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Two-point boundary value problems for strongly singular higher-order linear differential equations with deviating arguments

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For the the differential equations with deviating arguments

$$u^{(n)}(t)=\sum_{j=1}^{m}p_j(t)u^{(j-1)}(\tau_j(t))+q(t) \quad \text{for} \quad a<t<b \tag{1}$$

with the two-point boundary conditions

$$u^{(i-1)}(a)=0 \quad (j=1,\dots,m), \qquad u^{(i-1)}(b)=0 \quad (j=1,\dots,n-m), \tag{2}$$

$$u^{(i-1)}(a)=0 \quad (i=1,\dots,m), \qquad u^{(i-1)}(b)=0 \quad (j=m+1,\dots,n). \tag{3}$$

We are proved the Agarwal-Kiguradze type theorems which guarantee Fredholm's property for problems (1), (2), and (1), (3). Moreover, we establish in some sense, the optimal sufficient conditions of the solvability for problems (1), (2), and (1), (3).

Here $n\geq 2$, m is the integer part of $n/2$, $-\infty<a<b<+\infty$, $p_j,q\in L_{loc}(]a,b[)$ $(j=1,\dots,m)$, and $\tau_j\colon\ ]a,b[\,\to\,]a,b[$ are measurable functions. By $u^{(j-1)}(a)$ $(u^{(j-1)}(b))$ we denote the right (the left) limit of the function $u^{(j-1)}$ at the point a (b).

We use the following notations:

$L_{\alpha,\beta}(]a,b[)$ $(L^2_{\alpha,\beta}(]a,b[))$ is the space of integrable (square integrable) with the weight $(t-a)^{\alpha}(b-t)^{\beta}$ functions $y\colon\ ]a,b[\,\to\mathbb{R}$, with the norm

$$\|y\|_{L_{\alpha,\beta}}=\int_a^b(s-a)^{\alpha}(b-s)^{\beta}|y(s)|ds \quad \left(\|y\|_{L^2_{\alpha,\beta}}=\left(\int_a^b(s-a)^{\alpha}(b-s)^{\beta}y^2(s)ds\right)^{1/2}\right);$$

$M(]a,b[)$ is the set of the measurable functions $\tau\colon\ ]a,b[\,\to\,]a,b[$;

$\widetilde{L}^2_{\alpha,\beta}(]a,b[)$ $(\widetilde{L}^2_{\alpha}(]a,b[))$ is the space of functions $y\in L_{loc}(]a,b[)$ $(L_{loc}(]a,b]))$ such that $\widetilde{y}\in L_{\alpha,\beta}(]a,b[)$, where $\widetilde{y}(t)=\int_c^t y(s)ds$, $c=(a+b)/2$ $(\widetilde{y}\in L_{\alpha,0}(]a,b[)$, where $\widetilde{y}(t)=\int_t^b y(s)ds)$;

$\|\cdot\|_{\widetilde{L}^2_{\alpha,\beta}}$ ($\|\cdot\|_{\widetilde{L}^2_{\alpha}}$) denote the norm in $\widetilde{L}^2_{\alpha,\beta}(]a,b[)$ ($\widetilde{L}^2_{\alpha}(]a,b[)$, and are defined by the equalities

$$\|y\|_{\widetilde{L}^2_{\alpha,\beta}} = \max\left\{\left[\int_a^t (s-a)^\alpha \left(\int_s^t y(\xi)d\xi\right)^2 ds\right]^{1/2} : a \le t \le \frac{a+b}{2}\right\}$$
$$+ \max\left\{\left[\int_t^b (b-s)^\beta \left(\int_t^s y(\xi)d\xi\right)^2 ds\right]^{1/2} : \frac{a+b}{2} \le t \le b\right\}, \tag{4}$$
$$\|y\|_{\widetilde{L}^2_{\alpha}} = \max\left\{\left[\int_a^t (s-a)^\alpha \left(\int_s^t y(\xi)d\xi\right)^2 ds\right]^{1/2} : a \le t \le b\right\};$$

$\widetilde{C}^{n-1,\,m}(]a,b[)$ ($\widetilde{C}^{n-1,\,m}(]a,b])$) is the space of functions $y \in \widetilde{C}^{n-1}_{loc}(]a,b[)$ ($y \in \widetilde{C}^{n-1}_{loc}(]a,b])$) such that $\int_a^b |u^{(m)}(s)|^2 ds < +\infty$.

When problem (1), (2) is discussed, we assume that when $n = 2m$, the conditions $p_j \in L_{loc}(]a,b[)$ $(j = 1,\dots,m)$, are fulfilled, and when $n = 2m+1$, along with $p_j \in L_{loc}(]a,b[)$, the conditions

$$\limsup_{t\to b} \left|(b-t)^{2m-1}\int_{t_1}^t p_1(s)ds\right| < +\infty \quad \left(t_1 = \frac{a+b}{2}\right)$$

are fulfilled. Problem (1), (3) is discussed under the assumptions $p_j \in L_{loc}(]a,b])$ $(j = 1,\dots,m)$. A solution of problem (1), (2) ((1), (3)) is sought in the space $\widetilde{C}^{n-1,\,m}(]a,\,b[)$ ($\widetilde{C}^{n-1,\,m}(]a,\,b])$).

