Approximation Theory in Jacobi-weighted Spaces and Its Application to h-p FEM

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1 Approximation in Jacobi-weighted Spaces over $Q = (-1, 1)^n$

1.1 Jacobi-weighted Besov and Sobolev spaces

$H^{k,\beta}(Q), Q = (-1,1)^n$ with integer $k \geq 0$, real $\beta_\ell > -1, 1 \leq \ell \leq n$

$$\|u\|_{H^{k,\beta}(Q)} = \left\{ \sum_{|\alpha|=0}^{k} \int_Q |D^\alpha u|^2 W_{\alpha\beta}(x) dx \right\}^{1/2}$$

with Jacobi weight function :

$$W_{\alpha\beta}(x) = \prod_{i=1}^{n} (1 - {x_i}^2)^{\alpha_i + \beta_i}$$

Jacobi-weighted interpolation spaces

$$\mathcal{B}^{s,\beta}_{2,q}(Q) = \Big(H^{\ell,\beta}(Q), H^{k,\beta}(Q)\Big)_{\theta,q}$$

with $s = (1-\theta)\ell + \theta k, k > \ell \geq 0, \theta \in (0,1)$.

$\mathcal{B}^{s,\beta}_{2,2}(Q) = H^{s,\beta}(Q)$ is
Jacobi-weighted (fraction order) Sobolev space ,

$$\|u\|^2_{H^{s,\beta}(Q)} = \int_0^\infty |t^{-\theta}K(t,u)|^2 \frac{dt}{t}$$

where

$$K(t,u) = \inf_{u=v+w}\big(\|v\|_{H^{\ell,\beta}(Q)} + t\|w\|_{H^{k,\beta}(Q)}\big) .$$

Approximation in Jacobi-weighted Spaces

$\mathcal{B}^{s,\beta}_{2,\infty}(Q) = B^{s,\beta}(Q)$ is Jacobi-weighted Besov space,

$$\|u\|_{B^{s,\beta}(Q)} = \sup_{t>0} \; t^{-\theta} K(t,u)$$

Modified Jacobi-weighted spaces $B^{s,\beta}_{\nu}(Q), \nu > 0$

$$\|u\|_{B^{s,\beta}_{\nu}(Q)} = \sup_{t>0} \frac{K(t,u)\, t^{-\theta}}{(1 + |\, log t|)^{\nu}}.$$

Remark $B^{s,\beta}_{\nu}(Q), \nu > 0$ is not an exact interpolation space, and $B^{s,\beta}(Q)$ and $H^{s,\beta}(Q)$ are.

1.2 Approximation in the spaces $H^{k,\beta}(Q), H^{s,\beta}(Q), B_{\nu}^{s,\beta}(Q)$

Jacobi Projection Let $\mathcal{P}_p(Q)$ be set of all polynomials of degree (separate) $\leq p$. For $u \in H^{k,\beta}(Q)$ for $k \geq 0$,

$$u(x) = \sum_{i_1, i_2 \cdots i_n = 0}^{\infty} C_{i_1, i_2 \cdots i_n} P_{i_1}(x_1, \beta_1) P_{i_2}(x_2, \beta_2) \cdots P_{i_n}(x_n, \beta_n).$$

where $P_{i_1}(x_1, \beta_1)$ is Jacobi polynomial, etc. then the Jacobi projection on $\mathcal{P}_p(Q)$ is

$$u_p(x) = \sum_{i_1, i_2 \cdots i_n = 0}^{p} C_{i_1, i_2 \cdots i_n} P_{i_1}(x_1, \beta_1) P_{i_2}(x_2, \beta_2) \cdots P_{i_n}(x_n, \beta_n)$$

Approximation in Jacobi-weighted Spaces

Theorem 1.1 *Let u_p be the Jacobi projection of u on $\mathcal{P}_p(Q)$. Then*

(i) For $0 \leq l \leq k$ and $p > 0$

$$\|u - u_p\|_{H^{l,\beta}(Q)} \leq Cp^{-(k-l)}\|u\|_{H^{k,\beta}(Q)}, \tag{1.1}$$

and for $0 \leq l < s$ and $p > 0$

$$\|u - u_p\|_{H^{l,\beta}(Q)} \leq Cp^{-(s-l)}\,\|u\|_{H^{s,\beta}(Q)}, \tag{1.2}$$

$$\|u - u_p\|_{H^{l,\beta}(Q)} \leq Cp^{-(s-l)}(1 + \log p)^{\nu}\,\|u\|_{B_{\nu}^{s,\beta}(Q)}; \tag{1.3}$$

Approximation in Jacobi-weighted Spaces

(ii) If $k > n/2$ or $s > n/2$, and $\beta_\ell \le -1/2$ for $1 \le \ell \le n, 1 \le n \le 3$, then

$$|(u - u_p)(x)| \le Cp^{-(k-n/2)} \|u\|_{H^{k,\beta}(Q)}, \tag{1.4}$$

$$|(u - u_p)(x)| \le Cp^{-(s-n/2)} \|u\|_{H^{s,\beta}(Q)}, \tag{1.5}$$

$$|(u - u_p)(x)| \le Cp^{-(s-n/2)} (1 + \log p)^{\nu} \|u\|_{B_{\nu}^{s,\beta}(Q)}. \tag{1.6}$$

(iii) If $p \ge k - 1$ the estimations hold in terms of semi norms for integers l and k

$$|u - u_p|_{H^{l,\beta}(Q)} \le Cp^{-(s-l)} |u|_{H^{k,\beta}(Q)} \tag{1.7}$$

and if $k > n/2$, in addition

$$|(u - u_p)(x)| \le Cp^{-(k-n/2)} |u|_{H^{k,\beta}(Q)}. \tag{1.8}$$

Approximation in Jacobi-weighted Spaces

The constant C in the above inequalities is independent of p, u, but may depend on k.

Corollary 1.1 The above estimations can be easily generalized to for non-integer l.

