Limiting the size of a singular set for 3D Navier-Stokes equations

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Outline

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1. Navier-Stokes equations
2. Hausdorff and the box-counting dimensions
3. Classical results
4. Box-counting dimension of the set of singular times
5. Box-counting dimension of the singular set in space-time
6. Application: Lagrangian trajectories

3D Navier-Stokes equations

- The equations:

$$\begin{cases} u_t - \triangle u + u \cdot \nabla u + \nabla p = 0 \\ \nabla \cdot u = 0 \end{cases}$$

- Standard regularity of weak solutions

$$u \in L^2(0,T;V) \cap L^\infty(0,T;H)$$

- Regularity in $L^r L^s$ spaces:$u \in L^r(0,T;L^s(\Omega))$ for

$$\frac{2}{r} + \frac{3}{s} \leq \frac{3}{2}, \; 2 \leq s \leq 6$$

for $s = r = \frac{10}{3}$ we have

$$u \in L^{\frac{10}{3}}(Q_T)$$

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More properties of weak solutions

- Foias *et al.* 1981: $u \in L^{2/3}(0,T;D(A)) \cap L^2(0,T;V)$

$$\Rightarrow u \in L^1(0,T;L^\infty(\Omega))$$

- $p \in L^{5/3}(Q_T)$

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The gap

- Serrin's condition:
  If $u \in L^r(0, T; L^s(\Omega))$ with

$$\frac{2}{r} + \frac{3}{s} \leq 1, s \geq 3$$

  then u is regular.
  Local version: u and its space derivatives are uniformly bounded on compact subsets of Q
- The gap is small but

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Singular times and singular points

- A time $t \in (0, T)$ is singular if

$$||Du(t)|| = \infty$$

- A time t is regular if it is not singular
- A point (x, t) in space-time is regular if there exists a cylinder $Q_r(x, t)$ such that u is Hölder continuous on $Q_r(x, t)$
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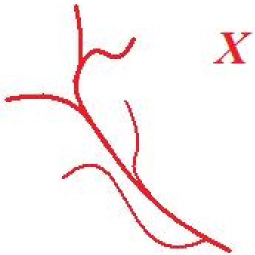
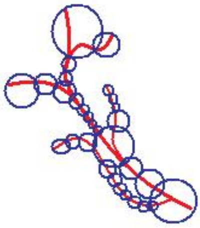
How can one measure smallness of a set?



Hausdorff dimension



We define

$$\mu^s(X) = \inf \left\{ \sum_{k=1}^{\infty} r_k^s : \ \textit{there exists a cover of } X \ \textit{by balls with radii } r_k \right\}$$

Then

$$dim_H(X) = inf\{s : \mu^s(X) = 0\}$$

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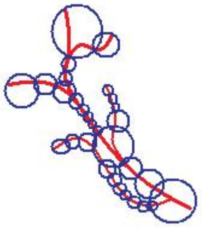

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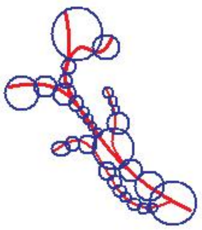

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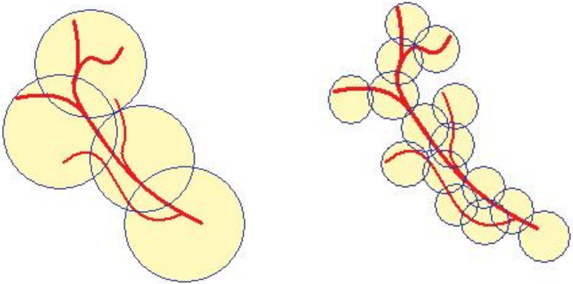
Box-counting dimension I



The upper box-counting dimension of a set X is given by

$$\limsup_{\epsilon \to 0} \frac{\log N(X,\epsilon)}{-\log \epsilon}.$$

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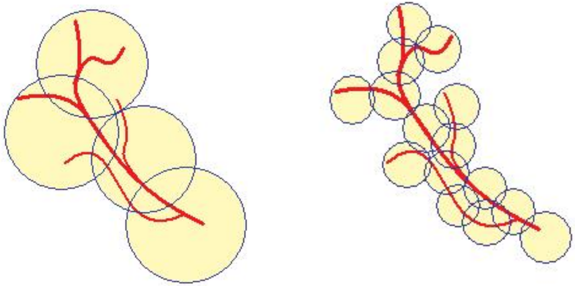

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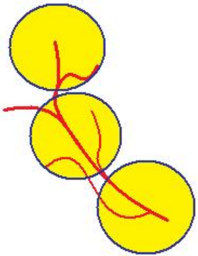
Box-counting dimension II

$N(X,\epsilon)$ can also be the maximum number of ϵ-separated points in X.



