

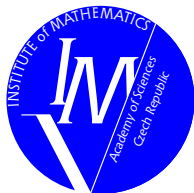
# A posteriori error estimates

## Part II – Complementary estimates

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# Toy problem

Classical formulation:

$$-\Delta u = f \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega$$

Weak formulation:  $V = H_0^1(\Omega)$

$$u \in V : \quad a(u, v) = \mathcal{F}(v) \quad \forall v \in V$$

Notation:

- ▶  $a(u, v) = (\nabla u, \nabla v)$
- ▶  $\mathcal{F}(v) = (f, v)$
- ▶  $(\varphi, \psi) = \int_{\Omega} \varphi \psi \, dx$
- ▶ Error:  $e = u - u_h$
- ▶ Energy norm:  $\|e\|^2 = a(e, e) = (\nabla e, \nabla e) = \|\nabla e\|_0^2$



# Derivation

Divergence thm.:  $(\operatorname{div} \mathbf{y}, v) + (\mathbf{y}, \nabla v) = 0 \quad \forall \mathbf{y} \in \mathbf{H}(\operatorname{div}, \Omega), v \in V$

Friedrichs' inequality:  $\|v\|_0 \leq C_F \|\mathbf{y}\| \quad \forall v \in V$

Theorem: Let  $u_h \in V$  be arbitrary then

$$\|u - u_h\| \leq \eta(u_h, \mathbf{y})$$

$$\eta(u_h, \mathbf{y}) = C_F \|f + \operatorname{div} \mathbf{y}\|_0 + \|\mathbf{y} - \nabla u_h\|_0 \quad \forall \mathbf{y} \in \mathbf{H}(\operatorname{div}, \Omega)$$

Proof:  $v \in V$

$$\begin{aligned} a(u - u_h, v) &= (f, v) - (\nabla u_h, \nabla v) = (f + \operatorname{div} \mathbf{y}, v) + (\mathbf{y} - \nabla u_h, \nabla v) \\ &\leq \|f + \operatorname{div} \mathbf{y}\|_0 \|v\|_0 + \|\mathbf{y} - \nabla u_h\|_0 \|\nabla v\|_0 \\ &\leq (C_F \|f + \operatorname{div} \mathbf{y}\|_0 + \|\mathbf{y} - \nabla u_h\|_0) \|v\| \end{aligned}$$

Set  $v = u - u_h$ . □



# Derivation

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Lemma: Let  $u \in V$  be the exact solution.

Then  $\|u - u_h\| = \eta(u_h, \nabla u)$ .

# Two options

## (A) Error majorant

$$\|u - u_h\| \leq \eta(u_h, \mathbf{y})$$

$$\eta(u_h, \mathbf{y}) = C_F \|f + \operatorname{div} \mathbf{y}\|_0 + \|\mathbf{y} - \nabla u_h\|_0 \quad \forall \mathbf{y} \in \mathbf{H}(\operatorname{div}, \Omega)$$

[S. Repin et al., 2000–]

## (B) Dual finite elements

$$\|u - u_h\| \leq \tilde{\eta}(u_h, \mathbf{y})$$

$$\tilde{\eta}(u_h, \mathbf{y}) = \|\mathbf{y} - \nabla u_h\|_0 \quad \forall \mathbf{y} \in \mathbf{Q}(f)$$

$$\mathbf{Q}(f) = \{\mathbf{y} \in \mathbf{H}(\operatorname{div}, \Omega) : f + \operatorname{div} \mathbf{y} = 0\}$$

[J. Haslinger, I. Hlaváček, M. Křížek, 1970s–80s]



## (A) Error majorant

$$\|u - u_h\| \leq \eta(u_h, \mathbf{y}) = C_F \|f + \operatorname{div} \mathbf{y}\|_0 + \|\mathbf{y} - \nabla u_h\|_0 \quad \forall \mathbf{y} \in \mathbf{H}(\operatorname{div}, \Omega)$$

$$\|u - u_h\|^2 \leq \hat{\eta}^2(u_h, \mathbf{y}, \beta) = \left(1 + \frac{1}{\beta}\right) C_F^2 \|f + \operatorname{div} \mathbf{y}\|_0^2 + (1 + \beta) \|\mathbf{y} - \nabla u_h\|_0^2$$

$\forall \beta > 0$

**Proof:**  $(A + B)^2 \leq \left(1 + \frac{1}{\beta}\right) A^2 + (1 + \beta) B^2 \quad \forall \beta > 0$

Equality for  $\beta = A/B$ . □



## (A) Error majorant

$$\|u - u_h\| \leq \eta(u_h, \mathbf{y}) = C_F \|f + \operatorname{div} \mathbf{y}\|_0 + \|\mathbf{y} - \nabla u_h\|_0 \quad \forall \mathbf{y} \in \mathbf{H}(\operatorname{div}, \Omega)$$

$$\|u - u_h\|^2 \leq \hat{\eta}^2(u_h, \mathbf{y}, \beta) = \left(1 + \frac{1}{\beta}\right) C_F^2 \|f + \operatorname{div} \mathbf{y}\|_0^2 + (1 + \beta) \|\mathbf{y} - \nabla u_h\|_0^2$$

