Discrete Maximum Principle for Higher-order Finite Elements in 1D

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Model problem

- $-u'' = f \quad \text{in } \Omega = (a, b); \quad u(a) = u(b) = 0$

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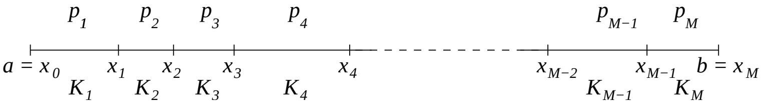

- $V_{hp} = \{v_{hp} \in H_0^1(\Omega) : v_{hp}|_{K_i} \in P^{p_i}(K_i)\}$
- $a(u,v) = \int_a^b u'v'\,\mathrm{d}x \qquad (u,v) = \int_a^b uv\,\mathrm{d}x$
- Find $u_{hp} \in V_{hp}$: $a(u_{hp}, v_{hp}) = (f, v_{hp})$ for all $v_{hp} \in V_{hp}$

Discrete Maximum Principle (DMP)

Definition (DMP)

$f \geq 0$ a.e. in Ω $\Rightarrow$ $u_{hp} \geq 0$ in Ω.

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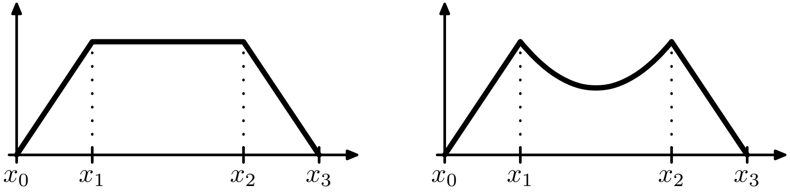

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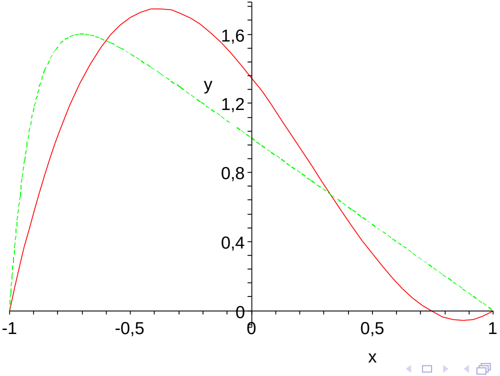
Not valid for $p = 3, 5$ and $p \geq 7$.

Example ($\Omega = (-1, 1)$, K_1, $p_1 = 3$, $f(x) = 200e^{-10(x+1)}$)



Discrete Green's function

Definition

For $z \in \overline{\Omega}$ find $G_{hp,z} \in V_{hp} : a(w_{hp}, G_{hp,z}) = \delta_z(w_{hp}) \quad \forall w_{hp} \in V_{hp}$

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- $G_{hp}(x,z) = \sum_{j=1}^{N} \sum_{k=1}^{N} A_{jk}^{-1} \phi_k(x) \phi_j(z),$

  where $\{\phi_1, \phi_2, \ldots, \phi_N\}$ is a basis of V_{hp} and $A_{ij} = a(\phi_j, \phi_i)$.

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- $a(\cdot,\cdot)$ symmetric $\Rightarrow G_{hp}(x,y) = G_{hp}(y,x)$
- $a(\phi_j, \phi_i) = \delta_{ij} \Rightarrow G_{hp}(x,z) = \sum_{i=1}^{N} \phi_i(x) \phi_i(z)$
- A symmetric positive definite $\Rightarrow G_{hp}(x,x) > 0 \ \forall x \in \Omega$

