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Generalized linear differential equations in a Banach space: Continuous dependence on a parameter

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In what follows, X is a Banach space and $L(X)$ is the Banach space of bounded linear operators on X. By $\|\cdot\|_X$ we denote the norm in a Banach space X. Further, $BV([a,b],X)$ is the set of X valued functions of bounded variation on $[a,b]$ and $G([a,b],X)$ is the set of X valued functions having on $[a,b]$ all one-sided limits (i.e. X valued functions regulated on $[a,b]$). A couple $P=(D,\xi)$ where $D=\{\alpha_0,\ \alpha_1,\dots,\ \alpha_m\}$ and $\xi=(\xi_1,\dots,\ \xi_m)\in[a,b]^m$ is said to be a partition of $[a,b]$ if $a=\alpha_0<\alpha_1<\ \dots\ <\alpha_m=b$ and $\alpha_{j-1}\le\xi_j\le\alpha_j$ for $j=1,2,\dots,m$. For such a partition P and functions $F\colon[a,b]\to L(X)$ and $g\colon[a,b]\to X$ we define

$$S(\mathrm{d}F,g,P)=\sum_{j=1}^{m}[F(\alpha_j)-F(\alpha_{j-1})]\,g(\xi_j)\quad\text{and}\quad S(F,\mathrm{d}g,P)=\sum_{j=1}^{m}F(\xi_j)\,[g(\alpha_j)-g(\alpha_{j-1})]\,.$$

For a gauge $\delta\colon[a,b]\to(0,\infty)$, the partition P is called δ-fine if

$$[\alpha_{j-1},\alpha_j]\subset(\xi_j-\delta(\xi_j),\xi_j+\delta(\xi_j))\quad\text{for all } j\in\mathbb{N}.$$

The integrals are the abstract *Kurzweil-Stieltjes integrals* (*KS-integrals*) defined as follows:

Definition 1. For $F\colon[a,b]\to L(X)$, $g\colon[a,b]\to X$ and $I\in X$ we say that $\int_a^b \mathrm{d}[F]\,g=I$ if for every $\varepsilon>0$ there exists a gauge δ on $[a,b]$ such that

$$\Big\|S(\mathrm{d}F,g,P)-I\Big\|_X<\varepsilon\quad\text{for all }\ \delta-\text{fine partitions }\ P\text{ of }\ [a,b]\,,$$

Similarly we define the KS-integral $\int_a^b F\,\mathrm{d}[g]$ using sums of the form $S(F,\mathrm{d}g,P)$.

It is known that the integrals $\int_a^b \mathrm{d}[F]\,g$, $\int_a^b F\,\mathrm{d}[g]$ exist if $F\in G([a,b],L(X))$, $g\in G([a,b],X)$ and at least one of the functions F, g has a bounded variation on $[a,b]$ (cf. [2]). Further basic

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properties of the abstract KS-integral, like e.g. the substitution theorem, the integration-by-parts theorem or the convergence theorems, have been described in [6] and [2].

Let $A, A_k \in BV([a,b], L(X))$, $\widetilde{x}$, $\widetilde{x}_k \in X$ and $f, f_k \in G([a,b], X)$ be given for $k \in \mathbb{N}$. Consider the generalized linear differential equations

$$x(t) = \widetilde{x} + \int_a^t \mathrm{d}[A(s)]\, x(s) + f(t) - f(a), \quad t \in [a,b]\,, \tag{1}$$

and

$$x_k(t) = \widetilde{x}_k + \int_a^t \mathrm{d}[A_k(s)]\, x_k(s) + f_k(t) - f_k(a), \quad t \in [a,b], \quad k \in \mathbb{N}\,. \tag{1_k}$$

The following assumptions are crucial for the existence of solutions to (1) and (1_k)

$$\big[I - \Delta^{-} A(t)\big]^{-1} \in L(X) \quad \text{for all } t \in (a,b\,]\,, \tag{2}$$

and

$$\big[I - \Delta^{-} A_k(t)\big]^{-1} \in L(X) \quad \text{for all } t \in (a,b\,],\ k \in \mathbb{N}\,. \tag{2_k}$$

For the basic properties of generalized linear differential equations in a Banach space, see [7].

Our first result extends that by M. Ashordia [1] valid for the case $X = \mathbb{R}^n$.

Theorem 1. *Let A, A_k satisfy* (1) *and* (1_k)*, and let*

$$A_k \rightrightarrows A \quad \text{on } [a,b]\,, \tag{3}$$

$$\alpha^* := \sup_{k \in \mathbb{N}} \big(\operatorname{var}_a^b A_k\big) < \infty\,, \tag{4}$$

$$f_k \rightrightarrows f \quad \text{on } [a,b]\,, \tag{5}$$

$$\widetilde{x}_k \to \widetilde{x} \quad \text{in } X\,. \tag{6}$$

Then (1) *has a unique solution x on $[a,b]$. Furthermore, for each $k \in \mathbb{N}$ large enough there is a unique solution x_k on $[a,b]$ to* (1_k) *and $x_k \rightrightarrows x$.*

The next result extends that by Z. Opial [5] to homogeneous generalized linear differential equations in a general Banach space X.

Theorem 2. *Let $f(t) \equiv f(a)$, $f_k(t) \equiv f_k(a)$ on $[a,b]$ for $k \in \mathbb{N}$ and let A, A_k satisfy* (1), (1_k)*. Let $\widetilde{x}, \widetilde{x}_k \in X$ satisfy* (6) *and let*

$$\lim_{k\to\infty} \Big(\sup_{t\in[a,b]} \|A_k(t) - A(t)\|_{L(X)} \Big) \Big(1 + \operatorname{var}_a^b A_k\Big) = 0 \tag{6}$$

Then the conclusions of Theorem 1 are true.

For the proofs of Theorems 1 and 2, see [3]. The case when (3) (and hence also (6)) is not satisfied is treated in [4].

References

- [1] M. Ashordia, *On the correctness of linear boundary value problems for systems of generalized ordinary differential equations.* Proc. Georgian Acad. Sci. Math. **1** (1993), 385–394.
- [2] G. Monteiro, M. Tvrdý, *On Kurzweil-Stieltjes integral in a Banach space.* Math. Bohem, to appear.
- [3] G. Monteiro, M. Tvrdý, *Generalized linear differential equations in a Banach space: Continuous dependence on a parameter.* Discrete Contin. Dyn. Syst., to appear.
- [4] Z. Halas, G. Monteiro, M. Tvrdý, *Emphatic convergence and sequential solutions of generalized linear differential equations.* Mem. Differential Equations Math. Phys. **54** (2011), 27–49.
- [5] Z. Opial, *Continuous Parameter Dependence in Linear Systems of Differential Equations.* J. Differential Equations **3** (1967), 571–579.
- [6] Š. Schwabik, *Abstract Perron-Stieltjes integral.* Math. Bohem. **121** (1996), 425–447.
- [7] Š. Schwabik, *Linear Stieltjes integral equations in Banach spaces.* Math. Bohem. **124** (1999), 433–457.