$\forall \beta > 0$

Notation:  $\mathbf{W} = \mathbf{H}(\operatorname{div}, \Omega)$

Complementary problem (equivalent formulations):

(i) Find  $\mathbf{y} \in \mathbf{W} : \eta(u_h, \mathbf{y}) \leq \eta(u_h, \mathbf{w}) \quad \forall \mathbf{w} \in \mathbf{W}$

(ii) Find  $\mathbf{y} \in \mathbf{W}$  and  $\beta > 0 : \hat{\eta}^2(u_h, \mathbf{y}, \beta) \leq \hat{\eta}^2(u_h, \mathbf{w}, \hat{\beta})$   
 $\forall \mathbf{w} \in \mathbf{W}, \hat{\beta} > 0$

If  $\beta > 0$  fixed:

(iii) Find  $\mathbf{y} \in \mathbf{W} : \hat{\eta}^2(u_h, \mathbf{y}, \beta) \leq \hat{\eta}^2(u_h, \mathbf{w}, \beta) \quad \forall \mathbf{w} \in \mathbf{W}$

(iv) Find  $\mathbf{y} \in \mathbf{W} : (\operatorname{div} \mathbf{y}, \operatorname{div} \mathbf{w}) + \frac{\beta}{C_F^2} (\mathbf{y}, \mathbf{w}) = \frac{\beta}{C_F^2} (\nabla u_h, \mathbf{w}) - (f, \operatorname{div} \mathbf{w})$

$\forall \mathbf{w} \in \mathbf{W}$





## (A) Error majorant

$$\|u - u_h\| \leq \eta(u_h, \mathbf{y}) = C_F \|f + \operatorname{div} \mathbf{y}\|_0 + \|\mathbf{y} - \nabla u_h\|_0 \quad \forall \mathbf{y} \in \mathbf{H}(\operatorname{div}, \Omega)$$

$$\|u - u_h\|^2 \leq \hat{\eta}^2(u_h, \mathbf{y}, \beta) = \left(1 + \frac{1}{\beta}\right) C_F^2 \|f + \operatorname{div} \mathbf{y}\|_0^2 + (1 + \beta) \|\mathbf{y} - \nabla u_h\|_0^2$$
$$\forall \beta > 0$$

Notation:  $\mathbf{W} = \mathbf{H}(\operatorname{div}, \Omega)$

Practical implementation:

- ▶  $\mathbf{W}_h \subset \mathbf{W}$ ,  $\dim \mathbf{W}_h < \infty$

e.g. Raviart-Thomas elements of degree  $p$ :

$$\mathbf{W}_h^p = \{\mathbf{w}_h \in \mathbf{H}(\operatorname{div}, \Omega) : \mathbf{w}_h|_K \in P^p(K) \forall K \in \mathcal{T}_h\}$$

- ▶ Set values for  $\beta$  and  $C_F$

- ▶ Find  $\mathbf{y}_h \in \mathbf{W}_h$  :

$$(\operatorname{div} \mathbf{y}_h, \operatorname{div} \mathbf{w}_h) + \frac{\beta}{C_F^2} (\mathbf{y}_h, \mathbf{w}_h) = \frac{\beta}{C_F^2} (\nabla u_h, \mathbf{w}_h) - (f, \operatorname{div} \mathbf{w}_h)$$

$$\forall \mathbf{w}_h \in \mathbf{W}_h$$

- ▶ Compute  $\eta(u_h, \mathbf{y}_h)$



# Friedrichs' constant $C_F$

$$\|v\|_0 \leq C_F \|v\| \quad \forall v \in V$$

(a) Analytical estimate (Mikhlin, 1986):  $V = H_0^1(\Omega)$

$$C_F \leq \frac{1}{\pi} \left( \frac{1}{|a_1|} + \dots + \frac{1}{|a_d|} \right)^{-1/2}, \quad \Omega \subset a_1 \times \dots \times a_d,$$

# Friedrichs' constant $C_F$

$$\|v\|_0 \leq C_F \|v\| \quad \forall v \in V$$

(b) Numerical upper bound:

$$C_F = \sup_{v \in V} \frac{\|v\|_0}{\|v\|} \Leftrightarrow \lambda_1 = \inf_{v \in V} \frac{\|v\|^2}{\|v\|_0^2}, \quad C_F^2 = 1/\lambda_1$$

Eigenvalue problem:  $u_i \in V : a(u_i, v) = \lambda_i(u_i, v) \quad \forall v \in V$