1.3 Regularity and Approximability of singular functions in Jacobi-weighted Besov spaces over $Q = (-1,1)^n, n = 2$

Consider typical singular function on $Q = (-1,1)^2$:

$$u(x) = r^\gamma \, \log^\nu r \, \chi(r) \, \Phi(\theta) \tag{1.9}$$

where real $\gamma > 0$, integer $\nu \geq 0$, $\chi(r)$ and $\Phi(r)$ are C^∞ functions such that for $0 < r_0 < 2$

$$\chi(r) = 1 \quad \text{for } 0 < r < r_0/2, \quad \chi(r) = 0 \quad \text{for } r > r_0$$

Approximation in Jacobi-weighted Spaces

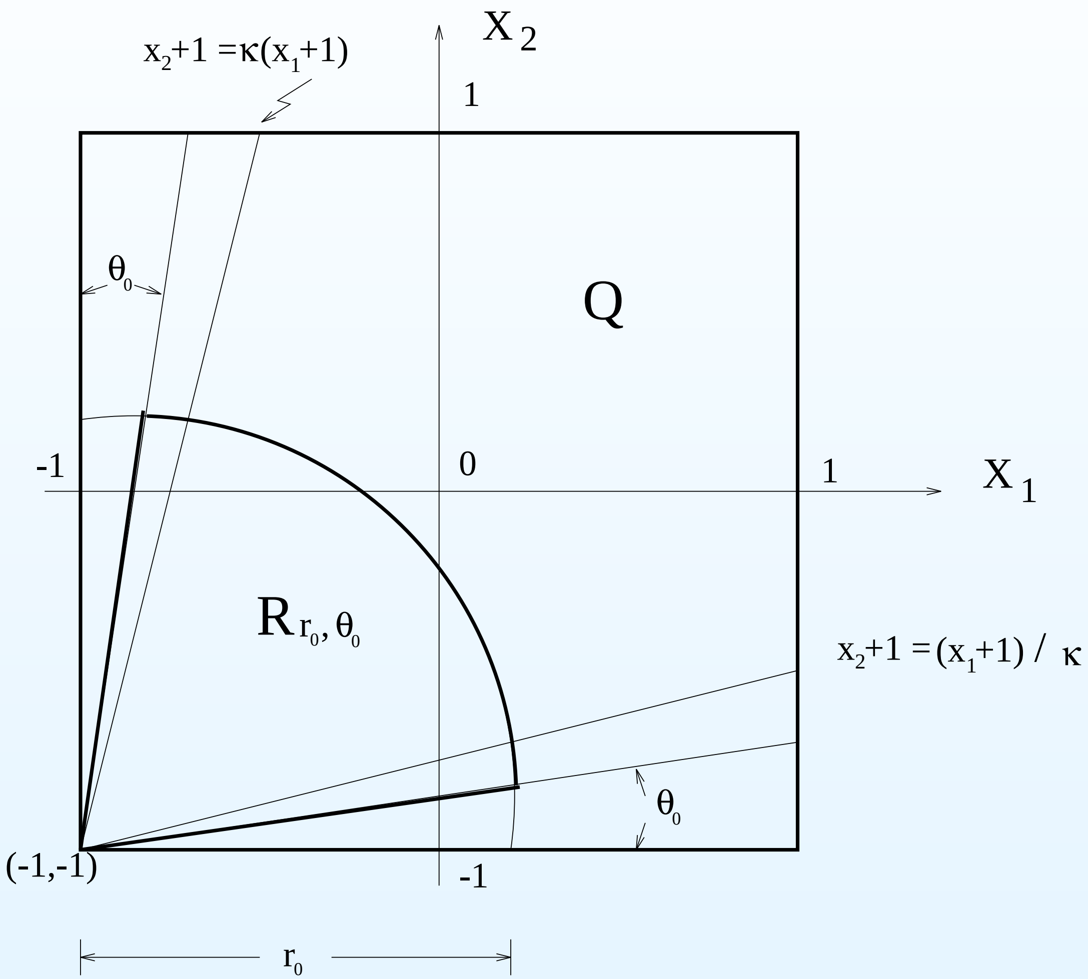

Fig. 1.1 Square Domain Q and sub region R_{r_0,θ_0}

Approximation in Jacobi-weighted Spaces

Theorem 1.2 *For* $\gamma > 0$ *and* $\nu \geq 0$, $u \in B^{s,\beta}_{\nu^*}(Q)$ *with* $2 + 2\gamma + \beta_1 + \beta_2$ *and*

$$\nu^* = \begin{cases} \nu & \text{if } \gamma \text{ is not an integer, or } \nu = 0 \\ \nu - 1 & \text{if } \gamma \text{ is an integer and } \nu \geq 1, \end{cases} \tag{1.10}$$

Theorem 1.3 *Let* $u(x)$ *be given in (1.9) with* $\gamma > 0$ *and integer* $\nu \geq 0$, *let* ψ *and* φ *are the Jacobi projection of* u *on* $\mathcal{P}_p(Q), p \geq 1$ *associated with* $\beta = (0, 0)$, *and* $\beta = (-1/2, -1/2)$, *respectively. Then,*

Approximation in Jacobi-weighted Spaces

$$\|u - \psi\|_{L^2(Q)} \leq C\, p^{-2-2\gamma} \log^{\nu^*}(1+p) \|u\|_{B^{2+2\gamma,\beta}_{\nu^*}(Q)} \tag{1.11}$$

with $\beta = (0, 0)$, and

$$\|u - \phi\|_{H^1(R_0)} \leq C\, p^{-2\gamma} \log^{\nu^*}(1+p) \|u\|_{B^{1+2\gamma,\beta}_{\nu^*}(Q)} \tag{1.12}$$

with $\beta = (-1/2, -1/2)$, where ν^* is given in and (1.10) and

$$R_0 = R_{r_0,\theta_0} = \left\{ x \in Q \;\middle|\; r < r_0, \quad \theta_0 < \theta < \pi/2 - \theta_0 \right\} \tag{1.13}$$

with $\theta_0 \in (0, \pi/4)$

2 Approximation in the Jacobi-weighted spaces on $Q_h = (-h, h)^n$

2.1 Jacobi-weighted Besov and Sobolev spaces over Q_h

Let $w_{h,\alpha,\beta}(x)$ be a weighted function on $Q_h = (-h, h)^n$, $1 \leq n \leq 3$:

$$w_{h,\alpha,\beta}(x) = \prod_{i=1}^{n} \Big(1 - (\frac{x_i}{h})^2\Big)^{\alpha_i+\beta_i}$$

with $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\alpha_i \geq 0$ integer, and $\beta = (\beta_i, 1 \leq i \leq n), \beta_i > -1$.

