On various aspects of the hp-FEM

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Outline

- 1D
  - *hp*-FEM
  - Basis functions
  - Exponential convergence
  - *hp*-adaptivity
  - A role of a posteriori error estimators
- 2D
  - H^1-conforming elements
  - *hp*-adaptivity
  - Arbitrary level hanging nodes
- Static condenstion of internal DOFs
- Maxwell's equations – $H(\text{curl})$-conforming elements
- Numerical examples

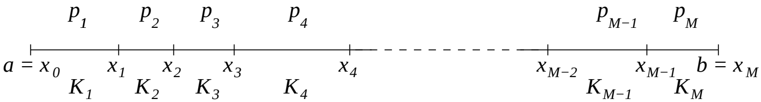
Model problem

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

Weak formulation: $V = H_0^1(\Omega)$

$$u \in V : \quad \underbrace{a(u,v)}_{\int_\Omega \nabla u \cdot \nabla v} = \underbrace{F(v)}_{\int_\Omega f v} \quad \forall v \in V.$$

hp-FEM



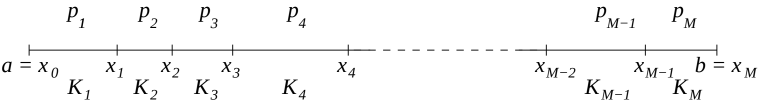
$$V_{hp} = \{v_{hp} \in V : v_{hp}|_{K_i} \in P^{p_i}(K_i)\}$$

$$u_{hp} \in V_{hp} : \quad a(u_{hp}, v_{hp}) = F(v_{hp}) \quad \forall v_{hp} \in V_{hp}$$

$\varphi_1, \varphi_2, \dots, \varphi_N$ – basis in V_{hp}

$$u_{hp}(x) = \sum_{j=1}^{N} y_j \varphi_j(x) \quad \Rightarrow \quad Ay = b, \quad \text{where} \quad \begin{array}{l} A_{ij} = a(\varphi_j, \varphi_i) \\ b_i = F(\varphi_i) \end{array}$$

hp-FEM



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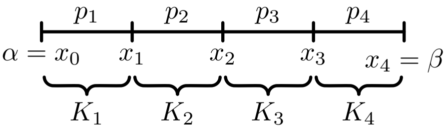
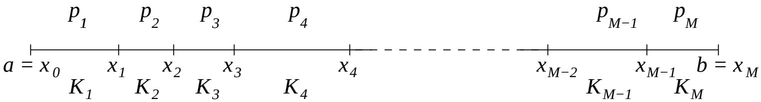


hp-FEM



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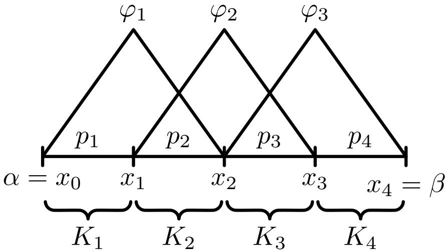
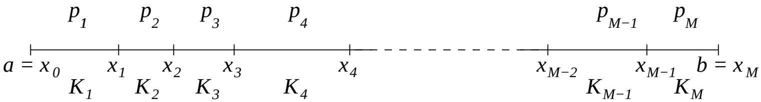


hp-FEM



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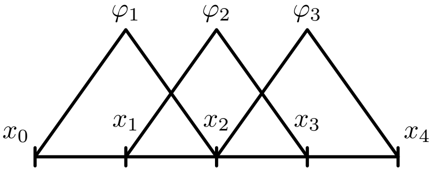
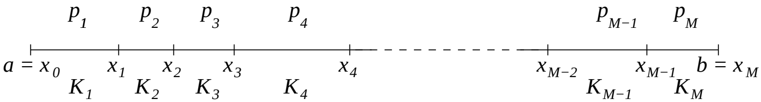


$p_1 = 1$

hp-FEM



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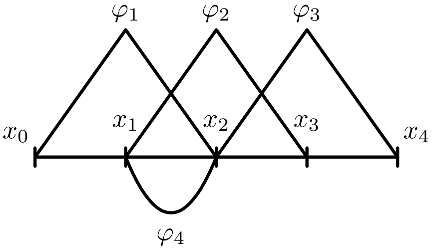
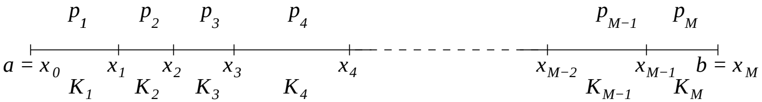


$p_1 = 1 \quad p_2 = 2$

hp-FEM



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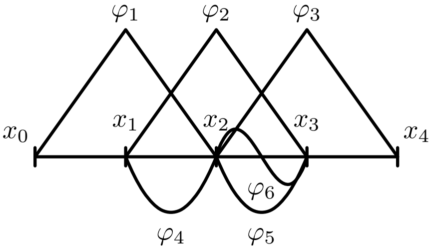
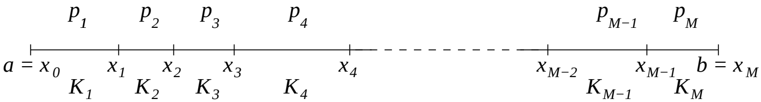
$$u_{hp} \in V_{hp} : \quad a(u_{hp}, v_{hp}) = F(v_{hp}) \quad \forall v_{hp} \in V_{hp}$$