Galerkin approxim.:  $u_i^h \in V_h : a(u_i^h, v^h) = \lambda_i^h(u_i^h, v^h) \quad \forall v^h \in V_h$   
 $V_h \subset V$

$$\Rightarrow \lambda_1^h = \inf_{v^h \in V_h} \frac{\|v^h\|^2}{\|v^h\|_0^2} \Rightarrow \lambda_1 \leq \lambda_1^h \Rightarrow 1/\lambda_1^h \leq C_F^2$$

Sigillito, Kuttler (1970s):  $\bar{\lambda}_1^h \leq \lambda_1 \Rightarrow C_F^2 \leq 1/\bar{\lambda}_1^h$



## (B) Dual finite elements

$$\|u - u_h\| \leq \tilde{\eta}(u_h, \mathbf{y}) = \|\mathbf{y} - \nabla u_h\|_0 \quad \forall \mathbf{y} \in \mathbf{Q}(f)$$

$$\mathbf{Q}(f) = \{\mathbf{y} \in \mathbf{H}(\operatorname{div}, \Omega) : f + \operatorname{div} \mathbf{y} = 0\}$$

Complementary problem:

(i) Find  $\mathbf{y} \in \mathbf{Q}(f) : \tilde{\eta}(u_h, \mathbf{y}) \leq \tilde{\eta}(u_h, \mathbf{w}) \quad \forall \mathbf{w} \in \mathbf{Q}(f)$

(ii) Find  $\mathbf{y} \in \mathbf{Q}(f) : \frac{1}{2} \|\mathbf{y}\|_0^2 \leq \frac{1}{2} \|\mathbf{w}\|_0^2 \quad \forall \mathbf{w} \in \mathbf{Q}(f)$

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Lemma 1: (i)  $\Leftrightarrow$  (ii)  $\Leftrightarrow$  (iii)

Proof: (i)  $\Leftrightarrow$  (ii)

$$\begin{aligned} \|\mathbf{y} - \nabla u_h\|_0^2 &\leq \|\mathbf{w} - \nabla u_h\|_0^2 \\ \|\mathbf{y}\|_0^2 - 2(\mathbf{y}, \nabla u_h) + \|\nabla u_h\|_0^2 &\leq \|\mathbf{w}\|_0^2 - 2(\mathbf{w}, \nabla u_h) + \|\nabla u_h\|_0^2 \end{aligned}$$



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$$\|u - u_h\| \leq \tilde{\eta}(u_h, \mathbf{y}) = \|\mathbf{y} - \nabla u_h\|_0 \quad \forall \mathbf{y} \in \mathbf{Q}(f)$$

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$$\begin{aligned} \|\mathbf{y} - \nabla u_h\|_0^2 &\leq \|\mathbf{w} - \nabla u_h\|_0^2 \\ \|\mathbf{y}\|_0^2 - 2(\mathbf{y}, \nabla u_h) &\leq \|\mathbf{w}\|_0^2 - 2(\mathbf{w}, \nabla u_h) \end{aligned}$$



## (B) Dual finite elements

$$\|u - u_h\| \leq \tilde{\eta}(u_h, \mathbf{y}) = \|\mathbf{y} - \nabla u_h\|_0 \quad \forall \mathbf{y} \in \mathbf{Q}(f)$$

$$\mathbf{Q}(f) = \{\mathbf{y} \in \mathbf{H}(\text{div}, \Omega) : f + \text{div } \mathbf{y} = 0\}$$

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Proof: (i)  $\Leftrightarrow$  (ii)

$$\begin{aligned} \|\mathbf{y} - \nabla u_h\|_0^2 &\leq \|\mathbf{w} - \nabla u_h\|_0^2 \\ \|\mathbf{y}\|_0^2 - 2(f, u_h) &\leq \|\mathbf{w}\|_0^2 - 2(f, u_h) \end{aligned}$$



## (B) Dual finite elements

$$\|u - u_h\| \leq \tilde{\eta}(u_h, \mathbf{y}) = \|\mathbf{y} - \nabla u_h\|_0 \quad \forall \mathbf{y} \in \mathbf{Q}(f)$$

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Proof: (i)  $\Leftrightarrow$  (ii)

$$\begin{aligned} \|\mathbf{y}\|_0^2 & \leq \|\mathbf{y} - \nabla u_h\|_0^2 \leq \|\mathbf{w} - \nabla u_h\|_0^2 \\ & \leq \|\mathbf{w}\|_0^2 \end{aligned}$$





## (B) Dual finite elements

$$\|u - u_h\| \leq \tilde{\eta}(u_h, \mathbf{y}) = \|\mathbf{y} - \nabla u_h\|_0 \quad \forall \mathbf{y} \in \mathbf{Q}(f)$$

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Lemma 1: (i)  $\Leftrightarrow$  (ii)  $\Leftrightarrow$  (iii)