Approximation in the Jacobi-weighted spaces on Q_h

The Jacobi-weighted Sobolev space $H^{k,\beta}(Q_h)$, $k \geq 0$, is the closure of C^∞ functions furnished with the norm

$$\|u\|^2_{H^{k,\beta}(Q_h)} = \sum_{0\leq|\alpha|\leq k} \int_Q |D^\alpha u(x)|^2 w_{h,\alpha,\beta}(x)dx$$

The Jacobi-weighted Sobolev spaces $H^{s,\beta}(Q_h)$ and Besov spaces $B^{s,\beta}(Q_h)$ are defined as usual interpolation spaces by the K-method,

$$H^{s,\beta}(Q_h) = \mathcal{B}^{s,\beta}_{2,2}(Q_h) = \Big(H^{\ell,\beta}(Q_h), H^{k,\beta}(Q_h)\Big)_{\theta,2}$$

and

$$B^{s,\beta}(Q_h) = \mathcal{B}^{s,\beta}_{2,\infty}(Q_h) = \Big(H^{\ell,\beta}(Q_h), H^{k,\beta}(Q_h)\Big)_{\theta,\infty}.$$

Approximation in the Jacobi-weighted spaces on Q_h

The space $B^{s,\beta}_{\nu}(Q_h)$ is an interpolation defined by the modified K-method,

$$B^{s,\beta}_{\nu}(Q_h) = \Big(H^{\ell,\beta}(Q_h), H^{k,\beta}(Q_h)\Big)_{\theta,\infty,\nu}.$$

Proposition 2.1 *Let $u(x)$ and $U(\xi) = u(h\xi)$ be functions defined on Q_h and Q, respectively.*

(i) $u \in H^{k,\beta}(Q_h)$ with integer $k \geq 0$ if $U(\xi) = u(h\xi) \in H^{k,\beta}(Q)$, visa versa. Furthermore, there holds for $\ell \leq k$

$$|u|^2_{H^{\ell,\beta}(Q_h)} = h^{n/2-\ell}|U|_{H^{\ell,\beta}(Q)}; \qquad (2.1)$$

Approximation in the Jacobi-weighted spaces on Q_h

(ii) $u \in H^{s,\beta}(Q_h)$ with noninteger $s > 0$ if $U(\xi) \in H^{s,\beta}(Q)$, visa versa. There holds for $\ell < s$

$$|u|_{H^{\ell,\beta}(Q_h)} = h^{n/2-\ell}|U|_{H^{\ell,\beta}(Q)}; \tag{2.2}$$

(iii) $u \in B_\nu^{s,\beta}(Q_h)$ with real $s > 0$ and interger $\nu \geq 0$ if $U(\xi) \in B_\nu^{s,\beta}(Q)$, visa versa.

Approximation in the Jacobi-weighted spaces on Q_h

Theorem 2.1 *Let u_p be the Jacobi projection of u on $\mathcal{P}_p(Q_h)$ with $p \geq 1$, Then for $0 \leq l \leq k$,*

$$\|u - u_p\|_{H^{l,\beta}(Q_h)} \leq C \frac{h^{\mu-l}}{p^{k-l}} \|u\|_{H^{k,\beta}(Q_h)}, \qquad (2.3)$$

for $0 \leq l < s$,

$$\|u - u_p\|_{H^{l,\beta}(Q_h)} \leq C \frac{h^{\mu-l}}{p^{s-l}} \|u\|_{H^{s,\beta}(Q_h)}. \qquad (2.4)$$

and

$$\|u - u_p\|_{H^{l,\beta}(Q_h)} \leq C \frac{h^{\mu-l}}{p^{s-l}} \log^{\nu}(1 + \frac{p}{h}) \|u\|_{B^{s,\beta}_{\nu}(Q_h)}. \qquad (2.5)$$

where $\mu = \min\{k, p+1\}$;

Approximation in the Jacobi-weighted spaces on Q_h

(ii) If $k > n/2$, or $s > n/2$ and $\beta_\ell \leq -1/2, 1 \leq \ell \leq n$, then for $x \in \bar{Q}_h$

$$|(u - u_p)(x)| \leq C \frac{h^{\mu - n/2}}{p^{k-n/2}} \|u\|_{H^{k,\beta}(Q_k)}, \tag{2.6}$$

$$|(u - u_p)(x)| \leq C p^{-(s-n/2)} h^{\mu_1 - n/2} \|u\|_{H^{s,\beta}(Q_h)}, \tag{2.7}$$

and

$$|(u - u_p)(x)| \leq C \frac{h^{\mu - n/2}}{p^{s-n/2}} \log^\nu (1 + \frac{p}{h}) \|u\|_{B^{s,\beta}_\nu (Q_h)} : \tag{2.8}$$

(iii) For $p \geq k - 1$ and $k \geq 1$, there hold

$$|u - u_p|_{H^{l,\beta}(Q_h)} \leq C \left(\frac{h}{p}\right)^{k-l} |u|_{H^{k,\beta}(Q_h)} \tag{2.9}$$

and

2.2 Regularity and Approximability of singular functions in Jacobi-weighted Besov spaces over $Q_h = (-h,h)^2$

Consider typical singular function on $Q_h = (-h,h)^2$:

$$u(x) = r^{\gamma}\, \log^{\nu} r\, \chi(r)\, \Phi(\theta) \tag{2.11}$$

Theorem 2.2 *For $\gamma > 0$ and $\nu \geq 0$, $u \in B^{s,\beta}_{\nu^*}(Q)$ with $s = 2 + 2\gamma + \beta_1 + \beta_2$ and ν^* given in (1.10).*

Approximation in the Jacobi-weighted spaces on Q_h

Theorem 2.3 *Let $u(x)$ be given in (2.11). Then there exist polynomials $\psi_{hp}(x)$ and $\varphi_{hp}(x)$ in $\mathcal{P}_p(Q), p \geq 1$ such that*

$$\|u - \psi_{hp}\|_{L^2(Q_h)} \leq C \left(\frac{h}{p^2}\right)^{1+\gamma} F_\nu(p,h) \|u\|^2_{B^{s,\beta}_{\nu*}(Q_h)} \tag{2.12}$$

with $\beta = (0,0), s = 2(1+\gamma)$, and

$$\|u - \varphi_{hp}\|_{H^1(R^h_0)} \leq C \left(\frac{h}{p^2}\right)^{\gamma} F_\nu(p,h) \|u\|^2_{B^{s,\beta}_{\nu*}(Q_h)} \tag{2.13}$$

with $\beta = (-1/2, -1/2), s = 1 + 2\gamma$, where $F_\nu(p,h)$ is a log-polynomial,