$\varphi_1, \varphi_2, \ldots, \varphi_N$ – basis in V_{hp}



$p_1 = 1 \quad p_2 = 2 \quad p_3 = 3$

hp-FEM



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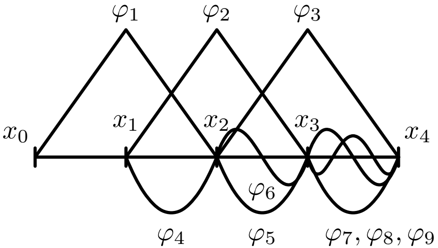


$p_1 = 1 \quad p_2 = 2 \quad p_3 = 3 \quad p_4 = 4$

h-FEM

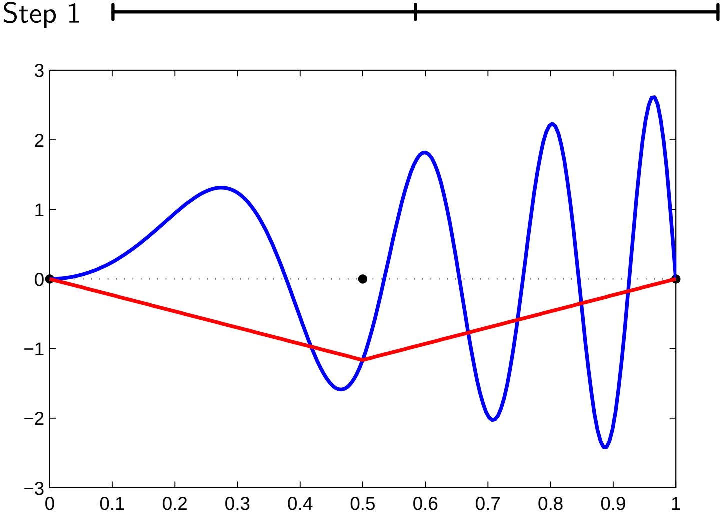
$$\|u - u_{hp}\| \leq C(u)h \qquad \begin{aligned} h &= \max \operatorname{diam}(K_i) \\ N &= N_{\mathrm{DOF}} = \dim V_{hp} \end{aligned}$$

Example: $\qquad u(x) = \sin(7\pi x^2)\exp(x) \qquad x \in (0, 1)$

h-FEM

$$\|u - u_{hp}\| \le C(u)h$$

$h = \max \operatorname{diam}(K_i)$
$N = N_{\mathrm{DOF}} = \dim V_{hp}$

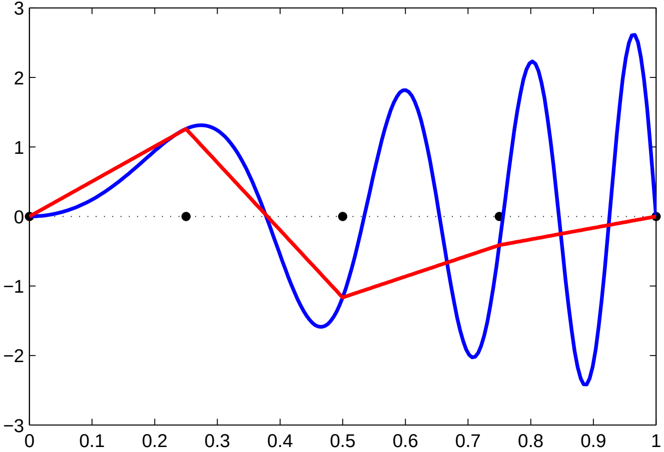


h-FEM

$$\|u - u_{hp}\| \leq C(u)h$$

$$h = \max \operatorname{diam}(K_i)$$
$$N = N_{\mathrm{DOF}} = \dim V_{hp}$$

Step 2 



h-FEM

$$\|u - u_{hp}\| \le C(u)h$$

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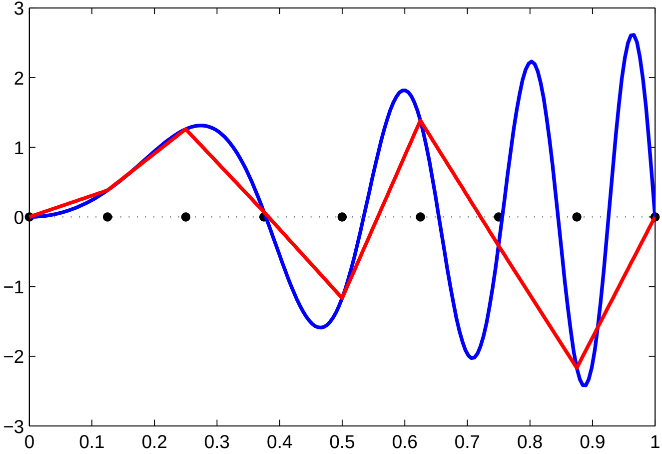
$$N = N_{\mathrm{DOF}} = \dim V_{hp}$$