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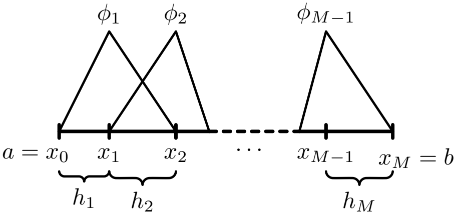
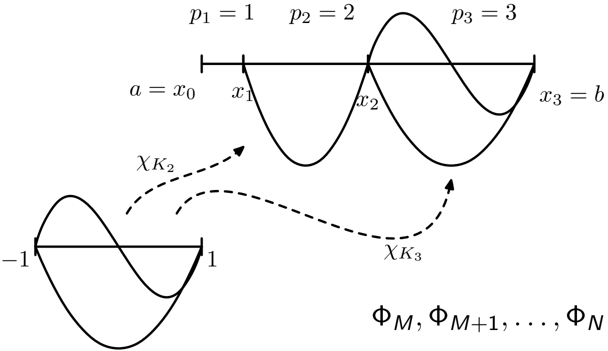
- A symmetric positive definite $\Rightarrow G_{hp}(x,x) > 0 \ \forall x \in \Omega$
- DMP $\Leftrightarrow G_{hp}(x,z) \geq 0$ in Ω^2

hp-FEM basis in 1D





$\Phi_M, \Phi_{M+1}, \ldots, \Phi_N$

hp-FEM basis in 1D

$$l_0(\xi) = (1-\xi)/2 \quad \xi \in \hat{K} = [-1,1]$$

$$l_1(\xi) = (1+\xi)/2$$

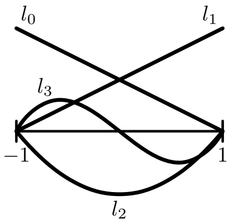
$$l_j(\xi) = \sqrt{\frac{2j-1}{2}} \int_{-1}^{\xi} P_{j-1}(x)\,\mathrm{d}x$$

$$\int_{-1}^{1} l_i'(\xi) l_j'(\xi)\,\mathrm{d}\xi = \delta_{ij} \quad i,j = 2,3,\dots$$

$$l_j(\xi) = l_0(\xi) l_1(\xi) \kappa_j(\xi)$$

$$\kappa_j(\xi) = \sqrt{\frac{2j-1}{2}} \frac{4}{j(1-j)} P_{j-1}'(\xi)$$

$$\int_{-1}^{1} \frac{(1-\xi^2)}{4} \kappa_i(\xi)\kappa_j(\xi)\,\mathrm{d}\xi = \begin{cases} 0, & i \neq j \\ \frac{4}{j(j-1)}, & i = j \end{cases}$$



Stiffness Matrix

$$A_{ij} = a(\phi_j', \phi_i') \qquad A = \begin{pmatrix} A^L & 0 \\ 0 & D \end{pmatrix}$$

$$A^L = \begin{pmatrix} \frac{1}{h_1} + \frac{1}{h_2} & -\frac{1}{h_2} & 0 & 0 & \cdots \\ -\frac{1}{h_2} & \frac{1}{h_2} + \frac{1}{h_3} & -\frac{1}{h_3} & 0 & \cdots \\ 0 & -\frac{1}{h_3} & \frac{1}{h_3} + \frac{1}{h_4} & -\frac{1}{h_4} & \cdots \\ 0 & 0 & -\frac{1}{h_4} & \frac{1}{h_4} + \frac{1}{h_5} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$$D = \operatorname{diag}\Big(\underbrace{\frac{2}{h_1}, \dots, \frac{2}{h_1}}_{(p_1-1)\text{ times}}, \underbrace{\frac{2}{h_2}, \dots, \frac{2}{h_2}}_{(p_2-1)\text{ times}}, \;\dots\;, \underbrace{\frac{2}{h_M}, \dots, \frac{2}{h_M}}_{(p_M-1)\text{ times}}\Big)$$

