Some developments on Dirichlet problems with discontinuous coefficients
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Let Ω be a bounded, open subset of $\mathbb{R}^N$, $N > 2$ and $M : \Omega \times \mathbb{R} \to \mathbb{R}^{N^2}$, be a bounded and measurable matrix such that

$$(1) \qquad \alpha|\xi|^2 \le M(x)\xi \cdot \xi, \quad |M(x)| \le \beta, \quad \text{a.e. } x \in \Omega, \quad \forall\, \xi \in \mathbb{R}^N .$$

Under the assumptions $|B|$, $|E| \in L^N(\Omega)$, $f \in L^m(\Omega)$ ($m \ge \frac{2N}{N+2}$) and $\mu > 0$ large enough, Guido Stampacchia proved that the boundary value problem

$$(2) \qquad \begin{cases} -\operatorname{div}(M(x)\nabla u - u\,E(x)) + B(x)\nabla u + \mu u = f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

has a unique weak solution u with some summability properties.

If we assume that $E(x)$ is a vector field and $f(x)$ is a function such that

$$(3) \qquad f \in L^m(\Omega), \;\; 1 \le m < \frac{N}{2},$$

$$(4) \qquad E \in (L^N(\Omega))^N,$$

and we consider the following Dirichlet problem [1]

$$(5) \qquad \begin{cases} -\operatorname{div}(M(x)\nabla u) = -\operatorname{div}(u\,E(x)) + f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

existence and summability properties (depending on m) of weak or distributional solutions are proved in [2].

In [3], equations with coefficients E which do not belong to $(L^N(\Omega))^N$ are considered. The most important aim is the study of the case $E \in (L^2(\Omega))^N$, where the main point is the definition of solution, since the distributional

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[1] related to the mathematical analysis of some models of flows in porous media (T. Gallouet)

definition of solution does not work. It is possible to give a meaning to solution for problem (5), using the concept of *entropy solutions* which has been introduced in [1]

An important difficulty is due to noncoercivity of the differential operator $-\operatorname{div}(M(x)\nabla v) + \operatorname{div}(v\, E(x))$.

Thus we assume

$$E \in (L^2(\Omega))^N \tag{6}$$

and

$$f \in L^1(\Omega). \tag{7}$$

We recall Stampacchia's definition of truncate

$$T_n(s) = \begin{cases} s, & \text{if } |s| \le n, \\ n\frac{s}{|s|}, & \text{if } |s| > n, \end{cases}$$

the definition of entropy solution and some results given in [1].

Proposition 0.1 *Let u be a measurable function such that $T_k(u)$ belongs to $W_0^{1,2}(\Omega)$ for every $k > 0$. Then there exists a unique measurable function $v : \Omega \to I\!R^N$ such that*

$$v\, \chi_{\{|u|<k\}} = \nabla T_k(u)\,, \quad \textit{almost everywhere in } \Omega,\ \forall k > 0.$$

If, moreover, u belongs to $W_0^{1,2}(\Omega)$, then v coincides with the standard distributional gradient of u.

Definition 0.2 *Let u be a measurable function such that $T_k(u)$ belongs to $W_0^{1,2}(\Omega)$ for every $k > 0$. We define ∇u, the weak gradient of u, as the function v given by Proposition 0.1.*

Definition 0.3 *Assume (1), (6), (7). A measurable function u is an entropy solution of the boundary value problem (5) if*

$$\begin{cases} T_k(u) \in W_0^{1,2}(\Omega), \\ \displaystyle\int_\Omega M(x)\nabla u \nabla T_k[u-\phi] \le \int_\Omega E(x)\nabla T_k[u-\phi] + \int_\Omega f(x) T_k[u-\phi], \\ \forall k \in I\!R^+, \forall \phi \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega), \end{cases} \tag{8}$$

Remark 0.4 *Note that in the previous inequality, any term is well defined.*

Theorem 0.5 *Assume (1), (6) and (7). Then there exists an entropy solution u of (5) in the sense of Definition 0.3. Moreover u satisfis the estimates*

$$\int_\Omega |\nabla \log(1+|u|)|^2 \le \frac{1}{2\alpha}\int_\Omega |E|^2 + \int_\Omega |f|, \tag{9}$$

$$\frac{\alpha}{2}\int_\Omega |\nabla T_k(u)|^2 \le \frac{k^2}{2\alpha}\int_{|u|<k} |E|^2 + k\int_\Omega |f| \le \frac{k^2}{2\alpha}\int_\Omega |E|^2 + k\int_\Omega |f|. \tag{10}$$