Approximation in the Jacobi-weighted spaces on Q_h

$$F_\nu(p,h) = \begin{cases} \log^\nu \frac{p}{h} & \text{for non-integer } \gamma, \\ \log^{\nu-1} \frac{p}{h} & \text{for integer } \gamma \text{ and } r^\gamma \Phi(\theta) \in \mathcal{P}_p \\ \max\left\{ \log^{\nu-1} \frac{p}{h},\ \log^\nu \frac{1}{h} \right\} & \text{for integer } \gamma \text{ and } r^\gamma \Phi(\theta) \notin \mathcal{P}_p \end{cases} \qquad (2.14)$$

Furthermore, for $x \in \bar{Q}_h$, there holds

$$|u(x) - \varphi_{hp}(x)| \leq C \left(\frac{h}{p^2}\right)^\gamma F_\nu(p,h). \qquad (2.15)$$

The constant C in (2.12) -(2.15) is independent of h and p.

3 The optimal convergence of the h-p version of FEM

Consider a boundary value problem:

$$\begin{cases} -\Delta u + u = f & \text{in } \Omega, \\ \frac{\partial u}{\partial n} = g & \text{on } \partial\Omega. \end{cases} \tag{3.1}$$

where Ω be a polygon.

The optimal convergence of the h-p -version of FEM

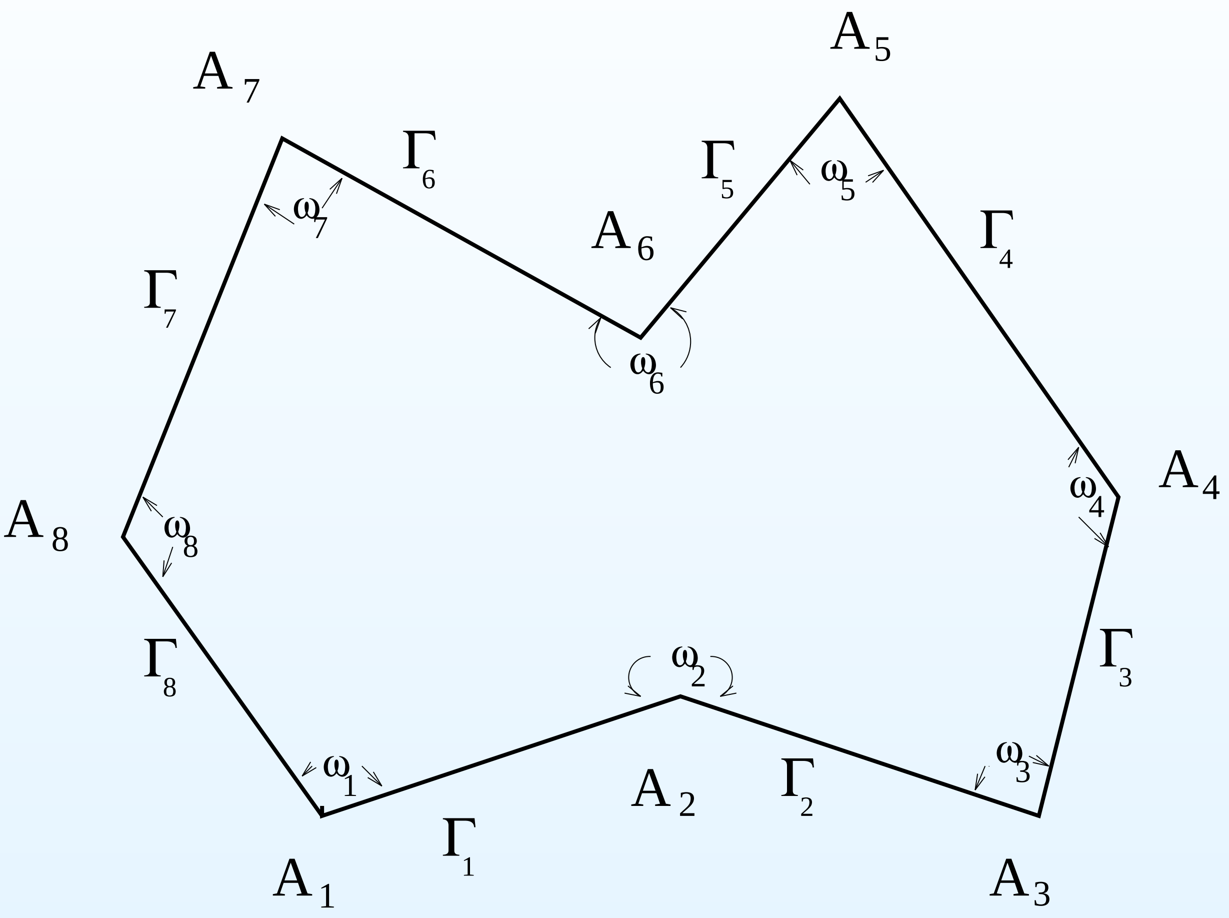

Fig. 3.1 Polygonal domain Ω

The optimal convergence of the h-p -version of FEM

Recent progresses

- h-p FEM

Babuska and Suri (1987): for $\gamma = \min_i = \frac{\pi}{\omega_i}, \ \nu = \max_i \nu_i \geq 0$

$$\|u - u_{h,p}\|_{H^1(\Omega)} \leq C h^\gamma p^{-2\gamma} \log^\nu(\frac{p}{h}).$$

Guo and Sun (2005): for $\gamma = \min_i \gamma_i = \frac{\pi}{\omega_i}, \ \nu = \max_i \nu_i \geq 0$

$$C_2 h^\gamma p^{-2\gamma} F_\nu(h,p) \leq \|u - u_{h,p}\|_{H^1(\Omega)} \leq C_1 h^\gamma p^{-2\gamma} F_\nu(h,p)$$

where

$$F_\nu(h,p) = \begin{cases} \log^\nu(\frac{p}{h}), & \gamma \text{ is not integer,} \\ \log^{\nu-1}(\frac{p}{h}), & \gamma \text{ is integer, } r^\gamma \Phi(\theta) \text{ is not polynomial} \\ \max\{\log^{\nu-1}(\frac{p}{h}), \log^\nu(\frac{1}{h})\}, & \text{otherwise .} \end{cases}$$

The optimal convergence of the h-p -version of FEM

Let $S^p_D(\Omega;\Delta_h;\mathcal{M})$ be the finite element spaces. Here $\mathcal{M}=\{M_j, 1\le j\le J\}$ denotes a mapping vector and M_j is an affine mapping of standard triangle T or square S onto Ω_j. Let $S^p(\Omega;\Delta_h,\mathcal{M})=\Big\{\phi(x)\in H^1(\Omega)\;\Big|\;\phi\;|_{\Omega_j}=\tilde{\phi}_j\circ M_j^{-1}, \tilde{\phi}_j\in$ $\mathcal{P}_p(T)$ or $\mathcal{P}_p(S), j=1,2,\ldots,J\Big\}$ and $S^p_D(\Omega;\Delta_h;\mathcal{M})=S^p(\Omega;\Delta_h;\mathcal{M})\cap H^1_D(\Omega)$.

