q}(f, u_h) = \left\{ \mathbf{y} \in [L^2(\Omega)]^d : \int_\Omega \mathbf{y} \cdot \nabla v \,\mathrm{d}x = \int_\Omega (f - \kappa^2 u_h) v \,\mathrm{d}x \quad \forall v \in V \right\}$$

Definition: $\tilde{\eta}(u_h, \tilde{\mathbf{y}}_h) = \|\tilde{\mathbf{y}}_h - \nabla u_h\|_0$ for $\tilde{\mathbf{y}}_h \in \mathbf{Q}(f, u_h)$

Theorem: $|\!|\!| u - u_h |\!|\!| \leq \tilde{\eta}(u_h, \tilde{\mathbf{y}}_h)$

Dual problem:

(A) Find $\tilde{\mathbf{y}} \in \mathbf{Q}(f, u_h) : \tilde{\eta}(u_h, \tilde{\mathbf{y}}) \leq \tilde{\eta}(u_h, \tilde{\mathbf{w}}) \quad \forall \tilde{\mathbf{w}} \in \mathbf{Q}(f, u_h)$

(B) Find $\tilde{\mathbf{y}} \in \mathbf{Q}(f, u_h) : \|\tilde{\mathbf{y}}\|_0 \leq \|\tilde{\mathbf{w}}\|_0 \quad \forall \tilde{\mathbf{w}} \in \mathbf{Q}(f, u_h)$

(C) Find $\tilde{\mathbf{y}} \in \mathbf{Q}(f, u_h) : \int_\Omega \tilde{\mathbf{y}} \cdot \tilde{\mathbf{w}} \,\mathrm{d}x = 0 \quad \forall \tilde{\mathbf{w}} \in \mathbf{Q}_0$

- $\mathbf{Q}_0 = \mathbf{Q}(0, 0)$

II. Complementarity: Properties ($\kappa \geq 0$)

- (A) $\Leftrightarrow$ (B) $\Leftrightarrow$ (C)
- $\exists ! \widetilde{\mathbf{y}} \in \mathbf{Q}(f, u_h)$
- $\widetilde{\mathbf{y}} = \nabla \tilde{z}$, where
  $$\tilde{z} \in V : \int_\Omega \nabla \tilde{z} \cdot \nabla v \, \mathrm{d}x = \int_\Omega (f - \kappa^2 u_h) v \, \mathrm{d}x \quad \forall v \in V$$
- $\|\widetilde{\mathbf{y}}_h - \nabla u\|_0^2 + \|u - u_h\|_\sim^2 = \widetilde{\eta}^2(u_h, \widetilde{\mathbf{y}}_h)$
  $$\|v\|_\sim^2 = \|v\|^2 + \|\kappa v\|_0^2 = \|\nabla v\|_0^2 + 2\|\kappa v\|_0^2 \quad \forall v \in V$$

u is primal solution; $\widetilde{\mathbf{y}}$ is dual solution; $u_h \in V$; $\widetilde{\mathbf{y}}_h \in \mathbf{Q}(f, u_h)$

II. Complementarity: Properties ($\kappa = 0$)

- $\tilde{z} = u, \quad \|v\|_\sim = \|v\|$
- $\widetilde{\mathbf{y}} = \boldsymbol{\nabla} u$
- $\widetilde{\eta}(u_h, \widetilde{\mathbf{y}}) = \|u - u_h\|$
- $\widetilde{\eta}(u, \widetilde{\mathbf{y}}_h) = \|\widetilde{\mathbf{y}} - \widetilde{\mathbf{y}}_h\|_0$
- $\|\widetilde{\mathbf{y}}_h - \widetilde{\mathbf{y}}\|_0^2 + \|u - u_h\|^2 = \widetilde{\eta}^2(u_h, \widetilde{\mathbf{y}}_h) \quad \forall \widetilde{\mathbf{y}}_h \in \mathbf{Q}(f)$
- $\widetilde{\eta}^2(u, \widetilde{\mathbf{y}}_h) + \widetilde{\eta}^2(u_h, \widetilde{\mathbf{y}}) = \widetilde{\eta}^2(u_h, \widetilde{\mathbf{y}}_h) \quad \forall \widetilde{\mathbf{y}}_h \in \mathbf{Q}(f)$

u is primal solution; $\widetilde{\mathbf{y}}$ is dual solution; $u_h \in V$; $\widetilde{\mathbf{y}}_h \in \mathbf{Q}(f)$

