Czech-Georgian Workshop on Boundary Value Problems
Institute of Mathematics, Academy of Sciences of the Czech Republic
5–9 December, 2011, Branch in Brno

Positive solutions of two-point boundary value problems for nonlinear differential equations with strong singularities

Ivan Kiguradze

A. Razmadze Mathematical Institute of I. Javakhishvili Tbilisi State University, Tbilisi, Georgia

e-mail: kig@rmi.ge

Let $-\infty < a < b < +\infty$, m be an arbitrary natural number, and $f\colon \,]a,b[\,\times\,]0,+\infty[\,\to \mathbb{R}$ be a continuous function. In the open interval $]a,b[$, we consider the nonlinear differential equation

$$u^{(2m)} = f(t,u) \tag{1}$$

with the boundary conditions of one of the following two types:

$$\lim_{t\to a} u^{(i-1)}(t) = 0, \quad \lim_{t\to b} u^{(i-1)}(t) = 0 \quad (i = 1,\dots,m); \tag{2}$$

$$\lim_{t\to a} u^{(i-1)}(t) = 0, \quad \lim_{t\to b} u^{(m+i-1)}(t) = 0 \quad (i = 1,\dots,m). \tag{3}$$

By $C^{2m,m}(]a,b[)$ we denote the space of $2m$-times continuously differentiable functions $u\colon \,]a,b[\,\to \mathbb{R}$, satisfying the condition $\int_a^b |u^{(m)}(t)|^2\,dt < +\infty$.

Theorem 1. *Let in the domain* $]a,b[\,\times\,]0,+\infty[$ *the inequality*

$$0 \le (-1)^m f(t,x) - h(t)x^\mu \le \ell\big((t-a)^{-2m} + (b-t)^{-2m}\big)x + q(t,x)$$

be satisfied, where $\mu \in [0,1[$ *and* $\ell \ge 0$ *are constants,* $h\colon \,]a,b[\,\to [0,+\infty[$ *is a continuous function, and* $q\colon \,]a,b[\,\times\,]0,+\infty[\,\to [0,+\infty[$ *is a continuous and nonincreasing in the second argument function. If, moreover,*

$$\ell < 4^{-m}\big[(2m-1)!!\big]^2, \tag{4}$$

$$h(t) \not\equiv 0, \qquad \int_a^b \big[(t-a)(b-t)\big]^{(1+\mu)(m-\frac{1}{2})} h(t)\,dt < +\infty,$$

$$\int_a^b \big[(t-a)(b-t)\big]^{m-\frac{1}{2}} q\big(t,(t-a)^m(b-t)^m x\big)\,dt < +\infty \quad \textit{for } x > 0,$$

then problem (1), (2) *in the space* $C^{2m,m}(]a,b[)$ *has at least one positive solution.*

Unlike the previous well-known results the Theorem 1 cover the case where equation (1), along with **strong singularities** with respect to the time variable at the points a and b, has **strong singularity** with respect to the phase variable, as well, i.e. the case where

$$\int_a^{t_0} (t-a)^{2m-1}|f(t,x)|\,dt = \int_{t_0}^{b} (t-a)^{2m-1}|f(t,x)|\,dt = +\infty \quad \text{for } a<t_0<b,\ x>0,$$

$$\limsup_{x\to 0}\big(x^k|f(t,x)|\big) = +\infty \quad \text{for arbitrary } t\in\,]a,b[\text{ and } k>0.$$

Theorem 2. *If*

$$(-1)^m\big[f(t,x)-f(t,y)\big] \le \ell\big((t-a)^{-2m}+(b-t)^{-2m}\big)(x-y) \quad \textit{for } a<t<b,\ x>y>0,$$

where ℓ is a nonnegative constant, satisfying (4)*, then problem* (1), (2) *in the space $C^{2m,m}(]a,b[)$ has at most one positive solution.*

As an example let us consider the differential equation

$$u^{(2m)} = (-1)^m\big[p_0(t)u+p_1(t)u^{\mu}+p_2(t)u^{-\nu}\big], \tag{5}$$

where $\mu\in[0,1[$, $\nu\ge 0$ and $p_i:\ ]a,b[\,\to[0,+\infty[\ \ (i=0,1,2)$ are continuous functions such that either $p_1(t)\not\equiv 0$, or $p_0(t)p_2(t)\not\equiv 0$. From Theorems 1 and 2 follow the following corollaries.

Corollary 1. *Let*

$$p_0(t) \le \ell\big((t-a)^{-2m}+(b-t)^{-2m}\big) \quad \textit{for } a<t<b,$$

where ℓ is a nonnegative constant satisfying inequality (4)*. If, moreover,*

$$\int_a^b \big[(t-a)(b-t)\big]^{(1+\mu)(m-\frac12)}p_1(t)\,dt < +\infty,$$

$$\int_a^b \big[(t-a)(b-t)\big]^{(1-\nu)m-\frac12}p_2(t)\,dt < +\infty, \tag{6}$$

then problem (5), (2) *in the space $C^{2m,m}(]a,b[)$ has at least one positive solution.*

Corollary 2. *Let*

$$p_0(t)+\mu p_1(t) \le \ell\big((t-a)^{-2m}+(b-t)^{-2m}\big) \quad \textit{for } a<t<b,$$

$$\mu p_1(t)\le \nu p_2(t) \quad \textit{for } a<t<b,$$

where ℓ is a nonnegative constant satisfying inequality (4)*. If, moreover, the condition* (6) *holds, then problem* (5), (2) *in the space $C^{2m,m}(]a,b[)$ has one and only one positive solution.*

The results analogous to theorems and corollaries formulated above are established for problems (1), (3) and (5), (3) as well.

Acknowledgement

Supported by the Shota Rustaveli National Science Foundation (Project # GNSF/ST09_175_3-101).