Box-counting dimension III

- Let $d_B(X) = d$ and $X \subset \mathbb{R}^n$. Then for any $d' > d$ there are ϵ_0 and $c > 0$ such that for all $0 < \varepsilon < \epsilon_0$ we have

$$\mu(O(X,\varepsilon)) \leq c\varepsilon^{n-d'}$$

- We always have

$$d_H(X) \leq d_B(X)$$

and for some X: $d_H(X) < d_B(X)$.

- Example: $X = \{n^{-1} : n \in N\}$.

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Classical results

Leray - 1934

1. The set $\mathcal{T}$ of singular times of a weak solution u has $1/2$-dimensional Hausdorff measure zero.

CKN - 1982

The set S of singular points of a suitable weak solution u has 1-dimensional parabolic Hausdorff measure zero: for any $n \in N$ it can be covered by cylinders $Q_k = (t_k, t_k + r_k^2) \times B(x_k, r_k)$ such that

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Key ingredients

There is an absolute constant $\varepsilon > 0$ such that a point (x, t) is regular if

- for some $Q_r(x, t)$ we have

$$\frac{1}{r^2} \int_{Q_r(x,t)} |u|^3 + |p|^{3/2} \leq \varepsilon,$$

  or

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The box-counting dimension of the set of singular times

Theorem (James Robisnon and WS, 2007)

The upper box-counting dimension of a putative singular times $\mathcal{T}$ is no greater than $1/2$.

Proof.

- For $t > s$ we have

$$||Du(t)||^2 \leq \frac{||Du(s)||^2}{\sqrt{1 - c(t-s)||Du(s)||^4}}$$

- Therefore if there is blow-up at time t:

$$||Du(s)||^2 \geq \frac{1}{\sqrt{c(t-s)}}$$

- So

$$\int_0^T ||Du||^2 \geq \int_{t_1-r}^{t_1} ||Du||^2 + \int_{t_2-r}^{t_2} ||Du||^2 + \ldots + \int_{t_N-r}^{t_N} ||Du||^2$$

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Box-counting dimension of a singular set

Theorem

The upper box-counting dimension of a putative singular set S is less or equal $5/3$.

Proof. It is enough to deduce that at a singular point

$$\int_{Q_r(x,t)} |u|^{10/3} + |p|^{5/3} > cr^{5/3}.$$

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A better bound

- Igor Kukavica showed that:

$$d_B(\mathcal{S}) \leq \frac{135}{82}$$

- We have

$$\frac{135}{82} \approx 1.646$$

- Notice that

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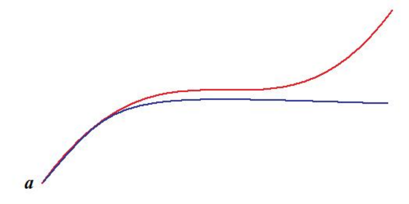
Application: Lagrange trajectories

$$\begin{cases} \frac{d\varphi}{dt} = u(\varphi(t), t) \\ \varphi(0) = a \end{cases}$$

1. Question: Are the particle trajectories unique for almost all $a \in \Omega$?



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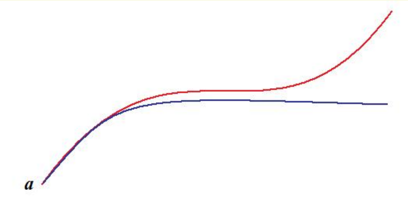

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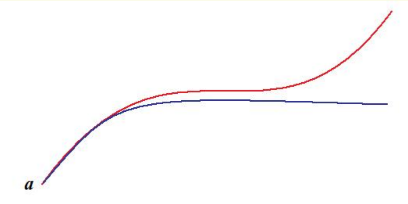