Proof: (ii)  $\Rightarrow$  (iii)

$$J(t) = \|\mathbf{y} + t\mathbf{w}^0\|_0^2, \quad J(t) \text{ has minimum at } t = 0$$

$$0 = J'(0) = \lim_{t \rightarrow 0} \frac{\|\mathbf{y} + t\mathbf{w}^0\|_0^2 - \|\mathbf{y}\|_0^2}{t}$$



## (B) Dual finite elements

$$\|u - u_h\| \leq \tilde{\eta}(u_h, \mathbf{y}) = \|\mathbf{y} - \nabla u_h\|_0 \quad \forall \mathbf{y} \in \mathbf{Q}(f)$$

$$\mathbf{Q}(f) = \{\mathbf{y} \in \mathbf{H}(\text{div}, \Omega) : f + \text{div } \mathbf{y} = 0\}$$

Complementary problem:

(i) Find  $\mathbf{y} \in \mathbf{Q}(f) : \tilde{\eta}(u_h, \mathbf{y}) \leq \tilde{\eta}(u_h, \mathbf{w}) \quad \forall \mathbf{w} \in \mathbf{Q}(f)$

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Proof: (ii)  $\Rightarrow$  (iii)

$$J(t) = \|\mathbf{y} + t\mathbf{w}^0\|_0^2, \quad J(t) \text{ has minimum at } t = 0$$

$$0 = J'(0) = \lim_{t \rightarrow 0} \frac{\|\mathbf{y}\|_0^2 + 2t(\mathbf{y}, \mathbf{w}^0) + t^2 \|\mathbf{w}^0\|_0^2 - \|\mathbf{y}\|_0^2}{t} = 2(\mathbf{y}, \mathbf{w}^0)$$



## (B) Dual finite elements

$$\|u - u_h\| \leq \tilde{\eta}(u_h, \mathbf{y}) = \|\mathbf{y} - \nabla u_h\|_0 \quad \forall \mathbf{y} \in \mathbf{Q}(f)$$

$$\mathbf{Q}(f) = \{\mathbf{y} \in \mathbf{H}(\text{div}, \Omega) : f + \text{div } \mathbf{y} = 0\}$$

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(i) Find  $\mathbf{y} \in \mathbf{Q}(f) : \tilde{\eta}(u_h, \mathbf{y}) \leq \tilde{\eta}(u_h, \mathbf{w}) \quad \forall \mathbf{w} \in \mathbf{Q}(f)$

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(iii) Find  $\mathbf{y} \in \mathbf{Q}(f) : (\mathbf{y}, \mathbf{w}^0) = 0 \quad \forall \mathbf{w}^0 \in \mathbf{Q}(0)$

Lemma 1: (i)  $\Leftrightarrow$  (ii)  $\Leftrightarrow$  (iii)

Proof: (iii)  $\Rightarrow$  (ii)

$$\mathbf{w} \in \mathbf{Q}(f), \quad \exists \mathbf{w}^0 \in \mathbf{Q}(0) : \mathbf{w} = \mathbf{y} + \mathbf{w}^0, \quad (\mathbf{y}, \mathbf{w}) = \|\mathbf{y}\|_0^2$$

$$0 \leq \|\mathbf{w} - \mathbf{y}\|_0^2 = \|\mathbf{w}\|_0^2 - 2(\mathbf{y}, \mathbf{w}) + \|\mathbf{y}\|_0^2 = \|\mathbf{w}\|_0^2 - \|\mathbf{y}\|_0^2 \quad \square$$



## (B) Dual finite elements

$$\|u - u_h\| \leq \tilde{\eta}(u_h, \mathbf{y}) = \|\mathbf{y} - \nabla u_h\|_0 \quad \forall \mathbf{y} \in \mathbf{Q}(f)$$

$$\mathbf{Q}(f) = \{\mathbf{y} \in \mathbf{H}(\text{div}, \Omega) : f + \text{div } \mathbf{y} = 0\}$$

Complementary problem:

(i) Find  $\mathbf{y} \in \mathbf{Q}(f) : \tilde{\eta}(u_h, \mathbf{y}) \leq \tilde{\eta}(u_h, \mathbf{w}) \quad \forall \mathbf{w} \in \mathbf{Q}(f)$

(ii) Find  $\mathbf{y} \in \mathbf{Q}(f) : \frac{1}{2} \|\mathbf{y}\|_0^2 \leq \frac{1}{2} \|\mathbf{w}\|_0^2 \quad \forall \mathbf{w} \in \mathbf{Q}(f)$

(iii) Find  $\mathbf{y} \in \mathbf{Q}(f) : (\mathbf{y}, \mathbf{w}^0) = 0 \quad \forall \mathbf{w}^0 \in \mathbf{Q}(0)$