II. Complementarity: Method of hypercircle ($\kappa = 0$)

Theorem: If

- $u \in V$ is primal solution
- $u_h \in V$, $\widetilde{\mathbf{y}}_h \in \mathbf{Q}(f)$ arbitrary

Then

$$\left\| \nabla u - \frac{\widetilde{\mathbf{y}}_h + \nabla u_h}{2} \right\|_0 = \underbrace{\frac{1}{2} \left\| \widetilde{\mathbf{y}}_h - \nabla u_h \right\|_0}_{\frac{1}{2} \widetilde{\eta}(u_h, \widetilde{\mathbf{y}}_h)}$$

Galerkin solutions

Approx. primal: $u_h \in V_h : \mathcal{B}(u_h, v_h) = \mathcal{F}(v_h) \quad \forall v_h \in V_h$

Approx. dual 1: $\mathbf{y}_h^{p_*} \in \mathbf{W}_h^{p_*} : \mathcal{B}^*(\mathbf{y}_h^{p_*}, \mathbf{w}_h) = \mathcal{F}^*(\mathbf{w}_h) \quad \forall \mathbf{w}_h \in \mathbf{W}_h^{p_*}$

Approx. dual 2: $\widehat{\mathbf{y}}_h^{p_*} \in \mathbf{W}_h^{p_*} : \widehat{\mathcal{B}}^*(\widehat{\mathbf{y}}_h^{p_*}, \mathbf{w}_h) = \widehat{\mathcal{F}}^*(\mathbf{w}_h) \quad \forall \mathbf{w}_h \in \mathbf{W}_h^{p_*}$

- $V_h = \{v \in V : v|_K \in P^1(K),\ K \in \mathcal{T}_h\}$
- $\mathbf{W}_h^{p_*} = \left\{\mathbf{w} \in \mathbf{W} : \mathbf{w}|_K \in [P^{p_*}(K)]^d,\ K \in \mathcal{T}_h\right\}$
- $\widehat{\mathcal{B}}^*(\mathbf{y}, \mathbf{w}) = C_\Omega^2 \int_\Omega \operatorname{div} \mathbf{y} \operatorname{div} \mathbf{w} \,\mathrm{d}x + \int_\Omega \mathbf{y} \cdot \mathbf{w} \,\mathrm{d}x$

  $\widehat{\mathcal{F}}^*(\mathbf{w}) = -C_\Omega^2 \int_\Omega f \operatorname{div} \mathbf{w} \,\mathrm{d}x$

Galerkin solutions

Dual problem 3: $\widetilde{\mathbf{y}} = \mathbf{q} + \mathbf{curl}\, z \in \mathbf{Q}(f, u_h)$:

$$\int_\Omega \widetilde{\mathbf{y}} \cdot \widetilde{\mathbf{w}} \,\mathrm{d}x = 0 \quad \forall \widetilde{\mathbf{w}} \in \mathbf{Q}_0$$

$$z \in H^1(\Omega) \ : \ \int_\Omega \mathbf{curl}\, z \cdot \mathbf{curl}\, v \,\mathrm{d}x = -\int_\Omega \mathbf{q} \cdot \mathbf{curl}\, v \,\mathrm{d}x \quad \forall v \in H^1(\Omega)$$

Approx. dual 3: $\widetilde{\mathbf{y}}_h^p = \mathbf{q} + \mathbf{curl}\, z_h$:

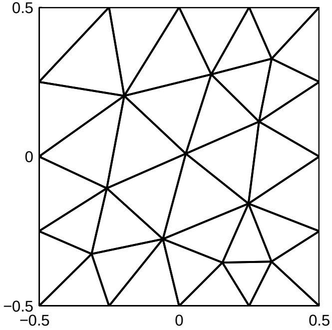
$$z_h \in \widetilde{V}_h^p \ : \ \int_\Omega \nabla z_h \cdot \nabla v_h \,\mathrm{d}x = -\int_\Omega \mathbf{q} \cdot \mathbf{curl}\, v_h \,\mathrm{d}x \quad \forall v_h \in \widetilde{V}_h^p$$

- $\mathbf{q} = -\mathbf{F} + \kappa^2 \mathbf{U}_h$: $\quad \operatorname{div} \mathbf{q} = -f + \kappa^2 u_h$
- $\mathbf{Q}(f, u_h) = \mathbf{q} + \mathbf{Q}_0, \quad \mathbf{Q}_0 = \mathbf{curl}\, H^1(\Omega), \quad \mathbf{curl} = (\partial_2, -\partial_1)^\top$
- $\widetilde{V}_h^p = \{v \in H^1(\Omega) : v|_K \in P^p(K),\ K \in \mathcal{T}_h\}$