Foias *et al* 1985

There exists at least one function $\Phi : \Omega \times [0; T] \to \Omega$ such that

1. the function $\phi_a(\cdot) = \Phi(a, \cdot)$ satisfies
$$\phi(t) = a + \int_0^t u(\phi(s), s)ds$$
2. $\varphi_a(\cdot) \in W^{1,1}(0, T)$
3. the mapping $a \to \Phi(a, \cdot)$ belongs to $L^\infty(\Omega; C([0, T], \bar{\Omega}))$
4. Φ is volume-preserving: for any Borel set $B \subset \Omega$,
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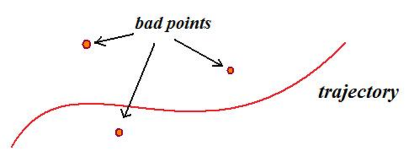
Conditional result

Theorem 1. James Robinson and WS 2008

If u is a suitable weak solution with $u \in L^{6/5}(0, T; L^\infty)$ corresponding to $u_0 \in H \cap H^{1/2}$, then almost every initial condition $a \in \Omega$ gives rise to a unique particle trajectory, which is C^1 function of time.

Idea of proof

Idea of the proof: show that particle trajectories "usually" avoid points where u behaves badly



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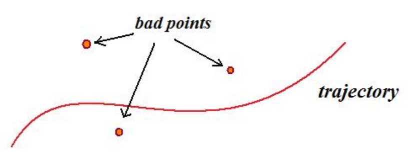


Avoiding "bad points"

Theorem 2

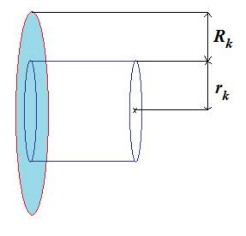
If u is a suitable weak solution with $u \in L^{6/5}(0, T; L^\infty)$ then the set of initial conditions $a \in \Omega$ that give rise to trajectories intersecting the singular set S is of Lebesgue measure zero.

Proof of Theorem 2 - part one

1. We cover the singular set by cylinders Q_k
2. We define the numbers R_k:

$$R_k = \int_{t_k}^{t_k + r_k^2} ||u||_\infty$$

3. Then we consider balls $\hat{B}_k = (x_k, r_k + R_k)$



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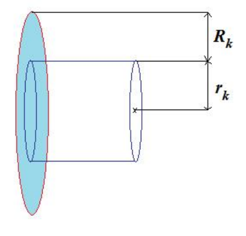

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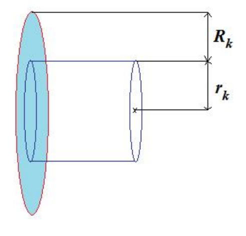

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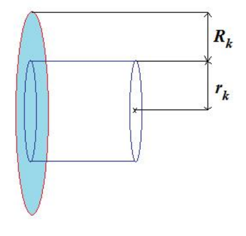

Proof of Theorem 2 - part one

1. We cover the singular set by cylinders Q_k
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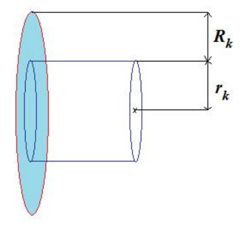

Proof of Theorem 2 - part two

1. using inequality:

$$|\varphi_a(t) - \varphi_a(t_k)| \leq R_k$$

   we prove that trajectories that do not meet $\hat{B}_k$ at time t_k cannot enter cylinder Q_k

2. we estimate the volume of the family of balls $\hat{B}_k$:

$$\sum_{k=1}^{\infty} \mu(\hat{B}_k) \leq \sum_{k=1}^{\infty} c(r_k + R_k)^3 \leq C[\sum_{k=1}^{\infty} r_k^3 + \sum_{k=1}^{\infty} R_k^3]$$

3. Proof

$$R_k \leq \left(\int_{t_k}^{t_k+r_k^2} ds\right)^{1/6} \left(\int_{t_k}^{t_k+r_k^2} ||u||_{\infty}^{6/5}\right)^{5/6} \leq Cr_k^{1/3}$$