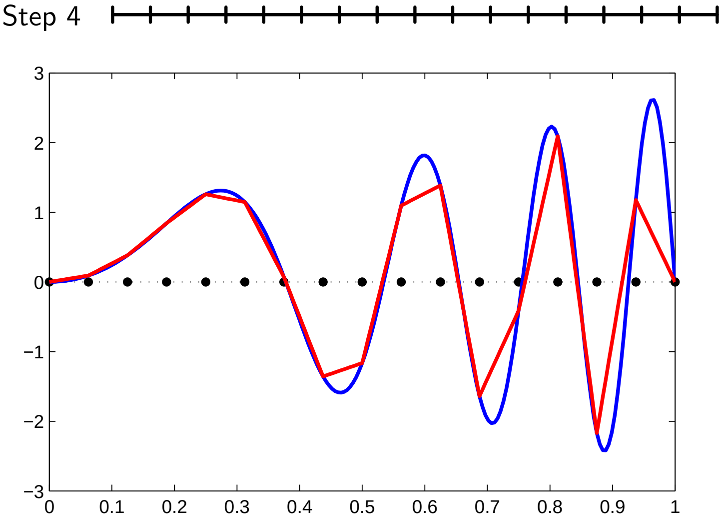
Step 3



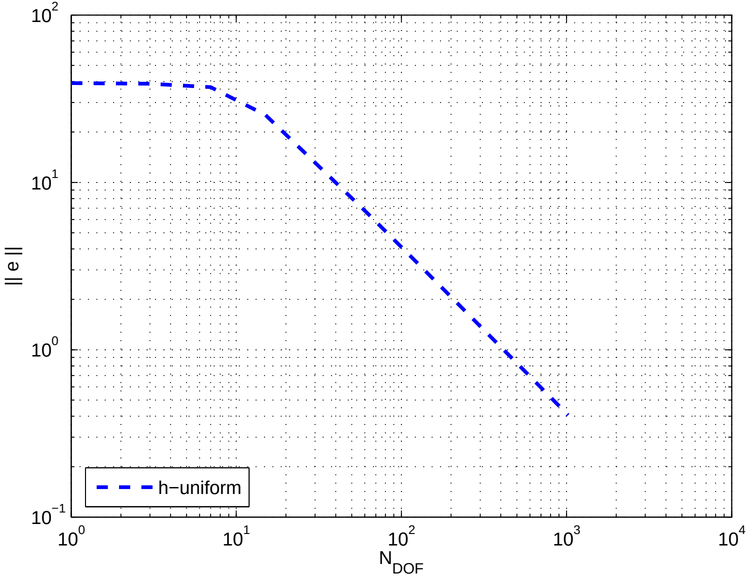
h-FEM

$$\|u - u_{hp}\| \le C(u)h$$

$$h = \max \operatorname{diam}(K_i)$$
$$N = N_{\mathrm{DOF}} = \dim V_{hp}$$



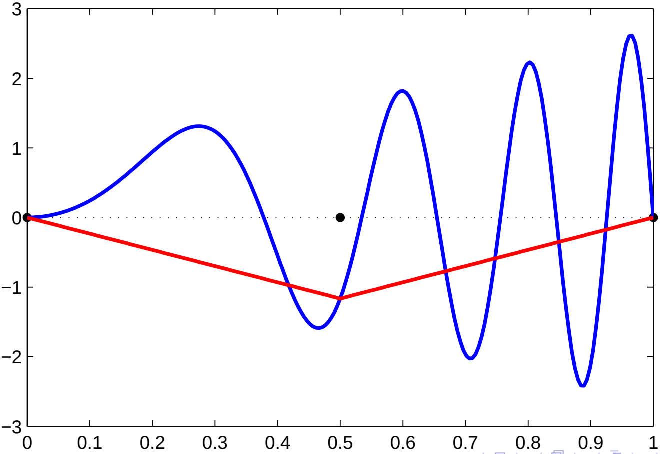
h-FEM convergence history



p-FEM

$$\|u - u_{hp}\| \leq C(u)h^p$$

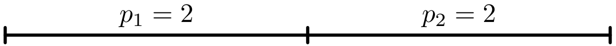
Step 1 $\quad p_1 = 1 \quad p_2 = 1$

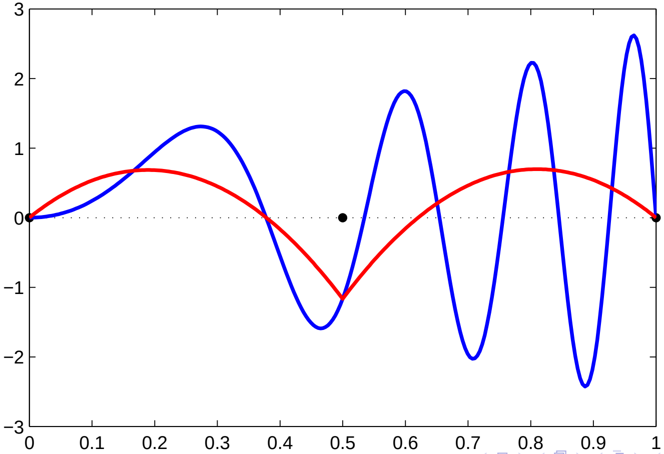


p-FEM

$$\|u - u_{hp}\| \leq C(u)h^p$$

Step 2

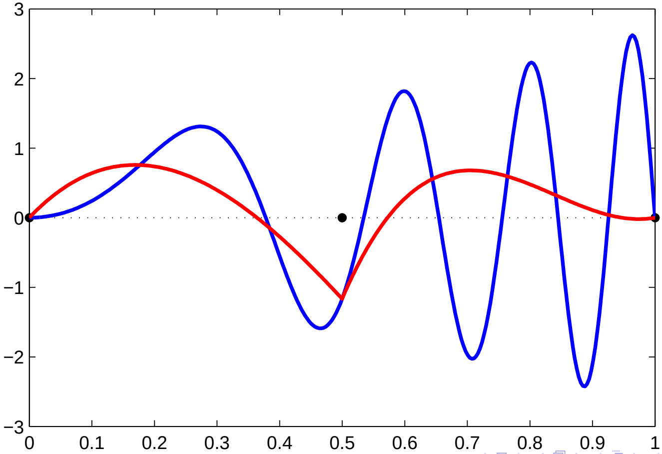




p-FEM

$$\|u - u_{hp}\| \leq C(u)h^p$$