Test 1

$$-\Delta u + \kappa^2 u = f \quad \text{in } \Omega$$
$$u = 0 \quad \text{on } \partial\Omega$$

- $\Omega = (-1/2, 1/2)^2$
- $f = \cos(\pi x_1)\cos(\pi x_2)$
- $u = \dfrac{\cos(\pi x_1)\cos(\pi x_2)}{\pi^2 + \kappa^2}$
- $C_\Omega = (\pi\sqrt{2})^{-1}$



Test 1: I_{eff} with $p_* = 1$, $p = 1$

κ	$\eta\left(u_h, \mathbf{y}_h^1\right)$	$\widehat{\eta}\left(u_h, \widehat{\mathbf{y}}_h^1\right)$	$\widetilde{\eta}\left(u_h, \widetilde{\mathbf{y}}_h^1\right)$	η^{comb}
0	—	1.782	1.410	1.782
10^{-3}	$3.513 \cdot 10^3$	1.782	1.410	1.782
10^{-2}	$3.513 \cdot 10^2$	1.782	1.409	1.782
10^{-1}	$3.514 \cdot 10^1$	1.782	1.429	1.782
1	3.650	1.784	$5.041 \cdot 10^1$	1.784
10	1.058	1.889	$5.343 \cdot 10^3$	1.058
10^2	1.001	$2.219 \cdot 10^1$	$9.066 \cdot 10^4$	1.001
10^3	1.000	$2.292 \cdot 10^2$	$1.458 \cdot 10^6$	1.000
10^4	1.000	$2.293 \cdot 10^3$	$1.705 \cdot 10^7$	1.000
10^5	1.000	$2.293 \cdot 10^4$	$1.359 \cdot 10^8$	1.000
10^6	1.000	$2.293 \cdot 10^5$	$1.112 \cdot 10^9$	1.000

$$\eta^{comb} = \begin{cases} \min\{\eta(u_h, \mathbf{y}_h), \widehat{\eta}(u_h, \mathbf{y}_h)\} & \text{for } C_\Omega^2 \kappa^2 \geq 1, \\ \min\{\eta(u_h, \widehat{\mathbf{y}}_h), \widehat{\eta}(u_h, \widehat{\mathbf{y}}_h)\} & \text{otherwise,} \end{cases}$$

Test 1: I_{eff} with $p_* = 2$, $p = 2, 3$

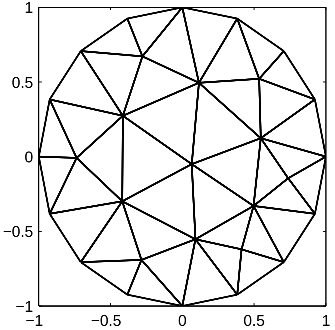
κ	$\eta\left(u_h, \mathbf{y}_h^2\right)$	$\widehat{\eta}\left(u_h, \widehat{\mathbf{y}}_h^2\right)$	$\widetilde{\eta}\left(u_h, \widetilde{\mathbf{y}}_h^2\right)$	$\widetilde{\eta}\left(u_h, \widetilde{\mathbf{y}}_h^3\right)$	η^{comb}
0	—	1.161	1.008	1.000	1.161
10^{-3}	$4.937 \cdot 10^2$	1.161	1.008	1.000	1.161
10^{-2}	$4.939 \cdot 10^1$	1.161	1.009	1.000	1.161
10^{-1}	5.038	1.161	1.036	1.000	1.161
1	1.115	1.166	$4.496 \cdot 10^1$	1.003	1.166
10	1.001	1.640	$4.763 \cdot 10^3$	1.131	1.001
10^2	1.000	$1.732 \cdot 10^1$	$8.082 \cdot 10^4$	5.752	1.000
10^3	1.000	$1.771 \cdot 10^2$	$1.300 \cdot 10^6$	$5.744 \cdot 10^1$	1.000
10^4	1.000	$1.771 \cdot 10^3$	$1.520 \cdot 10^7$	$5.744 \cdot 10^2$	1.000
10^5	1.000	$1.771 \cdot 10^4$	$1.212 \cdot 10^8$	$5.744 \cdot 10^3$	1.000
10^6	1.000	$1.771 \cdot 10^5$	$9.908 \cdot 10^8$	$5.744 \cdot 10^4$	1.000