**Lemma 2:**  $\mathbf{y} = \nabla u \in \mathbf{Q}(f)$  is the unique solution of (i)–(iii)

**Proof:**

If  $\mathbf{y}_1 \in \mathbf{Q}(f)$  and  $\mathbf{y}_2 \in \mathbf{Q}(f)$  satisfy (iii):

$$\Rightarrow \mathbf{y}_2 - \mathbf{y}_1 \in \mathbf{Q}(0) \text{ and } (\mathbf{y}_2 - \mathbf{y}_1, \mathbf{w}^0) = 0$$

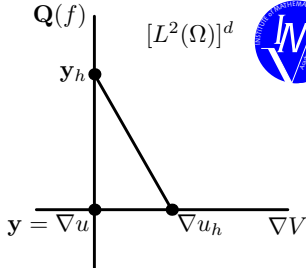
$$\Rightarrow \|\mathbf{y}_2 - \mathbf{y}_1\|_0^2 = 0 \quad \Rightarrow \quad \mathbf{y}_2 = \mathbf{y}_1$$



## (B) Dual finite elements

$$\|u - u_h\| \leq \tilde{\eta}(u_h, \mathbf{y}) = \|\mathbf{y} - \nabla u_h\|_0$$

$$\mathbf{Q}(f) = \{\mathbf{y} \in \mathbf{H}(\text{div}, \Omega) : f + \text{div } \mathbf{y} = 0\}$$



Complementary problem:

(i) Find  $\mathbf{y} \in \mathbf{Q}(f) : \tilde{\eta}(u_h, \mathbf{y}) \leq \tilde{\eta}(u_h, \mathbf{w}) \quad \forall \mathbf{w} \in \mathbf{Q}(f)$

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(iii) Find  $\mathbf{y} \in \mathbf{Q}(f) : (\mathbf{y}, \mathbf{w}^0) = 0 \quad \forall \mathbf{w}^0 \in \mathbf{Q}(0)$

**Lemma 3:**  $\tilde{\eta}^2(u, \mathbf{y}_h) + \tilde{\eta}^2(u_h, \mathbf{y}) = \tilde{\eta}^2(u_h, \mathbf{y}_h) \quad \forall u_h \in V, \mathbf{y}_h \in \mathbf{Q}(f)$

$$\|\mathbf{y}_h - \mathbf{y}\|_0^2 + \|\nabla u - \nabla u_h\|_0^2 = \|\mathbf{y}_h - \nabla u_h\|_0^2$$

**Proof:**

$$\|\mathbf{y}_h - \nabla u + \nabla u - \nabla u_h\|_0^2 = \|\mathbf{y}_h - \nabla u\|_0^2 + \|\nabla u - \nabla u_h\|_0^2$$

$$(\mathbf{y}_h - \nabla u, \nabla u - \nabla u_h) = 0$$



## (B) Dual finite elements

Practical implementation:  $d = 2$ ,  $\Omega$  simply connected

$$\blacktriangleright \bar{\mathbf{q}}(x_1, x_2) = - \left( \int_0^{x_1} f(s, x_2) ds, 0 \right)^\top \Rightarrow -\operatorname{div} \bar{\mathbf{q}} = f$$

$$\blacktriangleright \mathbf{Q}(f) = \bar{\mathbf{q}} + \mathbf{Q}(0) = \bar{\mathbf{q}} + \operatorname{curl} H^1(\Omega) \quad \operatorname{curl} = (\partial_2, -\partial_1)^\top$$

$\blacktriangleright$  Complementary problem:

$$(iii) \quad \mathbf{y} = \bar{\mathbf{q}} + \operatorname{curl} z \in \mathbf{Q}(f) : (\mathbf{y}, \mathbf{w}^0) = 0 \quad \forall \mathbf{w}^0 \in \mathbf{Q}(0)$$

$$(iv) \quad z \in H^1(\Omega) : \quad (\operatorname{curl} z, \operatorname{curl} v) = -(\bar{\mathbf{q}}, \operatorname{curl} v) \quad \forall v \in H^1(\Omega)$$

$$(v) \quad z \in H^1(\Omega) : \quad (\nabla z, \nabla v) = -(\bar{\mathbf{q}}, \operatorname{curl} v) \quad \forall v \in H^1(\Omega)$$

$\blacktriangleright$  Galerkin approximation:

$$z_h \in Z_h \subset H^1(\Omega) : \quad (\nabla z_h, \nabla v_h) = -(\bar{\mathbf{q}}, \operatorname{curl} v_h) \quad \forall v_h \in Z_h$$

$$\blacktriangleright \mathbf{y}_h = \bar{\mathbf{q}} + \operatorname{curl} z_h$$

$\blacktriangleright$  Compute  $\tilde{\eta}(u_h, \mathbf{y}_h)$



# Energy minimization

$$a(u, v) = (\nabla u, \nabla v), \quad \mathcal{F}(v) = (f, v), \quad a^*(\mathbf{y}, \mathbf{w}) = (\mathbf{y}, \mathbf{w})$$

