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Limit properties of positive solutions of fractional boundary value problems

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Let $\mathcal{A}$ denote the set of linear functionals $\Phi\colon C[0,1]\to\mathbb{R}$ which are nondecreasing (i.e., $x,y\in C[0,1]$, $x\le y$ on $[0,1]\Rightarrow\Phi(x)\le\Phi(y)$). Let $\mathcal{B}=\{\Phi\in\mathcal{A}:\Phi(1)<1\}$.

The Caputo fractional derivative ${}^cD^\gamma x$ *of order* $\gamma>0$, $\gamma\notin\mathbb{N}$, of a function $x\colon[0,1]\to\mathbb{R}$ is defined as

$$ {}^cD^\gamma x(t)=\frac{1}{\Gamma(n-\gamma)}\frac{\mathrm{d}^n}{\mathrm{d}t^n}\int_0^t (t-s)^{n-\gamma-1}\left(x(s)-\sum_{k=0}^{n-1}\frac{x^{(k)}(0)}{k!}s^k\right)\mathrm{d}s, $$

where $n=[\gamma]+1$ and $[\gamma]$ means the integral part of γ and where Γ is the Euler gamma function.

We investigate the sequence of fractional boundary value problems

$$ {}^cD^{\alpha_n}u(t)=\sum_{k=1}^m a_k(t)\,{}^cD^{\mu_{k,n}}u(t)+f(t,u(t),u'(t),{}^cD^{\beta_n}u(t)),\quad n\in\mathbb{N}, \tag{1} $$

$$ u'(0)=0,\ \ u(1)=\Phi(u)-\Lambda(u'),\quad \Lambda\in\mathcal{A},\ \Phi\in\mathcal{B}, \tag{2} $$

where $\alpha_n\in(1,2)$, $\beta_n,\mu_{k,n}\in(0,1)$, $\lim_{n\to\infty}\alpha_n=2$, $\lim_{n\to\infty}\beta_n=1$, $\lim_{n\to\infty}\mu_{k,n}=1$, $a_k\in C[0,1]$ ($k=1,2,\dots,m$) and $f\in C([0,1]\times\mathcal{D})$, $\mathcal{D}\subset\mathbb{R}^3$.

A function $u\colon[0,1]\to\mathbb{R}$ is called *a positive solution of problem* (1), (2) if $u\in C^1[0,1]$ (and then ${}^cD^{\mu_{k,n}}u,{}^cD^{\beta_n}u\in C[0,1]$), ${}^cD^{\alpha_n}u\in C[0,1]$, $u>0$ on $[0,1)$, u satisfies (2) and equality (1) holds for $t\in[0,1]$.

Together with (1) the differential equation

$$ u''(t)=u'(t)\sum_{k=1}^m a_k(t)+f\big(t,u(t),u'(t),u'(t)\big) \tag{3} $$

is investigated. A function $u\in C^2[0,1]$ is called *a positive solution of problem* (3), (2) if $u>0$ on $[0,1)$, u satisfies (2) and (3) holds for $t\in[0,1]$.

We investigate the relation between positive solutions of problems (1), (2) and (3), (2). It is proved that

- for each $n \in \mathbb{N}$, problem (1), (2) has a positive solution u_n,
- there exists a subsequence $\{u_{n'}\}$ of $\{u_n\}$ that converges to a positive solution u of problem (3), (2) (i.e., $\|u_{n'} - u\|_{C^1} \to 0$, $\|{}^cD^{\alpha_{n'}} u_{n'} - u''\| \to 0$, $\|{}^cD^{\mu_{k,n'}} u_{n'} - u'\| \to 0$ and $\|{}^cD^{\beta_{n'}} u_{n'} - u'\| \to 0$ as $n' \to \infty$).

The existence result for problem (1), (2) is proved by the Leray-Schauder degree theory.