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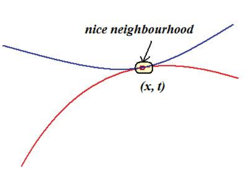
Theorem 2 implies Theorem 1

Proof

1. since $u_0 \in H^{1/2}$ trajectories are unique on some interval $[0, \varepsilon)$
2. assume that for some $t > 0$ and $x \in \Omega$ (x, t) is not singular and there are two trajectories passing through (x, t)



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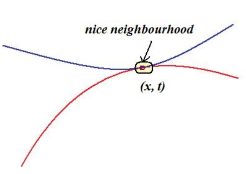

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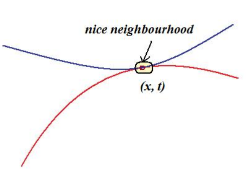

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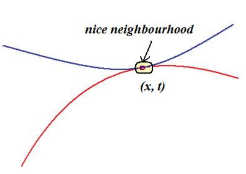

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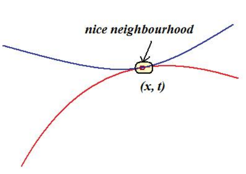

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Theorem 3. (James Robinson and WS 2009)

If u is a suitable weak solution with $p \in L^{5/3}((0,T) \times \Omega)$ corresponding to $u_0 \in H \cap H^{1/2}(\Omega)$ then almost every initial condition $a \in \Omega$ gives rise to a unique particle trajectory, which is a C^1 function of time.

Aizenman's result

Proposition

Let $\Omega \subset R^d$, and let $\Phi : \Omega \times [0, T] \to \Omega$ be a volume-preserving solution mapping corresponding to a vector field $u \in L^1(0, T; L^\infty(\Omega))$ for every $T > 0$. If X is a compact subset of Ω with $d_F(X) < d - 1$ then for almost every initial condition $a \in \Omega$, $\Phi(t, a)$ is not an element of X for all times t.

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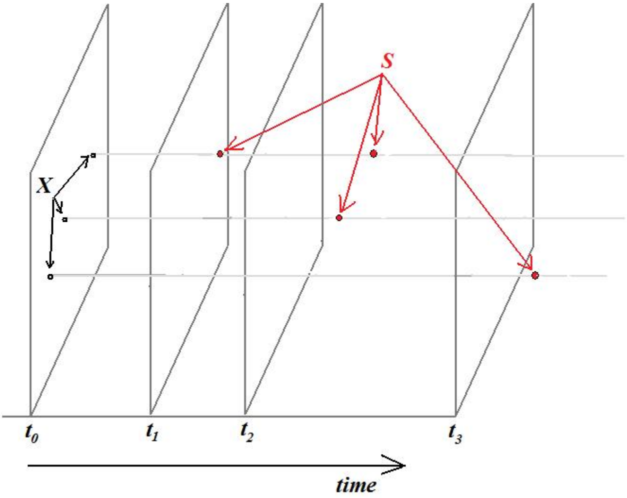
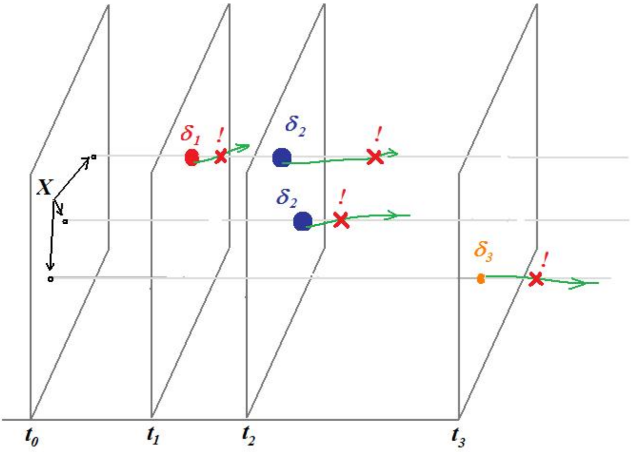
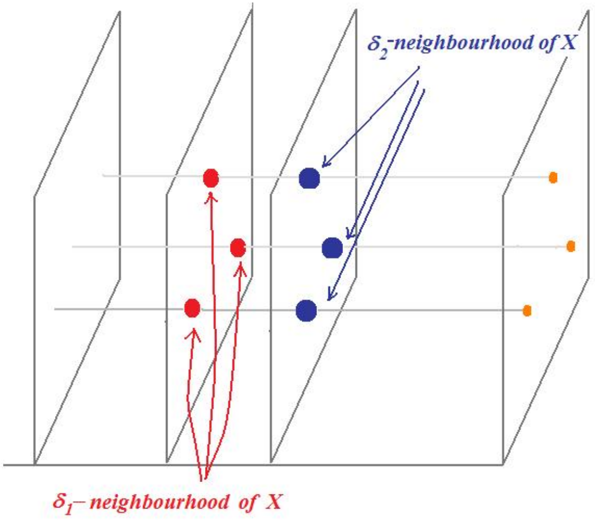