Stiffness Matrix

$$A_{ij} = a(\phi'_j, \phi'_i) \qquad A^{-1} = \begin{pmatrix} (A^L)^{-1} & 0 \\ 0 & D^{-1} \end{pmatrix}$$

$$(A^L)^{-1} = \frac{1}{b-a}$$

$$\begin{pmatrix} (x_1-a)(b-x_1) & (x_1-a)(b-x_2) & (x_1-a)(b-x_3) & \dots \\ (x_1-a)(b-x_2) & (x_2-a)(b-x_2) & (x_2-a)(b-x_3) & \dots \\ (x_1-a)(b-x_3) & (x_2-a)(b-x_3) & (x_3-a)(b-x_3) & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$$D^{-1} = \operatorname{diag}\Big(\underbrace{\frac{h_1}{2},\dots,\frac{h_1}{2}}_{(p_1-1)\text{ times}}, \underbrace{\frac{h_2}{2},\dots,\frac{h_2}{2}}_{(p_2-1)\text{ times}}, \quad \dots \quad, \underbrace{\frac{h_M}{2},\dots,\frac{h_M}{2}}_{(p_M-1)\text{ times}}\Big)$$

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$$G_{hp}(x,z) = \sum_{j=1}^{N}\sum_{k=1}^{N} A^{-1}_{jk}\phi_k(x)\phi_j(z)$$

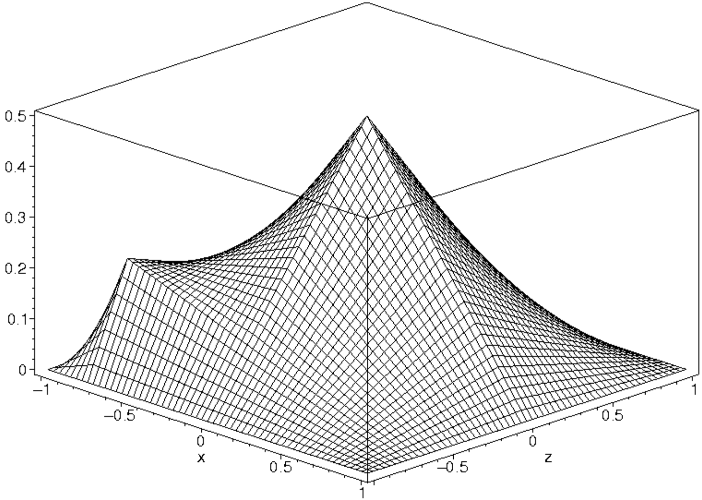
DGF explicitely

$$G_{hp}(x,z) = G_{hp}^L(x,z) + G_{hp}^B(x,z)$$

$$G_{hp}^L(x,z) = \frac{1}{b-a}\left(\sum_{i=1}^{M-1}(x_i-a)(b-x_i)\phi_i(x)\phi_i(z) \right.$$
$$\left. + \sum_{i=1}^{M-2}\sum_{j=i+1}^{M-1}(x_i-a)(b-x_j)[\phi_i(x)\phi_j(z)+\phi_j(x)\phi_i(z)]\right)$$
$$\geq 0 \quad \forall [x,z] \in \Omega^2$$

$$G_{hp}^B(x,z) = \sum_{k=M}^{N} D_{kk}^{-1}\phi_k(x)\phi_k(z) \not\geq 0 \text{ in } \Omega^2$$

DGF example



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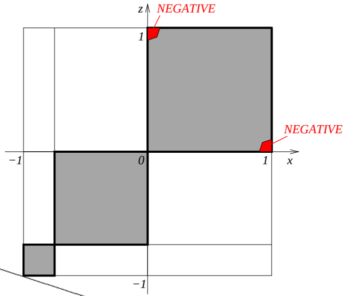
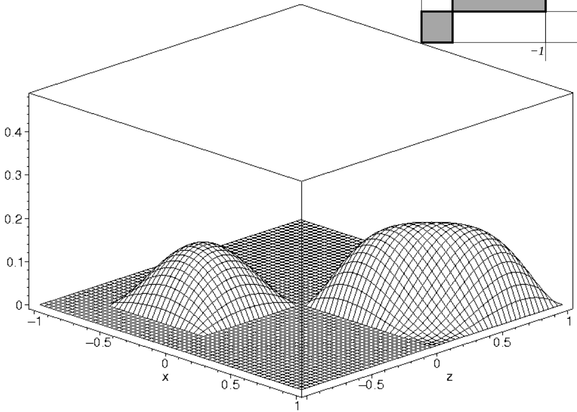