By $h_j\colon\ ]a,b[\times]a,b[\to \mathbb{R}_+$ and $f_j\colon \mathbb{R}\times M(]a,b[) \to C_{loc}(]a,b[\times]a,b[)$ $(j = 1,\dots,m)$ we denote the functions and operator, respectively defined by the equalities

$$h_1(t,s) = \left|\int_s^t (\xi-a)^{n-2m}[(-1)^{n-m}p_1(\xi)]_+ d\xi\right|,$$
$$h_j(t,s) = \left|\int_s^t (\xi-a)^{n-2m}p_j(\xi)d\xi\right| \quad (j = 2,\dots,m),$$

and

$$f_j(c,\tau_j)(t,s) = \left|\int_s^t (\xi-a)^{n-2m}|p_j(\xi)|\left|\int_\xi^{\tau_j(\xi)} (\xi_1-c)^{2(m-j)}d\xi_1\right|^{1/2} d\xi\right|.$$

1. Fredholm type theorems

Definition 1. We said that problem (1), (2) ((1), (3)) has the Fredholm's property in the space $\widetilde{C}^{n-1}(]a,\,b[)$ ($\widetilde{C}^{n-1}(]a,b])$), if the unique solvability of the corresponding homogeneous problem in this space, implies the unique solvability of problem (1), (2) ((1), (3)) for every $q \in \widetilde{L}^2_{2n-2m-2,\,2m-2}(]a,b[)$ ($q \in \widetilde{L}^2_{2n-2m-2}(]a,b])$).

Theorem 1. *Let there exist $a_0 \in]a,b[$, $b_0 \in]a_0,b[$, numbers $l_{kj} > 0$, $\gamma_{kj} > 0$, and functions $\tau_j \in M(]a,b[)$ $(k = 0,1,\ j = 1,\dots,m)$ such that*

$$(t-a)^{2m-j}h_j(t,s) \le l_{0j} \quad \textit{for } a \le t \le s < a_0,$$
$$\limsup_{t\to a}(t-a)^{m-\frac{1}{2}-\gamma_{0j}}f_j(a,\tau_j)(t,s) < +\infty, \tag{5}$$

$$
\begin{gathered}
(b-t)^{2m-j} h_j(t,s) \le l_{1j} \quad \textit{for } b_0 \le s \le t < b, \\
\limsup_{t\to b} (b-t)^{m-\frac{1}{2}-\gamma_{1j}} f_j(b,\tau_j)(t,s) < +\infty,
\end{gathered} \tag{6}
$$

and

$$
\sum_{j=1}^{m} \frac{(2m-j)2^{2m-j+1}}{(2m-1)!!(2m-2j+1)!!}\, l_{kj} < 1 \quad (k=0,1). \tag{7}
$$

Then problem (1), (2) *has the Fredholm's property in the space* $\widetilde{C}^{n-1,m}(]a,b[)$ *and exists the constant* r*, independent on* q*, such that* $\|u^{(m)}\|_{L^2} \le r\|q\|_{\widetilde{L}^2_{2n-2m-2,\,2m-2}}$.

Theorem 2. *Let there exist* $a_0 \in]a,b[$*, numbers* $l_{0j} > 0$*,* $\gamma_{0j} > 0$*, and functions* $\tau_j \in M(]a,b[)$ *such that condition* (5) *is fulfilled and*

$$
\sum_{j=1}^{m} \frac{(2m-j)2^{2m-j+1}}{(2m-1)!!(2m-2j+1)!!}\, l_{0j} < 1.
$$

Then problem (1), (3) *has the Fredholm's property in the space* $\widetilde{C}^{n-1,\,m}(]a,b[)$ *and exists the constant* r*, independent on* q*, such that* $\|u^{(m)}\|_{L^2} \le r\|q\|_{\widetilde{L}^2_{2n-2m-2}}$.