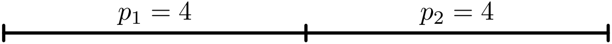
Step 3 $p_1 = 3$ $p_2 = 3$

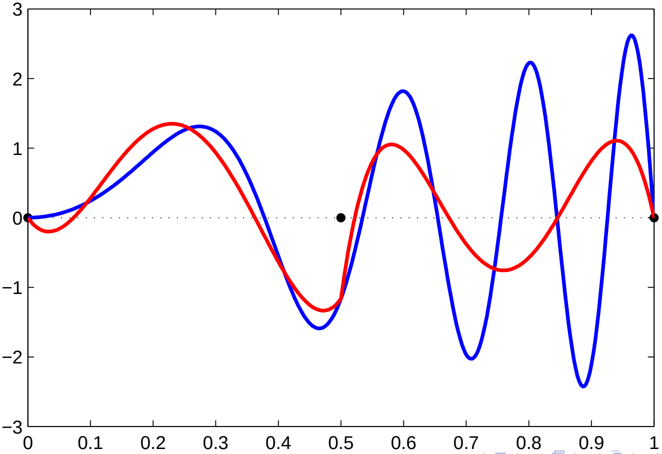


p-FEM

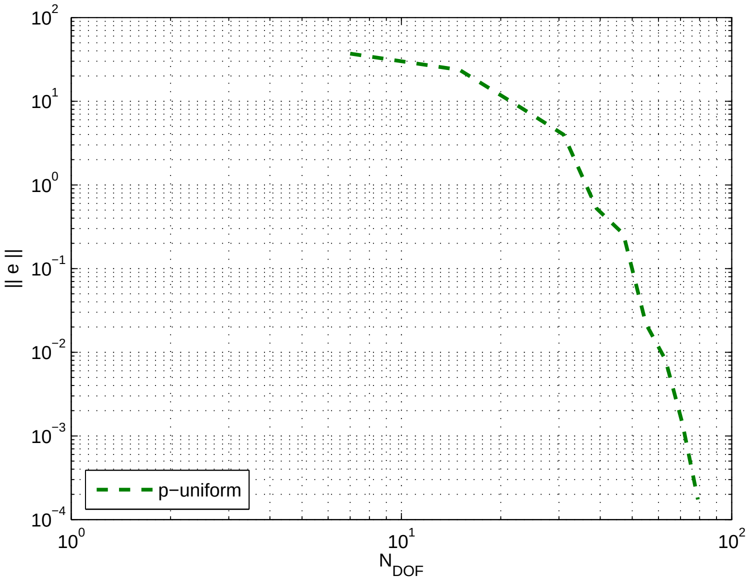
$$\|u - u_{hp}\| \leq C(u)h^p$$

Step 4

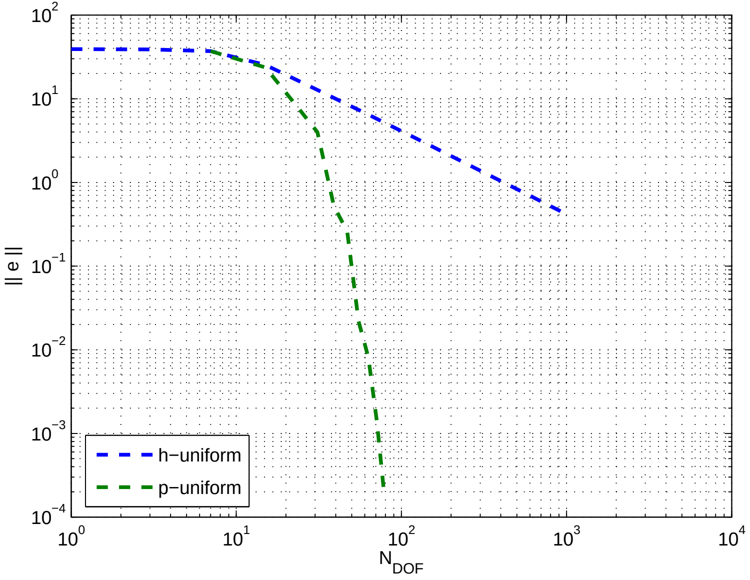




p-FEM convergence history



h-FEM and p-FEM convergence comparison



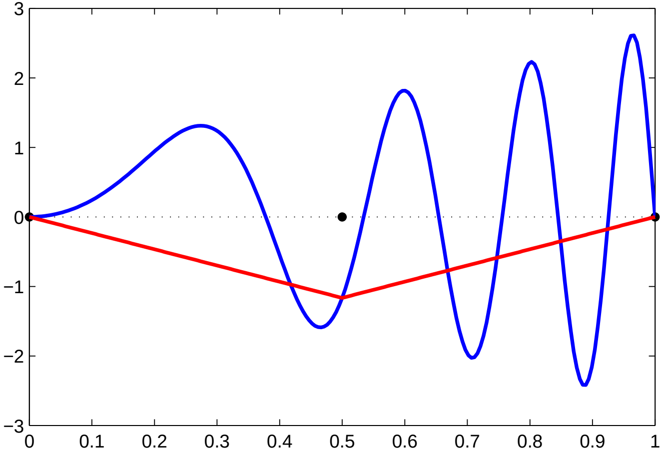
Adaptivity

1. Solve the problem on the given mesh.
2. Estimate error on each element.
3. If error tolerance achieved then stop.
4. Mark elements with big error.
5. Adapt marked elements.
6. Go to 1.