3.1 The h-p-version finite element method for problems with smooth solutions

Lemma 3.1 *Let $u \in H^{k,\beta}(Q_h), k \geq 0$, and let $U(\xi) = u(h\xi)$. Then*

$$\|U - U_p\|_{H^{k,\beta}(Q)} \leq Ch^{\mu-1}\|u\|_{H^{k,\beta}(Q_h)} \tag{3.2}$$

where $\mu = \min\{k, p+1\}$, and C depends on k, but is independent of p, h and u.

Lemma 3.2 *Let γ_h be an edge of T_h which is a triangle or a quadrilateral, and let ψ be a polynomial of degree p on γ_h vanishing at the ending points of γ_h. Then there exists an extension $\Psi(x) \in \mathcal{P}_p(T_h)$ such that $\Psi(x)\mid_{\gamma_h} = \psi$ and vanishes at other edges of T_h, and*

$$\|\Psi\|_{H^1(T_h)} \leq C\|\psi\|_{H^{1/2}_{00}(\gamma_h)}. \tag{3.3}$$

The optimal convergence of the h-p -version of FEM

Lemma 3.3 *Let $u \in H^k(\Omega_i), k > 1$, where Ω_i is a curved triangular or quadrilateral element of the mesh Δ_h with size h. Then there exists a polynomial $\phi \in \mathcal{P}_p(\Omega_i)$ such that*

$$\|u - \phi\|_{H^1(\Omega_i)} \leq C \, \frac{h^{\mu-1}}{p^{k-1}} \|u\|_{H^k(\Omega_i)} \tag{3.4}$$

with $\mu = \min\{p+1, k\}$, and $u(V_l) = \phi(V_l), 1 \leq l \leq 3$ or $4, V_l$ are the vertices of Ω_i.

Proof Assume that Ω_i is a curved quadrilateral. Let M_i be a mapping of $Q_{h/2} = (-h/2, h/2)^2$ onto Ω_i. If Ω_i is a curved triangle, the mapping M_i maps $T_{h/2} = \{x = (x_1, x_2) \mid -\frac{h}{2} + \frac{x_2+h/2}{\sqrt{3}} \leq x_1 \leq \frac{h}{2} - \frac{x_2+h/2}{\sqrt{3}}, -\frac{h}{2} \leq x_2 \leq \frac{\sqrt{3}-1}{2}h\}$ onto Ω_i.