2. Existence and uniqueness theorems

Theorem 3. *Let there exist numbers* $t^* \in]a,b[$ *,* $\ell_{kj} > 0$*,* $\bar{l}_{kj} \ge 0$*, and* $\gamma_{kj} > 0$ $(k=0,1;\ j=1,\cdots)$*, such that along with*

$$
\sum_{j=1}^{m} \left(\frac{(2m-j)2^{2m-j+1}\, l_{0j}}{(2m-1)!!(2m-2j+1)!!} + \frac{2^{2m-j-1}(t^*-a)^{\gamma_{0j}}\, \bar{l}_{0j}}{(2m-2j-1)!!(2m-3)!!\sqrt{2\gamma_{0j}}} \right) < \frac{1}{2}\,, \tag{8}
$$

$$
\sum_{j=1}^{m} \left(\frac{(2m-j)2^{2m-j+1}\, l_{1j}}{(2m-1)!!(2m-2j+1)!!} + \frac{2^{2m-j-1}(b-t^*)^{\gamma_{0j}}\, \bar{l}_{1j}}{(2m-2j-1)!!(2m-3)!!\sqrt{2\gamma_{1j}}} \right) < \frac{1}{2}\,, \tag{9}
$$

the conditions

$$
\begin{gathered}
(t-a)^{2m-j} h_j(t,s) \le l_{0j}, \quad (t-a)^{m-\gamma_{0j}-1/2} f_j(a,\tau_j)(t,s) \le \bar{l}_{0j} \quad \textit{for } a < t \le s \le t^*, \\
(b-t)^{2m-j} h_j(t,s) \le l_{1j}, \quad (b-t)^{m-\gamma_{1j}-1/2} f_j(b,\tau_j)(t,s) \le \bar{l}_{1j} \quad \textit{for } t^* \le s \le t < b
\end{gathered}
$$

hold. Then for every $q \in \widetilde{L}^2_{2n-2m-2,\,2m-2}(]a,b[)$ *problem* (1), (2) *is uniquely solvable in the space* $\widetilde{C}^{n-1,\,m}(]a,b[)$.

Theorem 4. *Let numbers* $\ell_{0j} > 0$*,* $\bar{\ell}_{0j} \ge 0$*, and* $\gamma_{0j} > 0$ $(j=1,\ldots,m)$ *be such that conditions*

$$
(t-a)^{2m-j} h_j(t,s) \le l_{0j}, \quad (t-a)^{m-\gamma_{0j}-1/2} f_j(a,\tau_j)(t,s) \le \bar{l}_{0j} \quad \textit{for } a < t \le s \le b,
$$

and

$$
\sum_{j=1}^{m} \left(\frac{(2m-j)2^{2m-j+1}\, l_{0j}}{(2m-1)!!(2m-2j+1)!!} + \frac{2^{2m-j-1}(t^*-a)^{\gamma_{0j}}\, \bar{l}_{0j}}{(2m-2j-1)!!(2m-3)!!\sqrt{2\gamma_{0j}}} \right) < 1
$$

hold. Then for every $q \in \widetilde{L}^2_{2n-2m-2}(]a,b[)$ *problem* (1), (3) *is uniquely solvable in the space* $\widetilde{C}^{n-1,\,m}(]a,b])$.

To illustrate Theorem 3, we consider the second order differential equation with a deviating argument

$$u''(t) = p(t)u(\tau(t)) + q(t) \tag{10}$$

under the boundary conditions

$$u(a) = 0, \quad u(b) = 0. \tag{11}$$

Corollary 1. *Let function* $\tau \in M(]a,b[)$ *be such that*

$$\begin{aligned} 0 \le \tau(t) - t &\le \frac{2^6}{(b-a)^6}(t-a)^7 \quad \text{for } a < t \le \frac{a+b}{2}, \\ -\frac{2^6}{(b-a)^6}(b-t)^7 &\le t - \tau(t) \le 0 \quad \text{for } \frac{a+b}{2} \le t < b. \end{aligned} \tag{12}$$

Moreover, let function $p\colon\ ]a,b[\to \mathbb{R}$ *and constants* κ_0, κ_1 *be such that*

$$-\frac{2^{-2}(b-a)^2\kappa_0}{[(b-t)(t-a)]^2} \le p_1(t) \le \frac{2^{-7}(b-a)^6\kappa_1}{[(b-t)(t-a)]^4} \quad \text{for } a < t \le b \tag{13}$$

and $4\kappa_0 + \kappa_1 < \frac{1}{2}$. *Then for every* $q \in \widetilde{L}^2_{0,0}(]a,b[)$ *problem* (10), (11) *is uniquely solvable in the space* $\widetilde{C}^{1,1}(]a,b[)$.

Corollary 2. *Let functions* $p\colon\ ]a,b[\to \mathbb{R}$, $\tau \in M(]a,b[)$ *and constants* $\kappa_0 > 0$, $\kappa_1 > 0$ *be such that along with* (12) *and* (13) *the inequalities*

$$\operatorname{sign}\left[\left(\tau(t) - \frac{a+b}{2}\right)\left(t - \frac{a+b}{2}\right)\right] \ge 0 \quad \text{for } a < t < b$$

and $4\kappa_0 + \kappa_1 < 1$ *hold. Then for every* $q \in \widetilde{L}^2_{0,0}(]a,b[)$ *problem* (10), (11) *is uniquely solvable in the space* $\widetilde{C}^{1,1}(]a,b[)$.

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