Primal problem:

$$u \in V : a(u, v) = \mathcal{F}(v) \quad \forall v \in V$$

$$u \in V : J(u) = \min_{v \in V} J(v), \quad J(v) = \frac{1}{2}a(v, v) - \mathcal{F}(v)$$

Complementary problem:

$$\mathbf{y} \in \mathbf{Q}(f) : a^*(\mathbf{y}, \mathbf{w}^0) = 0 \quad \forall \mathbf{w}^0 \in \mathbf{Q}(0)$$

$$\mathbf{y} \in \mathbf{Q}(f) : J^*(\mathbf{y}) = \min_{\mathbf{w} \in \mathbf{Q}(f)} J^*(\mathbf{w}), \quad J^*(\mathbf{w}) = \frac{1}{2}a^*(\mathbf{w}, \mathbf{w})$$

Complementarity of energies:

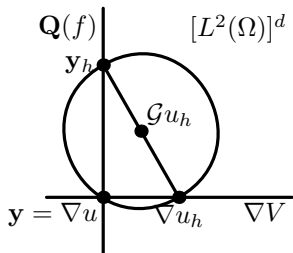
$$J(u) + J^*(\mathbf{y}) = -\frac{1}{2}a(u, u) + \frac{1}{2}a^*(\nabla u, \nabla u) = 0$$

# Method of hypercircle



Theorem: If

- ▶  $u \in V$  is primal solution
- ▶  $u_h \in V$ ,  $\mathbf{y}_h \in \mathbf{Q}(f)$  arbitrary
- ▶  $\mathcal{G}u_h = (\mathbf{y}_h + \nabla u_h)/2$



Then

$$\|\nabla u - \mathcal{G}u_h\|_0 = \frac{1}{2} \tilde{\eta}(u_h, \mathbf{y}_h).$$

Proof:

$$\begin{aligned} 4 \|\nabla u - \mathcal{G}u_h\|_0^2 &= \|\nabla u - \mathbf{y}_h + \nabla u - \nabla u_h\|_0^2 \\ &= \|\nabla u - \mathbf{y}_h\|_0^2 + \|\nabla u - \nabla u_h\|_0^2 = \|\nabla u_h - \mathbf{y}_h\|_0^2 \end{aligned}$$