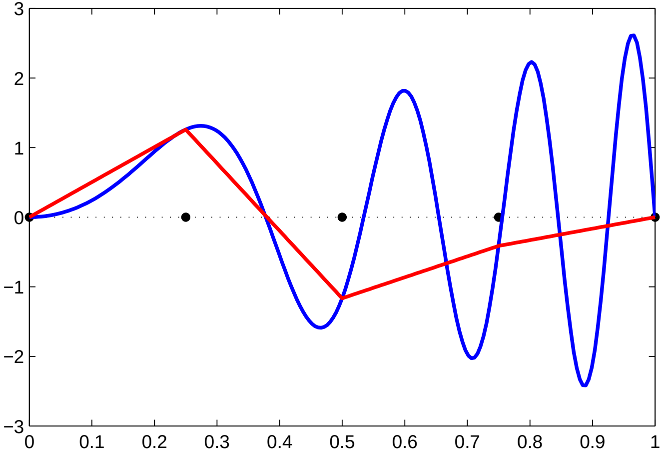
h-adaptivity

1. Solve the problem on the given mesh.
2. Estimate error on each element. A number on each element.
3. If error tolerance achieved then stop.
4. Mark elements with big error. If $\|e_k\| \geq \mathcal{E}^{\mathrm{avg}}$.
5. Adapt marked elements.
   Split the element into two!!!
   
6. Go to 1.

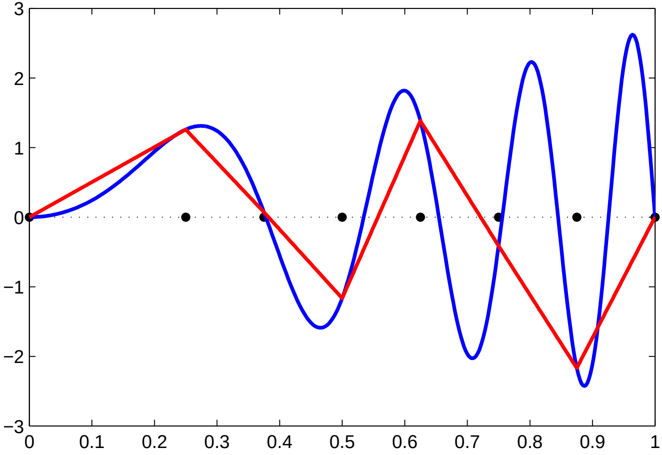
h-adaptivity



h-adaptivity



h-adaptivity



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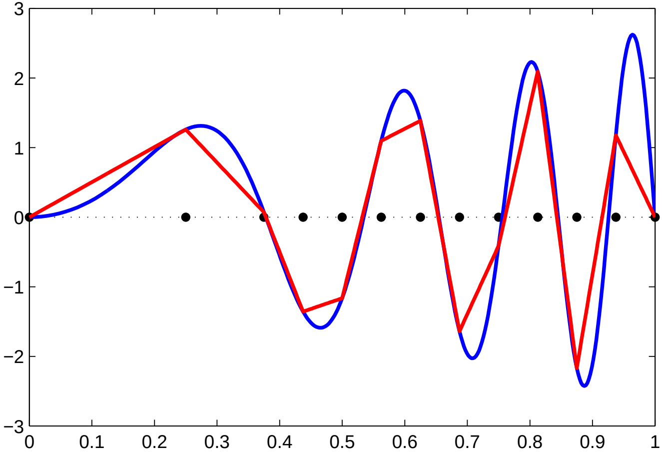

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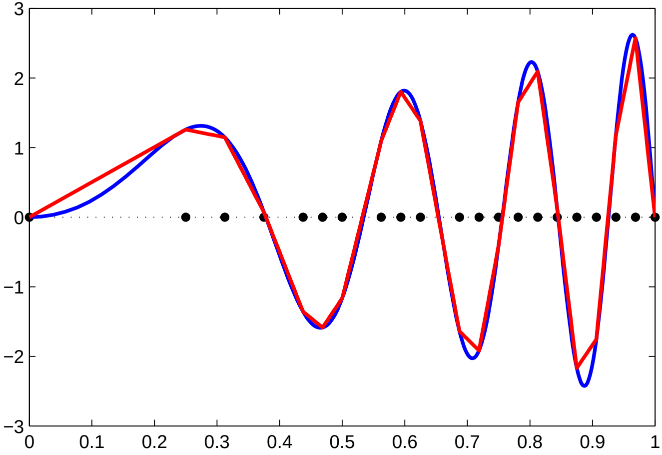

