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Impulsive differential equations

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Let $\mathbb{M}$ be a phase space of a certain evolution process. Let $x(t)$ denotes a point that describes the state of the process at the time t. A point $x(t)$ may be interpreted, for a fixed value of t, as an n-dimensional vector of the Euclidean space $\mathbb{R}^n$ and $\mathbb{M}$ may be considered as a subset of $\mathbb{R}^n$. The topological product of the phase space $\mathbb{M}$ and the real axis $\mathbb{R}$ will be called the extended phase space of the considered evolution process. Let the evolution of the process be described by:

a) the system of differential equations

$$\frac{dx}{dt} = f(t,x), \quad x \in M, \quad t \in \mathbb{R}; \tag{1}$$

b) a certain set $\mathcal{T}_t$ given in the extended phase space;

c) an operator $\mathcal{A}_t$ defined on the set $\mathcal{T}_t$ which is mapped into the set $\mathcal{T}_t' = \mathcal{A}_t\mathcal{T}_t$ in the extended phase space.

The set of conditions a)–c) that characterize the evolution process will be called an *impulsive differential system*. We will call the curve described by the point $\{t, x(t)\}$ in the extended phase space an *integral curve* and the function $x = x(t)$ that gives hat curve a *solution* of this system.

In the theory of impulsive systems, there are some problems similar to the ones considered in the theory of the ordinary differential equations but there are also problems that are specific to the theory of impulsive systems. These problems depend greatly on the properties of the operator $\mathcal{A}_t$. For example, if the operator $\mathcal{A}_t$ is not a single-valued, then there are the problems related to a study of trajectories, for which the moving point can “instantaneously” split into several points when it meets the set $\mathcal{T}_t$.

The talk is organized in the following way. In Section 1, we give a general description of the mathematical model and classification of impulsive differential system depending on the characteristics of the impulses. In Section 2, we consider a linear impulsive system and the basic properties and stability of solutions. A special attention are payed to the periodic impulsive systems.

A linear differential impulsive system

$$\frac{dx}{dt} = A(t)x, \quad t \neq \tau_i, \quad \Delta x|_{t=\tau_i} = B_i x \tag{2}$$

is called T-periodic if the matrix $A(t)$ is T-periodic and there is a natural number p such that

$$B_{i+p} = B_i, \quad \tau_{i+p} = \tau_i + T \tag{3}$$

for all $i \in \mathbb{Z}$. Assume that matrix $A(t)$ is continuous (piecewise continuous with the first kind discontinuities at $t = \tau_i$), the matrices $E + B_i$ are nonsingular, and the times τ_i are indexed by integers such that $0 < \tau_1 < \ldots < \tau_p < T$.

Theorem 1. *Linear periodic impulsive differential system* (2) *can be reduced to a system with constant coefficients by a linear nonsingular piecewise continuous periodic Lyapunov transformation of variables.*

In Section 3, we consider a system of differential equations, defined in the direct product of an m-dimensional torus $\mathcal{T}_m$ and an n-dimensional Euclidean space $\mathbb{R}^n$ that undergo impulsive perturbations at the moments when the phase point φ meets a given set in the phase space

$$\begin{aligned} \frac{d\varphi}{dt} &= a(\varphi), \\ \frac{dx}{dt} &= A(\varphi)x + f(\varphi), \quad \varphi \notin \Gamma, \\ \Delta x|_{\varphi\in\Gamma} &= B(\varphi)x + g(\varphi), \end{aligned} \tag{4}$$

where $\varphi = (\varphi_1, \ldots, \varphi_m)^T \in \mathcal{T}_m$, $x = (x_1, \ldots, x_n)^T \in \mathbb{R}^n$, $a(\varphi)$ is a continuous 2π-periodic with respect to each of the components φ_v, $v = 1, \ldots, m$ vector function that satisfies a Lipschitz condition with respect to φ. Functions $A(\varphi), B(\varphi)$ are continuous 2π-periodic with respect to each of the components φ_v, $v = 1, \ldots, m$ square matrices; $f(\varphi), g(\varphi)$ are continuous (piecewise continuous with first kind discontinuities in the set Γ) 2π-periodic with respect to each of the components φ_v, $v = 1, \ldots, m$ vector functions.

We assume that the set Γ is a subset of the torus $\mathcal{T}_m$, which is a manifold of dimension $m - 1$ defined by the equation $\Phi(\varphi) = 0$ for some continuous scalar 2π-periodic with respect to each of the components φ_v, $v = 1, \ldots, m$ function.

Denote by $\varphi_t(\varphi)$ the solution of the first equation of system (4) that satisfies the initial condition $\varphi_0(\varphi) = \varphi$. Let $t_i(\varphi)$, $i \in \mathbb{Z}$ are the solutions of the equation $\Phi(\varphi_t(\varphi)) = 0$ that are the moments of impulsive action in system (4). Let the function $\Phi(\varphi)$ be such that the solutions $t = t_i(\varphi)$ exist since otherwise system (4) would not be an impulsive system.

We call a point φ^* an ω-limit point of the trajectory $\varphi_t(\varphi)$ if there exists a sequence $\{t_n\}_{n\in\mathbb{N}}$ in $\mathbb{R}$ so that

$$\lim_{n\to+\infty} t_n = +\infty, \quad \lim_{n\to+\infty} \varphi_{t_n}(\varphi) = \varphi^*.$$

The set of all ω-limit points for a given trajectory $\varphi_t(\varphi)$ is called ω-limit set of the trajectory $\varphi_t(\varphi)$ and denoted by Ω_φ. Denote by

$$\Omega = \bigcup_{\varphi\in\mathcal{T}_m} \Omega_\varphi$$

and assume that the matrices $A(\varphi)$ and $B(\varphi)$ are constant in the domain Ω:

$$A(\varphi)|_{\varphi\in\Omega} = \widetilde{A}, \quad B(\varphi)|_{\varphi\in\Omega} = \widetilde{B}.$$

We have obtained sufficient conditions for the existence of the asymptotically stable invariant set of the system (4) in terms of the eigenvalues of the matrices $\widetilde{A}$ and $\widetilde{B}$. Denote by

$$\gamma = \max_{j=1,\dots,n} Re\lambda_j(\widetilde{A}), \quad \alpha^2 = \max_{j=1,\dots,n} \lambda_j((E+\widetilde{B})^T(E+\widetilde{B})).$$

Theorem 2. *Let the moments of impulsive perturbations $\{t_i(\varphi)\}$ be such that uniformly with respect to $t \in \mathbb{R}$ there exists a finite limit*

$$\lim_{\widetilde{T}\to\infty} \frac{i(t, t+\widetilde{T})}{\widetilde{T}} = p. \tag{5}$$

If the following inequality holds

$$\gamma + p\ln\alpha < 0, \tag{6}$$

then system (4) *has an asymptotically stable invariant set.*

References

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