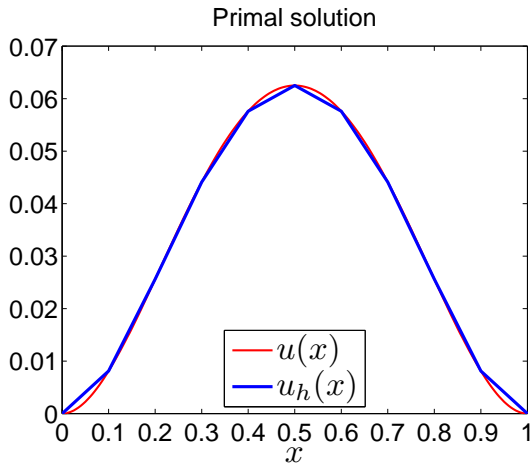




# Numerical illustration of hypercircle

$$-u''(x) = 2 - 12x + 12x^2 \text{ in } (0, 1), \quad u(0) = u(1) = 0$$

$$u(x) = x^2(1-x)^2$$

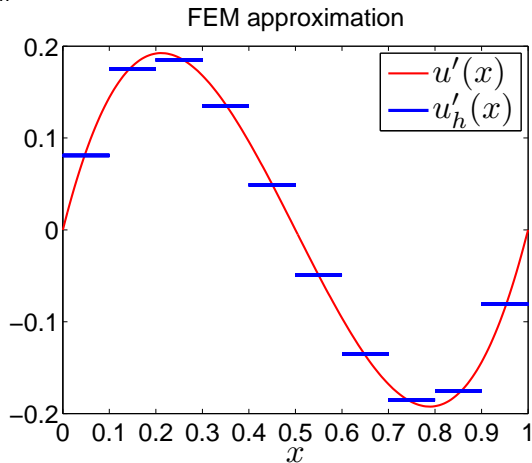


# Numerical illustration of hypercircle

$$-u''(x) = 2 - 12x + 12x^2 \text{ in } (0, 1), \quad u(0) = u(1) = 0$$

$$\mathbf{y}_h(x) = u'(x) = 2x(2x - 1)(x - 1)$$

$$\mathcal{G}u_h(x) = (\mathbf{y}_h(x) + u'_h)/2$$





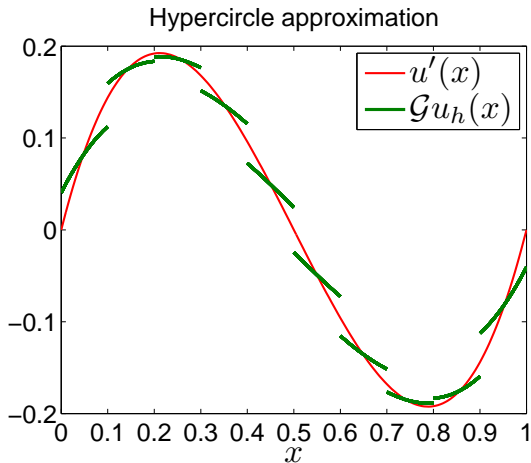
# Numerical illustration of hypercircle

$$-u''(x) = 2 - 12x + 12x^2 \text{ in } (0, 1), \quad u(0) = u(1) = 0$$

$$\mathbf{y}_h(x) = u'(x) = 2x(2x - 1)(x - 1)$$

$$\mathcal{G}u_h(x) = (\mathbf{y}_h(x) + u'_h)/2$$

$$\frac{\|\nabla u - \mathcal{G}u_h\|_0}{\|u_h\|_0} \doteq 9.3\%$$



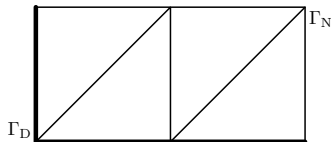
## Example 1



$$-\Delta u = f \text{ in } (0, 2) \times (0, 1)$$

$$u = 0 \text{ on } \Gamma_D$$

$$\mathbf{n}^\top \nabla u = 0 \text{ on } \Gamma_N$$

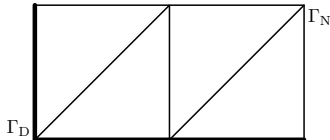


$$f = \frac{5\pi^2}{16} u$$

$$u = \sin \frac{\pi x_1}{4} \sin \frac{\pi x_2}{2}$$

# Example 1

$$\begin{aligned}
 -\Delta u &= f \text{ in } (0, 2) \times (0, 1) \\
 u &= 0 \text{ on } \Gamma_D \\
 \mathbf{n}^\top \nabla u &= 0 \text{ on } \Gamma_N
 \end{aligned}$$



$$f = \frac{5\pi^2}{16} u$$

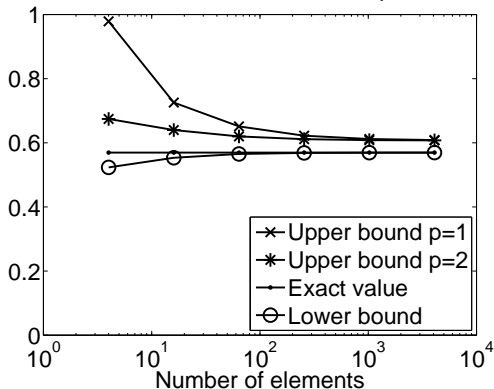
$$u = \sin \frac{\pi x_1}{4} \sin \frac{\pi x_2}{2}$$

$$C_F = \frac{4}{\sqrt{5}\pi} \doteq 0.5694$$

$$C_F^{\text{low}} = 0.5693$$

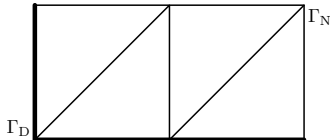
$$C_F^{\text{up}} = 0.6004$$

Friedrichs' constant – Example 1



# Example 1

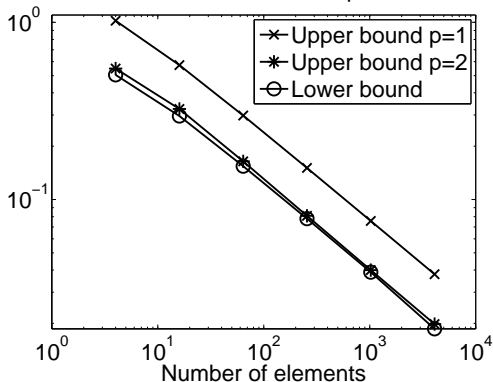
$$\begin{aligned}
 -\Delta u &= f \text{ in } (0, 2) \times (0, 1) \\
 u &= 0 \text{ on } \Gamma_D \\
 \mathbf{n}^\top \nabla u &= 0 \text{ on } \Gamma_N
 \end{aligned}$$



Lower bound:  
reference solution

Upper bound:  
error majorant

Error bounds – Example 1



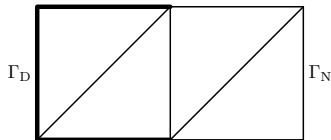
## Example 2



$$-\Delta u = f \text{ in } (0, 2) \times (0, 1)$$

$$u = 0 \text{ on } \Gamma_D$$

$$\mathbf{n}^\top \nabla u = 0 \text{ on } \Gamma_N$$



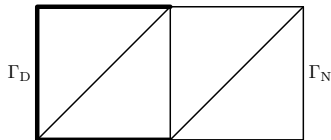
$$f = \frac{5\pi^2}{16} \sin \frac{\pi x_1}{4} \sin \frac{\pi x_2}{2}$$