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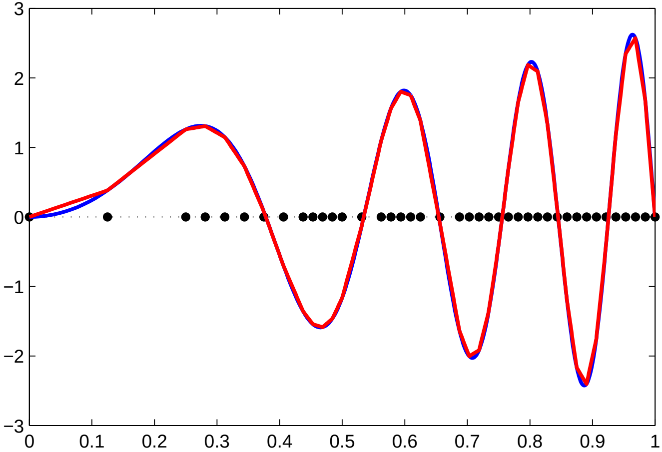
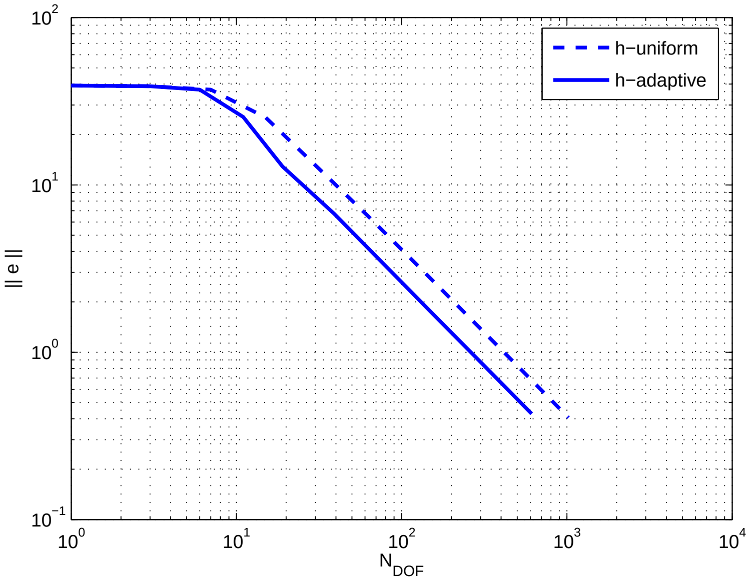


h-adaptive FEM convergence history

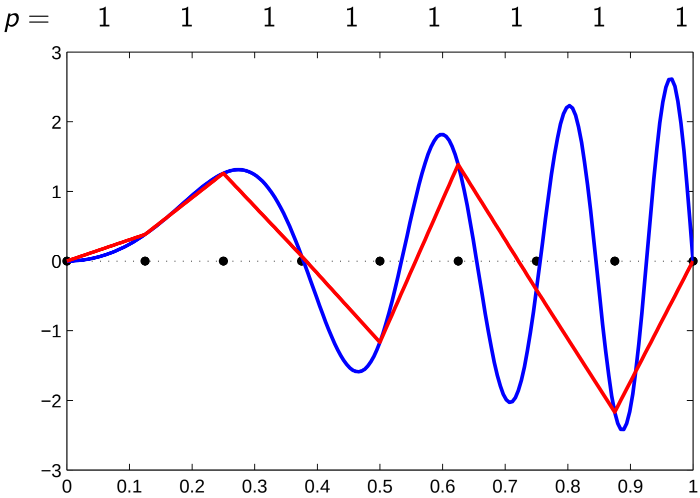


p-adaptivity

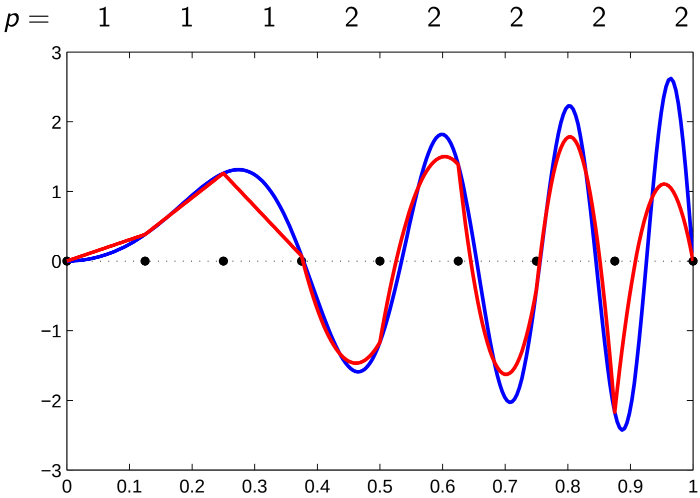
1. Solve the problem on the given mesh.
2. Estimate error on each element. A number on each element.
3. If error tolerance achieved then stop.
4. Mark elements with big error. If $\|e_k\| \geq \mathcal{E}^{\mathrm{avg}}$.
5. Adapt marked elements.
   Increase polynomial degree!!!
   