The optimal convergence of the h-p -version of FEM

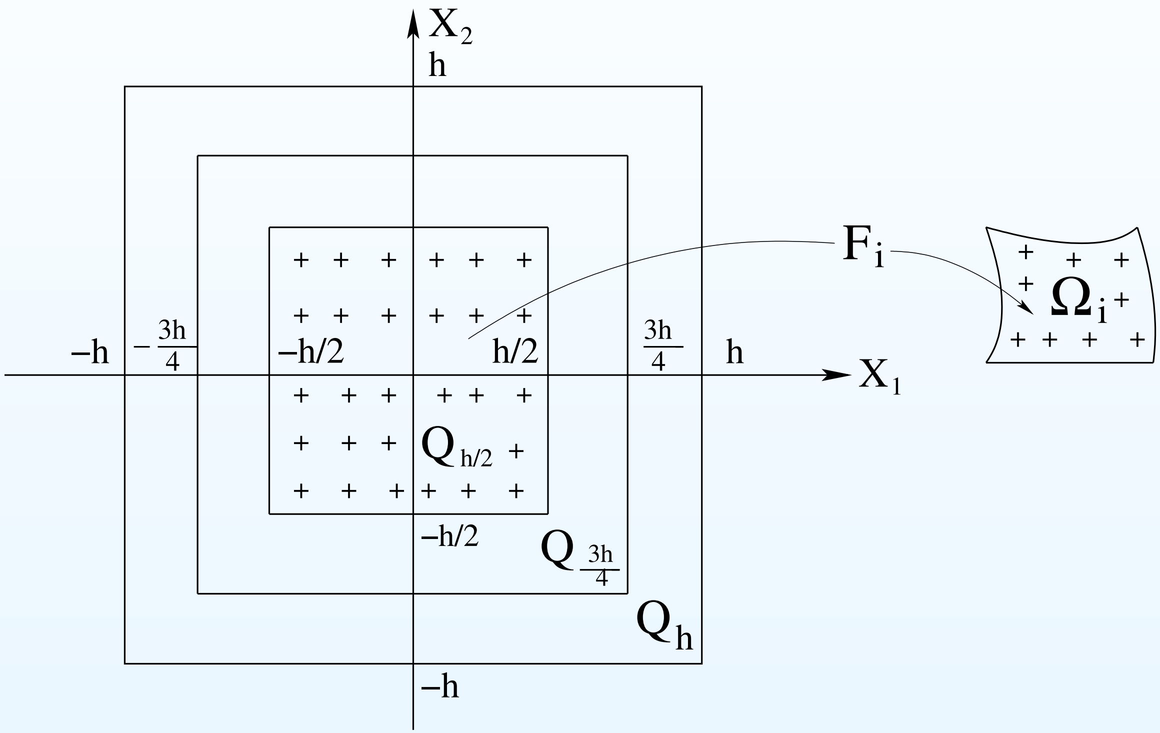

Fig. 3.2 Mapping of quadrilateral

The optimal convergence of the h-p -version of FEM

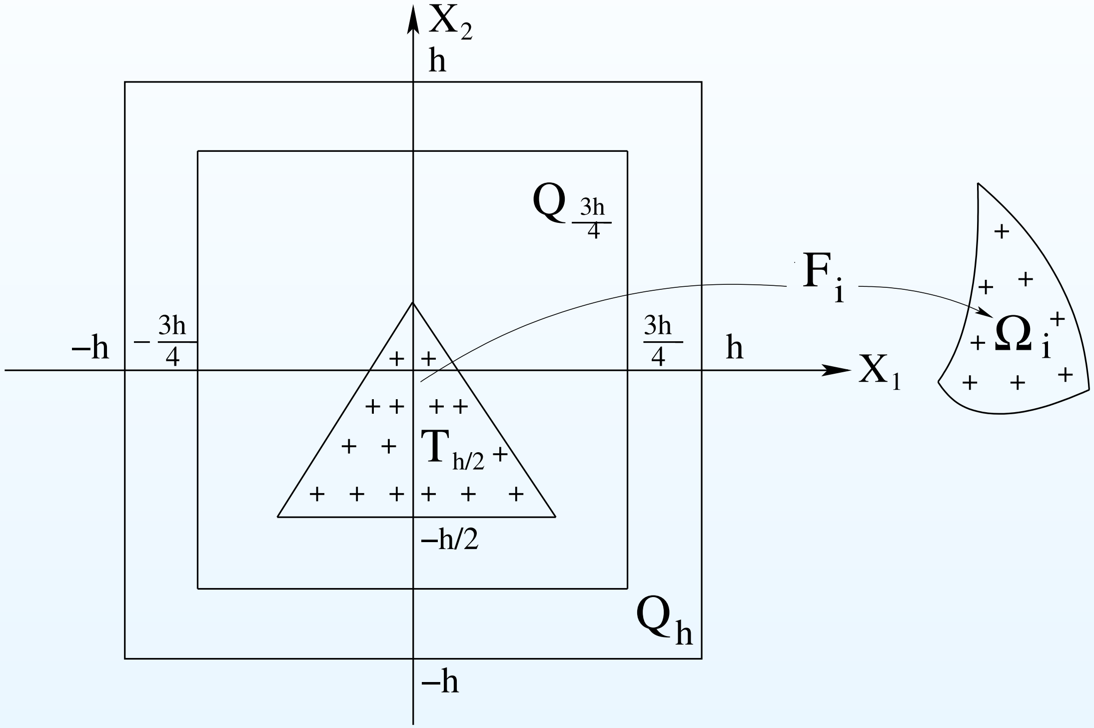

Fig. 3.3 Mapping of triangle

The optimal convergence of the h-p -version of FEM

Then $\tilde{u} = u \circ M_i \in H^K(Q_{h/2})$, and it can be extended to Q_h such that the extended function has a support contained in $Q_{2h/3}$ and preserves the norm. Furthermore, $\tilde{u} \in H^{k,\beta}(Q_h)$ with The Jacobi weight $\beta = (-1/2, -1/2)$, and

$$\|\tilde{u}\|_{H^{k,\beta}(Q_h)} \leq C\|\tilde{u}\|_{H^k(Q_h)} \leq C\|u\|_{H^k(\Omega_i)}. \tag{3.5}$$

Then using approximation in $H^{k,\beta}(\Omega_i)$, we get the results. $\square$

The optimal convergence of the h-p -version of FEM

Theorem 3.4 *Let* $\Delta_h = \left\{\Omega_j,\ 1 \le j \le J\right\}$ *be a quasi-uniform mesh with element size* h *over* Ω *containing triangular and quadrilateral elements, and let* $S_D^p(\Omega; \Delta_h; \mathcal{M})$ *be the finite element space defined as above. The data functions* f *and* g *are assumed such that the solution* u *of (3.1) is in* $H^k(\Omega)$ *with* $k > 1$*. Then the finite element solution* $u_{hp} \in S_D^p(\Omega; \Delta_h; \mathcal{M})$ *with* $p \ge 1$ *satisfies*

$$\|u - u_{hp}\|_{H^1(\Omega)} \le C\ \frac{h^{\mu-1}}{p^{k-1}} \|u\|_{H^k(\Omega)} \tag{3.6}$$

where $\mu = \min\{p+1, k\}$ *and the constant* C *is independent of* p *and* u*.*

3.2 The h-p version finite element method for problems with singular solutions

We assume that f and g are such that the solution u of (3.1) is in $H^k(\Omega_0), k \geq 1$, and in each neighborhood S_{δ_i}, u have an expansion in terms of singular functions of $r^\gamma \log^\nu r$ -type

$$u = u_1 + u_0 = \sum_{0<\gamma_m^{[i]}\leq k-1} C_m^{[i]}\, r_i^{\gamma_m^{[i]}}\, |\log r_i|^{\nu_m^{[i]}}\, \Phi_m^{[i]}(\theta_i)\, \chi(r_i) + u_0^{[i]} \quad (3.7)$$

where (r_i, θ_i) are polar coordinates with the vertex A_i, $u_0^{[i]} \in H^k(S_{\delta_i})$ is the smooth part of u, $\gamma_m^{[i]} > 0$, and $\nu_m^{[i]} \geq 0$ are integers.