Test 2

$$-\Delta u + \kappa^2 u = f \quad \text{in } \Omega$$
$$u = 0 \quad \text{on } \partial\Omega$$

- $\Omega = \{(x_1, x_2) : r < 1\}$
- $f = 1 \qquad r = \sqrt{x_1^2 + x_2^2}$
- $u = \dfrac{1}{\kappa^2}\left(1 - \dfrac{I_0(\kappa r)}{I_0(\kappa)}\right) \quad \text{for } \kappa > 0$

  $u = \dfrac{1 - x_1^2 - x_2^2}{4} \quad \text{for } \kappa = 0$
- $C_\Omega = 1/\pi$



Test 2: I_{eff} with $p_* = 1$, $p = 1$

κ	$\eta\left(u_h, \mathbf{y}_h^1\right)$	$\widehat{\eta}\left(u_h, \widehat{\mathbf{y}}_h^1\right)$	$\widetilde{\eta}\left(u_h, \widetilde{\mathbf{y}}_h^1\right)$	η^{comb}
0	—	1.092	1.708	1.092
10^{-3}	1.000	1.092	1.708	1.092
10^{-2}	1.000	1.092	1.708	1.092
10^{-1}	1.001	1.092	1.711	1.092
1	1.086	1.166	7.789	1.148
10	1.223	3.712	$7.051 \cdot 10^1$	1.223
10^2	1.021	$2.641 \cdot 10^1$	$4.406 \cdot 10^2$	1.021
10^3	1.000	$2.579 \cdot 10^2$	$6.811 \cdot 10^3$	1.000
10^4	1.000	$2.579 \cdot 10^3$	$6.739 \cdot 10^4$	1.000
10^5	1.000	$2.579 \cdot 10^4$	$9.389 \cdot 10^5$	1.000
10^6	1.000	$2.579 \cdot 10^5$	$8.363 \cdot 10^6$	1.000

Test 2: I_{eff} with $p_* = 2$, $p = 2, 3$

κ	$\eta\left(u_h, \mathbf{y}_h^2\right)$	$\widehat{\eta}\left(u_h, \widehat{\mathbf{y}}_h^2\right)$	$\widetilde{\eta}\left(u_h, \widetilde{\mathbf{y}}_h^2\right)$	$\widetilde{\eta}\left(u_h, \widetilde{\mathbf{y}}_h^3\right)$	η^{comb}
0	—	1.083	1.000	0.978	1.083
10^{-3}	0.978	1.083	1.000	0.978	1.083
10^{-2}	0.978	1.083	1.000	0.978	1.083
10^{-1}	0.978	1.083	1.002	0.978	1.083
1	0.976	1.093	6.642	0.978	1.049
10	1.013	1.674	$6.098 \cdot 10^1$	1.402	1.013
10^2	1.011	9.805	$3.821 \cdot 10^2$	8.219	1.011
10^3	1.000	$9.539 \cdot 10^1$	$5.906 \cdot 10^3$	$7.996 \cdot 10^1$	1.000
10^4	1.000	$9.539 \cdot 10^2$	$5.845 \cdot 10^4$	$7.996 \cdot 10^2$	1.000
10^5	1.000	$9.539 \cdot 10^3$	$8.100 \cdot 10^5$	$7.996 \cdot 10^3$	1.000
10^6	1.000	$9.539 \cdot 10^4$	$7.250 \cdot 10^6$	$7.996 \cdot 10^4$	1.000

Postprocessing: local algorithm for $\mathbf{y}_h$

$$\mathcal{B}(u-u_h,v) = \int_\Omega (f - \kappa^2 u_h + \operatorname{div}\mathbf{y}) v \,\mathrm{d}x + \int_\Omega (\mathbf{y} - \nabla u_h)\cdot \nabla v \,\mathrm{d}x$$

$$= \sum_{K\in\mathcal{T}_h} \left[\int_K \kappa^{-1} \left(f - \kappa^2 u_h + \operatorname{div}\mathbf{y}_K\right) \kappa v \,\mathrm{d}x + \int_K (\mathbf{y}_K - \nabla u_h)\cdot \nabla v \,\mathrm{d}x \right]$$

$$|\!|\!| u - u_h |\!|\!|^2 \le \left(\eta^{loc}(u_h,\mathbf{y}_K) \right)^2 \equiv \sum_{K\in\mathcal{T}_h} \eta_K^2(u_h,\mathbf{y}_K)$$

$$\eta_K^2(u_h,\mathbf{y}_K) = \left\| \kappa^{-1}(f - \kappa^2 u_h + \operatorname{div}\mathbf{y}_K) \right\|_{0,K}^2 + \|\mathbf{y}_K - \nabla u_h\|_{0,K}^2$$