6. Go to 1.

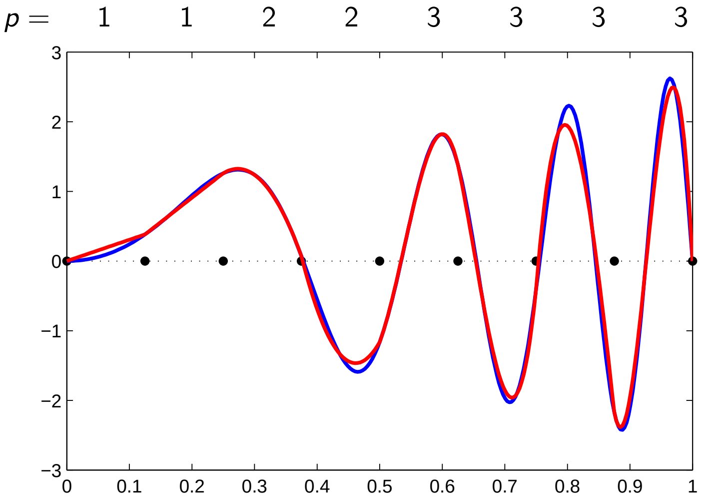
p-adaptivity



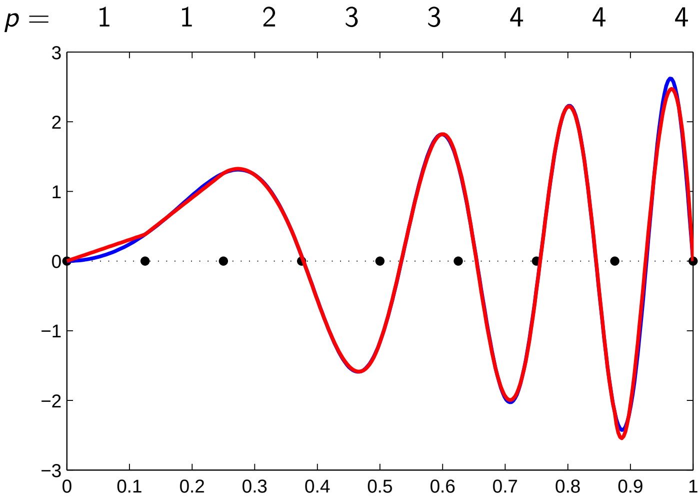
p-adaptivity



p-adaptivity

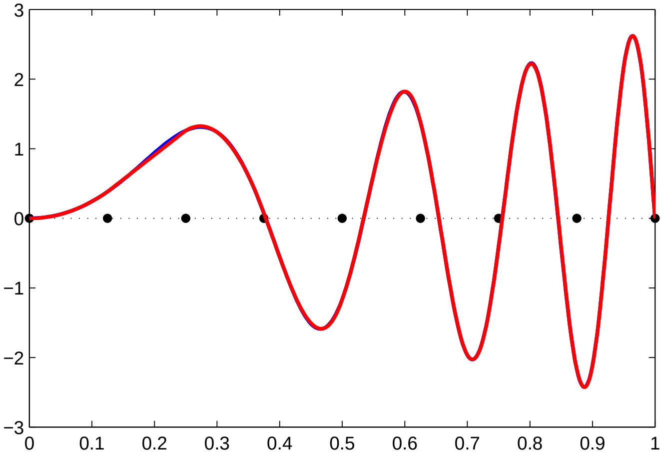


p-adaptivity

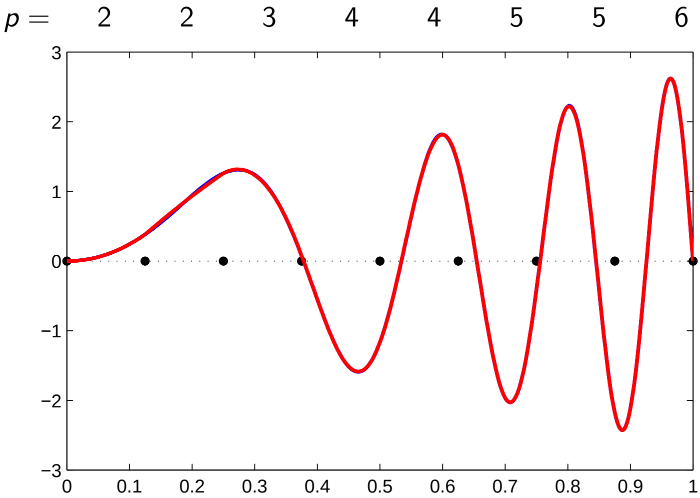


p-adaptivity

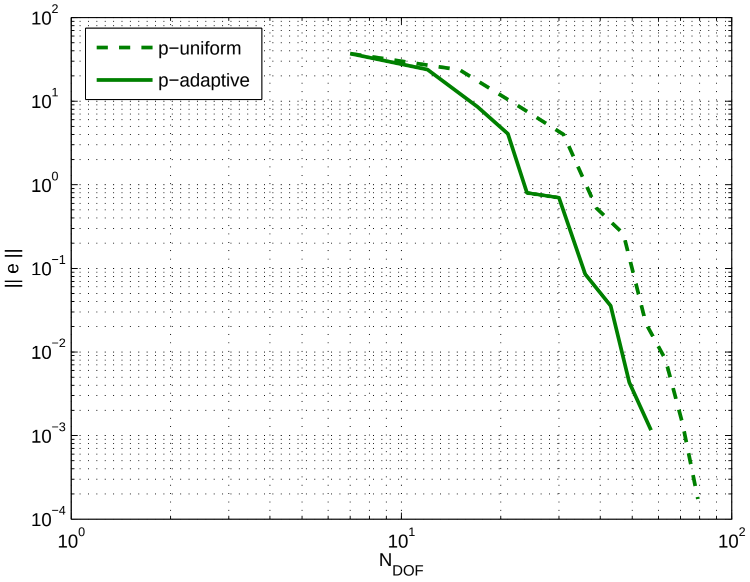
$p = \quad 2 \quad 1 \quad 2 \quad 3 \quad 3 \quad 5 \quad 4 \quad 5$