Remark 0.6 *The estimate (10) gets the uniqueness of the solution u of Theorem 0.5, if $f=0$. Let $h \to 0$ and $0<h<\delta$. Indeed, now (10) says*

$$S^2\Big[\int_{\delta<|u|} \frac{|T_h(u)|^{2^*}}{h^{2^*}}\Big]^{\frac{2}{2^*}} \le S^2\Big[\int_\Omega \frac{|T_h(u)|^{2^*}}{h^{2^*}}\Big]^{\frac{2}{2^*}} \le \int_\Omega \frac{|\nabla T_h(u)|^2}{h^2} \le \frac{1}{\alpha^2}\int_{0<|u|<h} |E|^2$$

which implies

$$S^2 \text{meas}\,\{\delta<|u|\}^{\frac{2}{2^*}} \le \frac{1}{\alpha^2}\int_{0<|u|<h} |E|^2.$$

Since $|E| \in L^2(\Omega)$, the right hand side goes to 0, as $h \to 0$. Thus $\text{meas}\,\{\delta<|u|\} = 0$, *for every $\delta>0$.*

We poin out that independently, with a similar approach, T. Gallouet ([13]) proved that if $f(x)\ge 0$ then $u(x) \ge 0$.

A borderline case: we start with two radial problems, where the data f and E are smooth enough, but E does not belong (as in in [2]) to $(L^N(\Omega))^N$, but to $(L^q(\Omega))^N$, for any $q<N$. With this slightly weaker assumptions the following examples show how all the existence and summability results about the solutions can be lost.

Remark 0.7 *Let $0<B<N-2$ and consider the boundary value problem*

$$\begin{cases} -\Delta u = -B\,\text{div}\Big(u\dfrac{x}{|x|^2}\Big) - B\,\dfrac{N-2}{|x|^2} & \text{in } \{x : |x|<1\}, \\ u = 0 & \text{on } \{x : |x| = 1\}. \end{cases}$$

Then the function $u_B(x) = \frac{1}{|x|^B} - 1$ is a weak solution in $W_0^{1,2}(\Omega)$ if $B < 1+N/2$ and it is a distributional solution $1+N/2 \le B < N-2$.

Note that $E = -B\frac{x}{|x|^2}$ belongs to $(L^q(\Omega))^N$, for any $q < N$, and the right hand side belongs to $L^m(\Omega)$, for any $m < \frac{N}{2}$. Nevertheless the solution u does not belong to any L^p space; that is: it is not possible to apply the results of [2], where the assumption is $|E| \in L^N(\Omega)$.

REMARK 0.8 *The function $u_D = r^{-D} - r^2$, $D \in I\!R$, is solution of the boundary value problem*

$$\begin{cases} -\Delta u = D \operatorname{div}\Big(u\frac{x}{|x|^2}\Big) + (2+D)N & \text{in } \{x : |x| < 1\}, \\ \\ u = 0 & \text{on } \{x : |x| = 1\}. \end{cases}$$

If $D > 0$, u_D is unbounded solution of a Dirichlet problem with bounded datum the real number $(2+D)N$; u_D is a weak solution if $D < 1 + N/2$ and it is a distributional solution $1 + N/2 \leq D < N - 2$.

Now, on the vector field E we assume

$$|E| \leq \frac{A}{|x|}, \ A > 0, \quad 0 \in \Omega, \tag{11}$$

(which is slightly weaker than (4)) ans we use the following inequality.

PROPOSITION 0.9 [HARDY-SOBOLEV INEQUALITY] *The Hardy inequality states that*

$$\mathcal{H}\Big(\int_\Omega \frac{|v|^2}{|x|^2}\Big)^{\frac{1}{2}} \leq \Big(\int_\Omega |\nabla v|^2\Big)^{\frac{1}{2}}, \quad \forall v \in W_0^{1,2}(\Omega). \tag{12}$$

Moreover $\mathcal{H} = \frac{N-2}{2}$ is optimal.

THEOREM 0.10 *Assume (1), (3), with $\frac{2N}{N+2} < m < \frac{N}{2}$, (11), with $|A| < \frac{\alpha N}{m^{**}}$. Then there exists a weak solution $u \in W_0^{1,2}(\Omega) \cap L^{m^{**}}(\Omega)$ of the Dirichlet problem (5).*

THEOREM 0.11 *Assume (1), (3), with $1 < m < \frac{2N}{N+2}$, (11), with $|A| < \frac{\alpha N}{m^{**}}$. Then there exists a distributional solution $u \in W_0^{1,m^*}(\Omega)$ of the Dirichlet problem (5).*

Let us recall the definition of Marcinkiewicz spaces $M^p(\Omega)$, we shall use later.