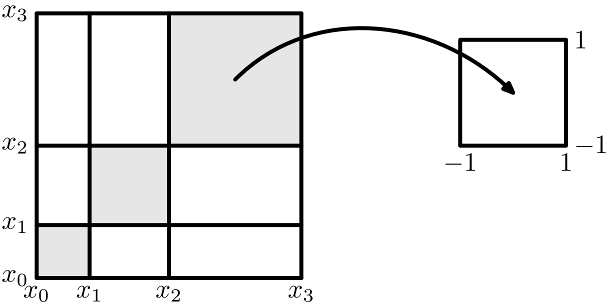
Restriction to K_i^2

$$K_i = [x_{i-1}, x_i]$$

$$G_{hp}(x,z)\big|_{K_i^2} = \frac{(x_{i-1}-a)(b-x_{i-1})}{b-a}\phi_{i-1}(x)\phi_{i-1}(z) + \frac{(x_i-a)(b-x_i)}{b-a}\phi_i(x)\phi_i(z)$$
$$+ \frac{(x_{i-1}-a)(b-x_i)}{b-a}\left[\phi_i(x)\phi_{i-1}(z) + \phi_{i-1}(x)\phi_i(z)\right]$$
$$+ \frac{x_i - x_{i-1}}{2} G^B_{hp}(x,z)\big|_{K_i^2}$$

$$G^B_{hp}(x,z)\big|_{K_i^2} = \sum_{k=0}^{p-2} \phi_{k_i^B+k}(x)\phi_{k_i^B+k}(z)$$

Technical modifications

- Transformation $(x, z) \in K_i^2 \mapsto (\xi, \eta) \in [-1, 1]^2 = \hat{K}^2$
- 

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- $H = x_i - x_{i-1}$
- $H_{\mathrm{rel}} = \dfrac{H}{b - a}$

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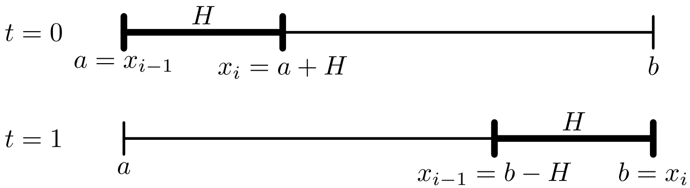
- $H = x_i - x_{i-1}$
- $H_{\mathrm{rel}} = \dfrac{H}{b - a}$
- $\exists t \in [0, 1]$:

$$\begin{aligned} x_{i-1} &= (1-t)a + t(b - H) \\ x_i &= (1-t)(a + H) + tb \quad t \in [0, 1] \end{aligned}$$



Relative Critical Element Length

$$\frac{\hat{G}_{hp}(\xi,\eta)}{H} = t(1-t)\frac{(1-H_{\mathrm{rel}})^2}{H_{\mathrm{rel}}} + t l_0(\xi) l_0(\eta)\left[1 - H_{\mathrm{rel}} + \frac{1}{2} l_1(\xi) l_1(\eta) \sum_{k=2}^{p} \kappa_k(\xi)\kappa_k(\eta)\right] + (1-t) l_1(\xi) l_1(\eta)\left[1 - H_{\mathrm{rel}} + \frac{1}{2} l_0(\xi) l_0(\eta) \sum_{k=2}^{p} \kappa_k(\xi)\kappa_k(\eta)\right]$$