S
X
t_0
t_1
t_2
t_3
time


X
δ_1
!
δ_2
!
δ_2
!
δ_3
!
t_0
t_1
t_2
t_3

Definition of deltas

$$\delta_k = \int_{t_k}^{t_{k+1}} ||u(s)||_\infty ds$$


δ_2-neighbourhood of X
δ_1– neighbourhood of X

Proposition

If $X \in R^n$ has a box-counting dimension d, then for any $d' > d$ there exists an $\epsilon_0 > 0$ such that

$$\mu(O(X,\epsilon)) \leq c_n \epsilon^{n-d'} \quad for \quad all \quad 0 < \epsilon < \epsilon_0$$

Corollary

1. $V_1 =$ total volume of δ_1-neighbourhood of $X \leq c_n \delta_1^r$
2. $V_2 =$ total volume of δ_2-neighbourhood of $X \leq c_n \delta_2^r$
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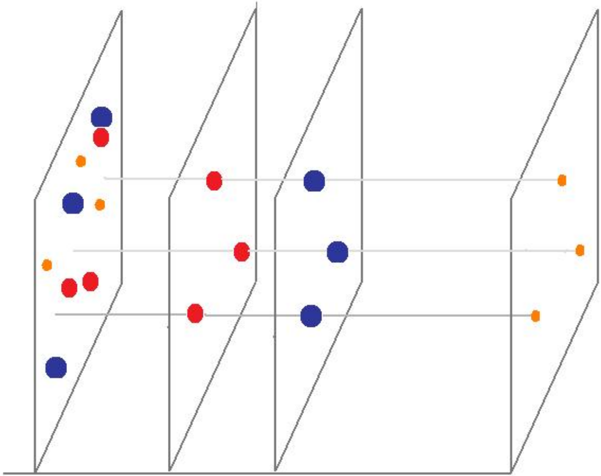

$$\mu(K) \leq V_1 + V_2 + V_3 + ... + V_N \leq c(\delta_0^r + \delta_1^r + \delta_2^r + ... + \delta_N^r) \leq$$

$$\leq c[\epsilon^{r-1} \int_{t_0}^{t_1} ||u||_\infty + \epsilon^{r-1} \int_{t_1}^{t_2} ||u||_\infty + ... + \epsilon^{r-1} \int_{t_{N-1}}^{t_N} ||u||_\infty] \leq$$

$$\leq c\epsilon^{r-1} ||u||_{L^1(0,T;L^\infty(\Omega))}$$

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1. Aizenman M. *A sufficient condition for the avoidance of sets by measure preservng flows in R^n*, 1978
2. Caffarelli, Kohn and Nirenberg *Partial regularity of suitable weak solutions of the Navier-Stokes equations*, Commu. Pure Appl. Math. 35 771-831, 1982
3. Foias, Guillope, Temam, *Lagrangian representation of a flow*, Journal of Differential Equations 57, 440-449 (1985)
4. Robinson, Sadowski *Decay of weak solutions and the singular set of the 3D N-S equations*, Nonlinearity, 2007
5. Robinson, Sadowski *A criterion for uniqueness of Lagrangian trajectories for weak solutions of the 3D Navier-Stokes equations*, Commun. Math. Phys., 2009
6. Robinson, Sadowski *Almost everywhere uniqueness of Lagrangian trajectories for suitable weak solutions of the three dimensional Navier-Stokes equations*, Nonlinearity, 2009.

THANK YOU