The optimal convergence of the h-p -version of FEM

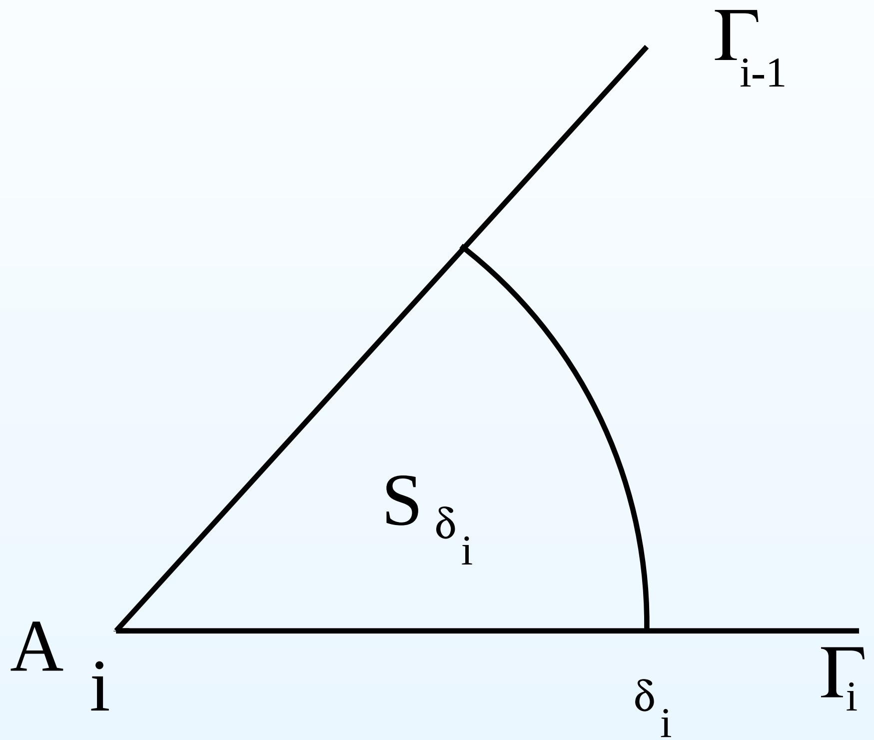

Fig. 3.4 A neighborhood of the vertex A_i

The optimal convergence of the h-p -version of FEM

We assume that $\nu_m^{[i]} > \nu_{m+1}^{[i]}$ and $\gamma_m^{[i]} \le \gamma_{m+1}^{[i]}$, $\chi(r_i)$ and $\Phi_m^{[i]}(\theta_i)$ are C^∞ functions, $\chi(r_i) = 1$ for $0 < r_i < \delta_i < \frac{1}{2}$, $\chi(r_i) = 0$ for $r_i > \delta_i$. Let

$$\gamma = \min_i \gamma_1^{[i]}, \qquad \nu_\gamma = \max_{i, \gamma_1^{[i]} = \gamma} \nu_1^{[i]}. \tag{3.8}$$

There exists i_0 such that $\gamma_1^{[i_0]} = \gamma$ and $\nu_\gamma = \nu_1^{[i_0]}$.

The optimal convergence of the h-p -version of FEM

Theorem 3.5 *Let $\Omega_h = \{\Omega_j,\ 1 \le j \le J\}$ be a quasi-uniform mesh over Ω containing triangular and parallelogram elements, and let $S_D^p(\Omega;\Delta_h;\mathcal{M})$ with $p > \gamma$ be the finite element space defined as above. The data functions f and g are assumed such that the solution u of (3.1) is in $H^k(\Omega_0)$ with $k > 1 + 2\gamma$, and u has the expansion (3.7) with $u_0^{[i]} \in H^k(S_{\delta_i})$ in each neighborhood S_{δ_i}. Then the finite element solution $u_{hp} \in S_D^p(\Omega;\Delta;\mathcal{M})$ for the problem (3.1) satisfies*

$$\|u - u_{hp}\|_{H^1(\Omega)} \le C_1 \frac{h^\gamma}{p^{2\gamma}}\, F_{\nu_\gamma}(p,h). \tag{3.9}$$

with the constant C_1 depending on u, γ and ν_γ, but not on p and h, where γ and and ν_γ are given in (3.8), and $F_{\nu_\gamma}(p,h)$ given in (2.14).

The optimal convergence of the h-p -version of FEM

proof For elements Ω_i contains no vertices, by Lemma 3.3, there exist a polynomial $\varphi^{[i]} \in \mathcal{P}_p(\Omega_i)$ such that $\varphi^{[i]} = u$ at the vertices of Ω_i, and

$$\|u - \varphi^{[i]}\|_{H^1(\Omega_i)} \leq C\,\frac{h^{\tilde{\mu}-1}}{p^{k-1}} \leq C\,\frac{h^{\gamma}}{p^{2\gamma}}$$

with $\tilde{\mu} = \min\{p+1, k\} \geq 1 + \gamma$. Let the element Ω_j contain a vertex A_1 of Ω. Then (3.7) holds with $i = 1$ in S_{δ_1}. By Lemma 3.3, there exist a polynomial $\psi_0 \in \mathcal{P}_p(\Omega_j)$ such that $\psi_0 = u$ at the vertices of Ω_j, and

$$\|u_0 - \psi_0\|_{H^1(\Omega_j)} \leq C\,\frac{h^{\mu-1}}{p^{k-1}}$$

The optimal convergence of the h-p -version of FEM

with $\mu = \min\{p+1, k\} \geq 1+\gamma$. For a sharp approximation to u_1, we map Ω_j onto $R_{0,h} \subset Q_h$ by an affine mapping F_j such that $A_1 \circ F_j = (-h, -h)$ and that Ω_j is contained in $R_{0,h}$. Due to Theorem 2.3, there exist polynomials $\psi_m \in \mathcal{P}_p(\Omega_j)$ such that $v_m = \psi_m$ at the vertices of Ω_j, and

$$\|v_m - \psi_m\|_{H^1(\Omega_j)} \leq C \, \frac{h^{\gamma^{[1]}}}{p^{2\gamma_m^{[1]}}} \, F_{\nu_m^{[1]}}(p,h).$$

where $v_m = r_1^{\gamma_m^{[1]}} \, |\log r_i|^{\nu_m^{[i]}} \Phi_m^{[i]}(\theta_1) \, \chi(r_i)$. Let $\psi = \sum_{0<\gamma_m^{[1]} \leq k-1} C_m^{[1]} \psi_m$ and $\varphi^{[j]} = \psi + \psi_0$. Then $u_1 = \psi$ at the vertices of Ω_j, and

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$$\|u_1-\psi\|_{H^1(\Omega_j)} \le C \sum_{0<\gamma_m^{[i]}\le k-1} \frac{h^{\gamma^{[1]}}}{p^{2\gamma_m^{[1]}}} F_{\nu_m^{[1]}}(p,h) \le C \frac{h^{\gamma}}{p^{2\gamma}} F_{\nu_\gamma}(p,h).$$

which implies that $u=\varphi^{[j]}$ at the vertices of Ω_j, and

$$\|u_-\varphi^{[j]}\|_{H^1(\Omega_j)} \le C\Big(\frac{h^{\gamma}}{p^{2\gamma}} F_{\nu_\gamma}(p,h) + \frac{h^{\tilde{\mu}-1}}{p^{k-1}}\Big) \le C \frac{h^{\gamma}}{p^{2\gamma}} F_{\nu_\gamma}(p,h)$$

Adjust $\varphi^{[j]}$ as in the proof of Theorem 3.4 to achieve the continuity across internal edges γ of elements and homogeneous Dirichlet boundary condition on the edges $\gamma \subset \Gamma_D$. Let $\varphi = \varphi^{[i]}$ on each $\Omega_i, 1 \le i \le J$, then $\varphi_p \in S_D^p(\Omega;\Delta;\mathcal{M})$ and satisfies (3.9). $\square$