Definition 0.12 *Let p be a positive number. The Marcinkiewicz space $M^p(\Omega)$ is the set of all measurable functions $v : \Omega \to I\!R$ such that*

$$\text{meas}\,\{x \in \Omega \,:\, |E(x)| > k\} \le \frac{c}{k^p}\,, \quad \textit{for every } k > 0\,,$$

for some constant $c > 0$. Moreover, for any $p \ge 1$, $L^p(\Omega) \subset M^p(\Omega)$ and, $p > 1$, $M^p(\Omega) \subset L^{p-\epsilon}(\Omega)$, $\epsilon > 0$.

Theorem 0.13 *Assume (1), $f \in L^1(\Omega)$, (11), with $|A| < \alpha(N-2)$. Then there exists a distributional solution u of the Dirichlet problem (5). The function u belongs to the Marcinkiewicz space $M^{\frac{N}{N-2}}(\Omega)$ and ∇u belongs to the Marcinkiewicz space $M^{\frac{N}{N-1}}(\Omega)$.*

Remark 0.14 *Let $E = \frac{(N-1)\,x}{|x|^2}$, so that $|E|$ belongs to $L^q(\Omega)$ for every $q < N$, but is not in $L^N(\Omega)$. Then the function (see [10]) $u(x) = u(|x|) = [\mathrm{e}^{|x|}|x|^{1-N} - \mathrm{e}]$ is a solution of the boundary value problem*

$$(13)\begin{cases} -\operatorname{div}\Big[\nabla u + u\dfrac{(N-1)\,x}{|x|^2}\Big] + u = \dfrac{\mathrm{e}(N-1)(N-2)}{|x|^2} - \mathrm{e}, & \textit{in } B_1(0); \\ u = 0, \quad \textit{on } \partial B_1(0). \end{cases}$$

The above example (13) shows that, for some values of $m > 1$, it is not true that u belongs to $L^m(\Omega)$, if f belongs to $L^m(\Omega)$, as usual if $E = 0$. Furthermore, even if E and f are quite regular, the summability of ∇u is poor.

Now we will show how, in the differential equation (5), the presence of a lower order term improves a little bit the regularity properties of the solutions, under the basic assumptions (1), (6), (3).

Let $\lambda > 0$ and $p \ge 1$. We consider here the following boundary value problem

$$(14) \quad \begin{cases} -\operatorname{div}(M(x)\nabla u) + \lambda|u|^{p-1}u = -\operatorname{div}(u\,E(x)) + f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

Theorem 0.15 *Assume (1), (3) with $m = 1$,*

$$(15) \qquad E \in (L^{\frac{2p}{p-1}}(\Omega))^N, \quad p > \frac{N}{N-2}.$$

Then there exists a distributional solution u of (14) such that $u \in L^p(\Omega)$ and $\nabla u \in M^{\frac{2p}{p+1}}(\Omega)$.

THEOREM 0.16 *Assume (1), (3) with* $m \geq \frac{p+1}{p}$,

$$E \in (L^{\frac{2(p+1)}{p-1}}(\Omega))^N, \quad p > \frac{N+2}{N-2}. \tag{16}$$

Then there exists a weak solution $u \in W_0^{1,2}(\Omega)$ *of (14) such that* $u \in L^{p+1}(\Omega)$.

REMARK 0.17 *Let* $0 < \epsilon < N-2$. *It is possible to state the previous theorem in the following way. Assume (1), (3) with* $m \geq 1 + \frac{\epsilon}{2+\epsilon}$, $E \in (L^{2+2\epsilon}(\Omega))^N$. *Then there exists a weak solution* $u \in W_0^{1,2}(\Omega)$ *of (14) such that* $u \in L^{\frac{2+2\epsilon}{\epsilon}}(\Omega)$.

Here, we shall prove, by duality, the exixtence of weak solutions for the boundary value problem problem

$$\begin{cases} -\operatorname{div}(M(x)\nabla u) + E\nabla u + \lambda u = f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{17}$$

under minimal assumptions on E.

THEOREM 0.18 *Assume (1), (6),*

$$\lambda > 0, \tag{18}$$

$$f \in L^\infty(\Omega). \tag{19}$$

Then there existe a weak solution u *in* $W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$ *of (17).*

REMARK 0.19 *If* $\lambda = 0$, *the problem (17) has been studied in [14], even if the principal part is nonlinear.*

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