## Example 2

$$-\Delta u = f \text{ in } (0, 2) \times (0, 1)$$

$$u = 0 \text{ on } \Gamma_D$$

$$\mathbf{n}^\top \nabla u = 0 \text{ on } \Gamma_N$$



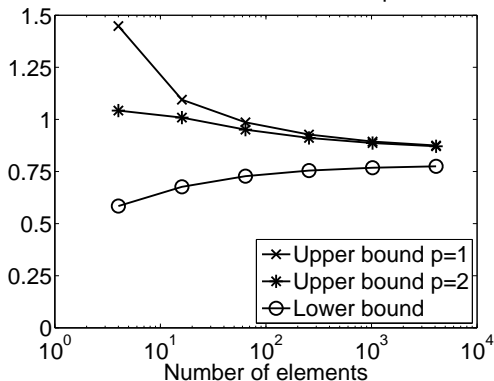
$$f = \frac{5\pi^2}{16} \sin \frac{\pi x_1}{4} \sin \frac{\pi x_2}{2}$$

$$C_F = ?$$

$$C_F^{\text{low}} = 0.7750$$

$$C_F^{\text{up}} = 0.8712$$

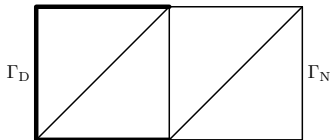
Friedrichs' constant – Example 2





## Example 2

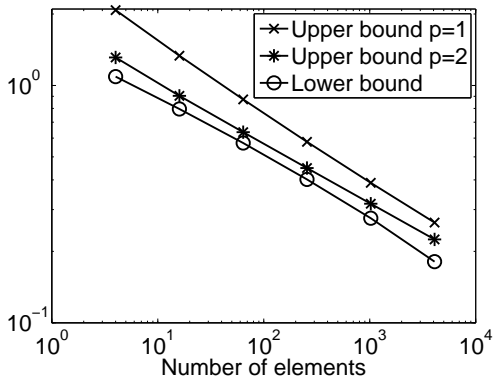
$$\begin{aligned}
 -\Delta u &= f \text{ in } (0, 2) \times (0, 1) \\
 u &= 0 \text{ on } \Gamma_D \\
 \mathbf{n}^\top \nabla u &= 0 \text{ on } \Gamma_N
 \end{aligned}$$



Lower bound:  
reference solution

Upper bound:  
error majorant

Error bounds – Example 2







- ▶ Guaranteed upper bounds
  - ▶ if exact arithmetic
  - ▶ if exact quadrature
- ▶ Arbitrary numerical method for  $u_h \in V$ 
  - ▶ solving complementary problem is expensive
- ▶ Particular numerical method for  $u_h \in V$ 
  - ▶ postprocessing of  $\nabla u_h \Rightarrow$  fast  $\mathbf{y}_h$
- ▶ Total error
  - ▶ including quadrature and algebraic errors
- ▶ Technical difficulties
  - ▶ Friedrichs' constant
  - ▶ handling of  $\mathbf{Q}(f)$







# History

- 2000– S. Repin (S. Korotov, J. Valdman, S. Sauter, M. Frolov, . . . )  
M. Vohralík (R. Fučík, I. Cheddadi, M.I. Prieto, . . . )
- 1976– I. Hlaváček (J. Haslinger, M. Křížek, J. Vacek, J. Weisz, . . . )
- 1971 J.P. Aubin and H.G. Burchard
- 1957 J.L. Synge

## Books:

-  P. Neittaanmäki, S. Repin, Reliable methods for computer simulation, error control and a posteriori estimates, Elsevier, Amsterdam, 2004.
-  S. Repin, A posteriori estimates for partial differential equations, de Gruyter, Berlin, 2008.

-  T. Vejchodský: Complementary error bounds for elliptic systems and applications, in press Appl. Math. Comput., 2011. (Preprint 232.)
-  T. Vejchodský: Complementarity - the way towards guaranteed error estimates, in: Programs and Algorithms of Numerical Mathematics 15, Institute of Mathematics, Prague, 2010, pp. 205–220. (Preprint 231.)
-  M. Ainsworth, T. Vejchodský: Fully computable robust a posteriori error bounds for singularly perturbed reaction-diffusion problems, Numer. Math. 119 (2011) 219–243. (Preprint 208.)
-  T. Vejchodský: Complementarity based a posteriori error estimates and their properties, in press Math. Comput. Simulation, 2011. (Preprint 190.)

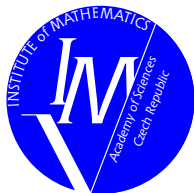
# Thank you for your attention

## Part II – Complementary estimates

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