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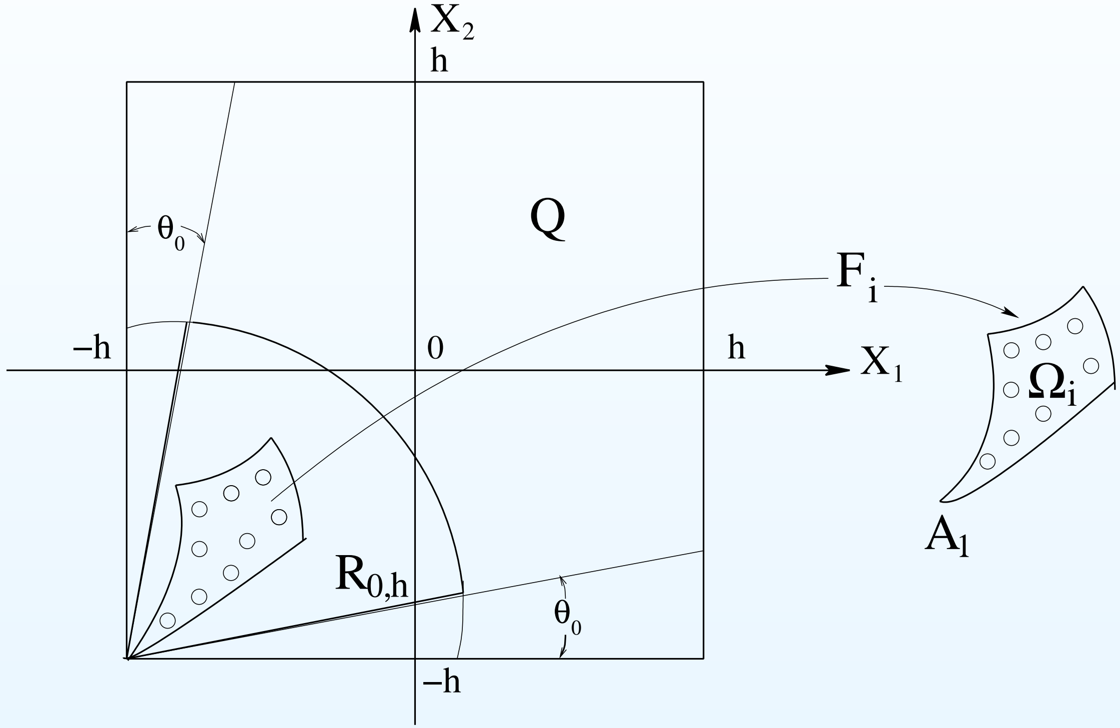

Fig. 3.5 Mapping of element with a vertex of Ω

4 Concluding Remarks

1. Effectiveness of functional spaces for approximation

Table 4.1. The value of k and s in Sobolev, Besov and weighted Besov spaces for functions of r^γ-type and $r^\gamma \log^\nu r$-type

Space	$H^k(Q)$	$H^{k,\beta}(Q)$	$H^s(Q)$	$B^s(Q)$	$B^{s,\beta}(Q)$	$B_\nu^{s,\beta}(Q)$
r^γ	$1+[\gamma]$	$1+[2\gamma]$	$1+\gamma-\epsilon$	$1+\gamma$	$1+2\gamma$	$1+2\gamma$
$r^\gamma \log^\nu r$	$1+[\gamma]$	$1+[2\gamma]$	$1+\gamma-\epsilon$	$1+\gamma-\epsilon$	$1+2\gamma-\epsilon$	$1+2\gamma$

Concluding Remarks

Table 4.2. Accuracy of approximation of the h- and p-version to functions of $r^\gamma \log^\nu r$-type based on Sobolev, Besov and weighted Besov spaces

	h version		h-p version			
Space	$H^s(Q)$	$B^s(Q)$	$H^s(Q)$	$B^s(Q)$	$B^{s,\beta}(Q)$	$B^{s,\beta}_\nu(Q)$
r^γ	$h^{\gamma-\epsilon}$	h^γ	$\left(\frac{h}{p}\right)^{\gamma-\epsilon}$	$\left(\frac{h}{p}\right)^{\gamma}$	$\left(\frac{h}{p^2}\right)^{\gamma}$	$\left(\frac{h}{p^2}\right)^{\gamma}$
$r^\gamma \log^\nu r$	$h^{\gamma-\epsilon}$	$h^{\gamma-\epsilon}$	$\left(\frac{h}{p}\right)^{\gamma-\epsilon}$	$\left(\frac{h}{p}\right)^{\gamma-\epsilon}$	$\left(\frac{h}{p^{2-\epsilon}}\right)^{\gamma} \lvert \log \frac{p}{h} \rvert^\nu$	$\left(\frac{h}{p^2}\right)^{\gamma} F_\nu(h,p)$

2. Optimal Convergence :

If u be the solution of the model problem in a polygonal domain, $u_p \in S^{p,1}(\Omega, \Delta)$ be FEM solution. Then

$$\|u - u_p\|_{H^1(\Omega)} \leq Cp^{-2\gamma} (1 + \log p)^{\nu^*}$$

is the optimal rate, i.e. $\exists C_1$, s.t.

$$\|u - u_p\|_{H^1(\Omega)} \geq C_1 p^{-2\gamma} (1 + \log p)^{\nu^*}$$

Concluding Remarks

3. Generalization and Application :

Jacobi-weighted Besov spaces can be generalized to all dimensions :

In One Dimensions : $\beta = 0$;

In Two Dimensions : $\beta = (-1/2, -1/2)$;

In One Dimension :
$\beta = (-1/3, -1/3, -1/3)$, in nbhd of vertex;
$\beta = (-1/2, -1/2, 0)$, in nbhd of vertex-edge;
$\beta = (-1/2, -1/2, \beta_3)$, in nbhd of edge, $\beta_3 > -1$, arbitrary.

Concluding Remarks

- Applicable to the p and h-p (with quasiuniform mesh) version of FEM/BEM for problems with singular as well smooth solutions.
- Approximation theory in the framework of the Jacobi-weighted spaces provides a theoretical foundation of the modern p and h-p (with quasiuniform mesh) version of FEM/BEM.