- $\mathbf{y}_K \cdot \mathbf{n}_K + \mathbf{y}_{K^*}\cdot \mathbf{n}_{K^*} = 0 \quad$ for $K \cap K^* = \gamma$
- $\mathbf{y}_K$ constructed form ∇u_h by
  - equilibration of residuals (κ small)
  - robust modification (κ big)

Test 1: I_{eff} η vs. η^{loc}

κ	$\eta\left(u_h, \mathbf{y}_h^1\right)$	$\eta^{loc}\left(u_h, \mathbf{y}_K^1\right)$	$\eta\left(u_h, \mathbf{y}_h^2\right)$	$\eta^{loc}\left(u_h, \mathbf{y}_K^2\right)$
0	—	—	—	—
10^{-3}	$3.513 \cdot 10^3$	$3.513 \cdot 10^3$	$4.937 \cdot 10^2$	$4.937 \cdot 10^2$
10^{-2}	$3.513 \cdot 10^2$	$3.513 \cdot 10^2$	$4.939 \cdot 10^1$	$4.939 \cdot 10^1$
10^{-1}	$3.514 \cdot 10^1$	$3.516 \cdot 10^1$	5.038	5.12
1	3.650	3.78	1.115	1.46
10	1.058	1.60	1.001	1.49
10^2	1.001	1.52	1.000	1.24
10^3	1.000	1.37	1.000	1.17
10^4	1.000	1.35	1.000	1.17
10^5	1.000	1.35	1.000	1.17
10^6	1.000	1.35	1.000	1.17

Test 2: I_{eff} η vs. η^{loc}

κ	$\eta\left(u_h, \mathbf{y}_h^1\right)$	$\eta^{loc}\left(u_h, \mathbf{y}_K^1\right)$	$\eta\left(u_h, \mathbf{y}_h^2\right)$	$\eta^{loc}\left(u_h, \mathbf{y}_K^2\right)$
0	—	—	—	—
10^{-3}	1.000	1.05	0.978	1.05
10^{-2}	1.000	1.05	0.978	1.05
10^{-1}	1.001	1.05	0.978	1.05
1	1.086	1.14	0.976	1.05
10	1.223	1.85	1.013	1.54
10^{2}	1.021	1.64	1.011	1.37
10^{3}	1.000	1.66	1.000	1.41
10^{4}	1.000	1.67	1.000	1.42
10^{5}	1.000	1.67	1.000	1.42
10^{6}	1.000	1.67	1.000	1.42

Conclusions

$$-\Delta u + \kappa^2 u = f \quad \text{in } \Omega$$
$$u = 0 \quad \text{on } \partial\Omega$$

$$\eta^2(u_h, \mathbf{y}_h) = \left\| \kappa^{-1}(f - \kappa^2 u_h + \operatorname{div} \mathbf{y}_h) \right\|_0^2 + \|\mathbf{y}_h - \nabla u_h\|_0^2 \quad \forall \mathbf{y}_h \in \mathbf{W}$$

$$\widehat{\eta}(u_h, \widehat{\mathbf{y}}_h) = C_\Omega \left\| f - \kappa^2 u_h + \operatorname{div} \widehat{\mathbf{y}}_h \right\|_0 + \|\widehat{\mathbf{y}}_h - \nabla u_h\|_0 \quad \forall \widehat{\mathbf{y}}_h \in \mathbf{W}$$

$$\widetilde{\eta}(u_h, \widetilde{\mathbf{y}}_h) = \|\widetilde{\mathbf{y}}_h - \nabla u_h\|_0 \quad \forall \widetilde{\mathbf{y}}_h \in \mathbf{Q}(f, u_h)$$

- Guaranteed upper bounds
- $\mathbf{y}_h$, $\widehat{\mathbf{y}}_h$, $\widetilde{\mathbf{y}}_h$ approximations of dual problems
- Postprocessing: fast algorithms for $\mathbf{y}_h$, $\widehat{\mathbf{y}}_h$, $\widetilde{\mathbf{y}}_h$
- Efficient and robust (for $\kappa \to \infty$, $\kappa \to 0$, $h \to 0$)
- $u_h \in V$ arbitrary (including algebraic error)

Thank you for your attention

Tomáš Vejchodský

Institute of Mathematics, Academy of Sciences
Žitná 25, 115 67 Praha 1
Czech Republic

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