Definition

$$H^*_{\mathrm{rel}}(1) = 1$$

$$H^*_{\mathrm{rel}}(p) = 1 + \frac{1}{2} \min_{(\xi,\eta)\in\hat{K}^2} l_0(\xi) l_0(\eta) \sum_{k=2}^{p} \kappa_k(\xi)\kappa_k(\eta) = 1 + \frac{1}{2} \min_{(\xi,\eta)\in\hat{K}^2} l_1(\xi) l_1(\eta) \sum_{k=2}^{p} \kappa_k(\xi)\kappa_k(\eta) \quad \text{for } p \geq 2$$

Subcritical Elements $\Rightarrow G_{hp} \geq 0$

Theorem

If $a \leq x_{i-1} < x_i \leq b$ *and* $\dfrac{x_i - x_{i-1}}{b-a} = H_{\mathrm{rel}} \leq H^*_{\mathrm{rel}}(p)$,
then $\hat{G}_{hp}(\xi,\eta) \geq 0$ *for all* $[\xi,\eta] \in \hat{K}^2 = [-1,1]^2$.

Main Result

Theorem

If the partition $a = x_0 < x_1 < \ldots < x_M = b$ of the domain $\Omega = (a, b)$ satisfies the condition

$$\frac{x_i - x_{i-1}}{b - a} \leq H^*_{\mathrm{rel}}(p_i) \quad \textit{for all } i = 1, 2, \ldots, M,$$

where $p_i \geq 1$ is the polynomial degree assigned to the element $K_i = [x_{i-1}, x_i]$, then the problem satisfies the discrete maximum principle (i.e., $u_{hp} \geq 0$ in Ω for arbitrary $f \in L^2(\Omega)$ which is nonnegative a.e. in Ω).

Value of $H^*_{\mathrm{rel}}(p)$

p	$H^*_{\mathrm{rel}}(p)$
1	1
2	1
3	9/10
4	1
5	0.919731
6	1
7	0.935127
8	0.987060
9	0.945933
10	0.973952

p	$H^*_{\mathrm{rel}}(p)$
11	0.953759
12	0.969485
13	0.959646
14	0.968378
15	0.964221
16	0.968695
17	0.967874
18	0.969629
19	0.970855
20	0.970814

$$H^*_{\mathrm{rel}}(p) = 1 + \frac{1}{2} \min_{(\xi,\eta)\in\hat{K}^2} l_0(\xi) l_0(\eta) \sum_{k=2}^{p} \kappa_k(\xi)\kappa_k(\eta)$$

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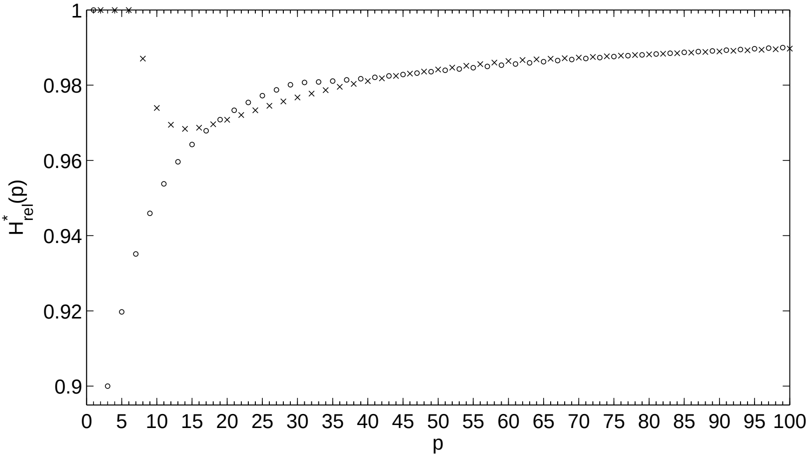

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- 1D Poisson equation + mixed BC $\Rightarrow$ DMP on arbitrary *hp*-mesh.
- Generalization

$$-(au')' = f, \quad a \text{ is piecewise constant}$$

Thank you for your attention

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