p-adaptivity

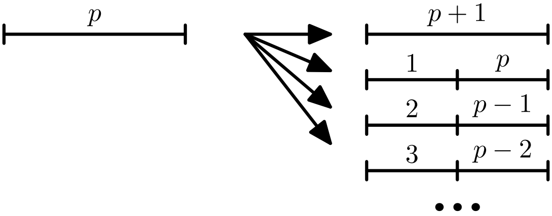


p-adaptive FEM convergence history



hp-adaptivity

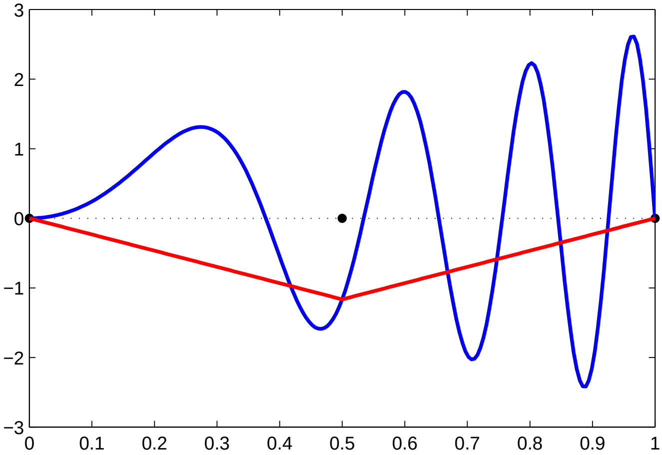
1. Solve the problem on the given mesh.
2. Estimate error on each element. A function on each element!!!
3. If error tolerance achieved then stop.
4. Mark elements with big error. If $\|e_k\| \geq \mathcal{E}^{\mathrm{avg}}$.
5. Adapt marked elements.
   Split the element and modify polynomial degrees!!!



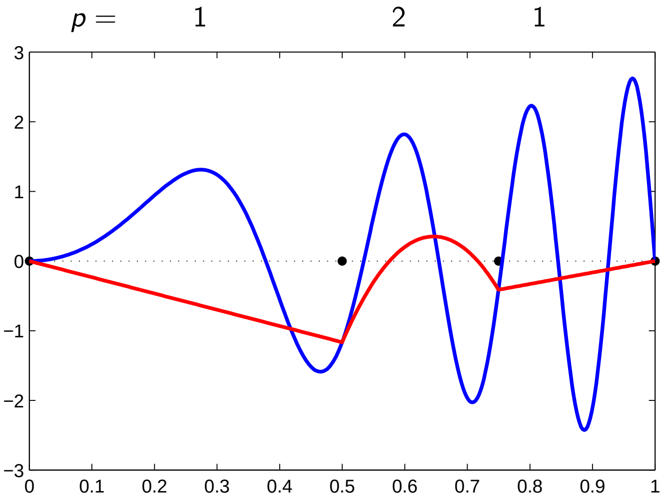
6. Go to 1.

hp-adaptivity

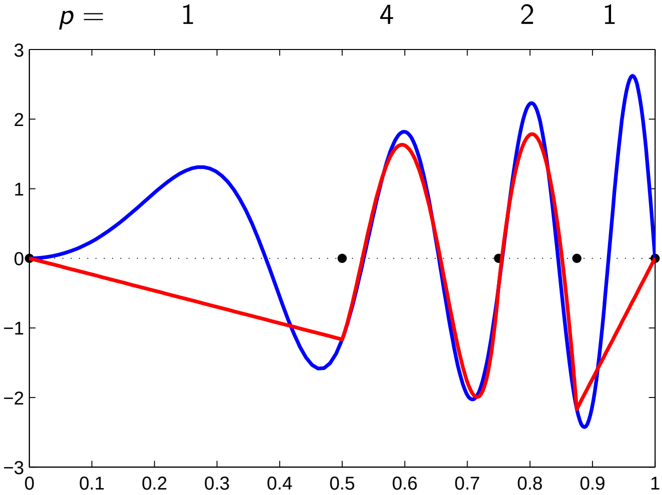
$p = \quad 1 \qquad\qquad 1$



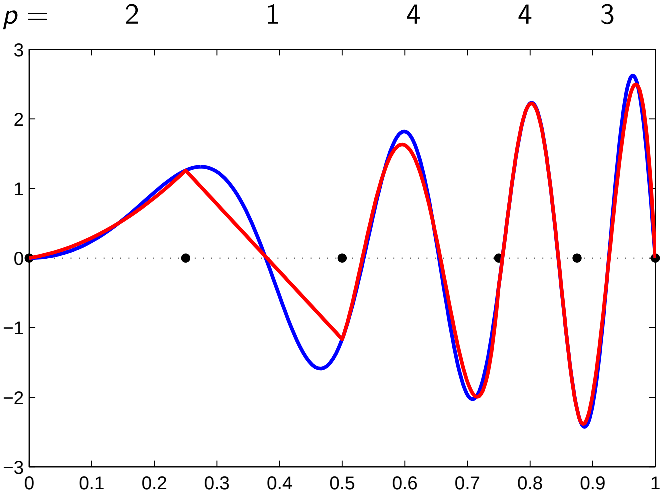
hp-adaptivity



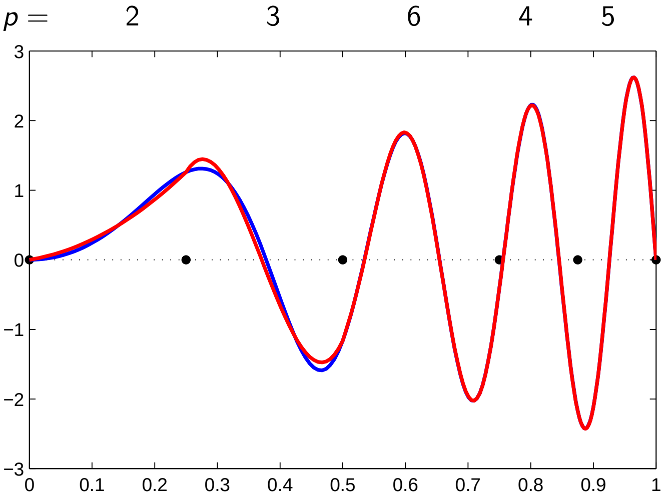
hp-adaptivity



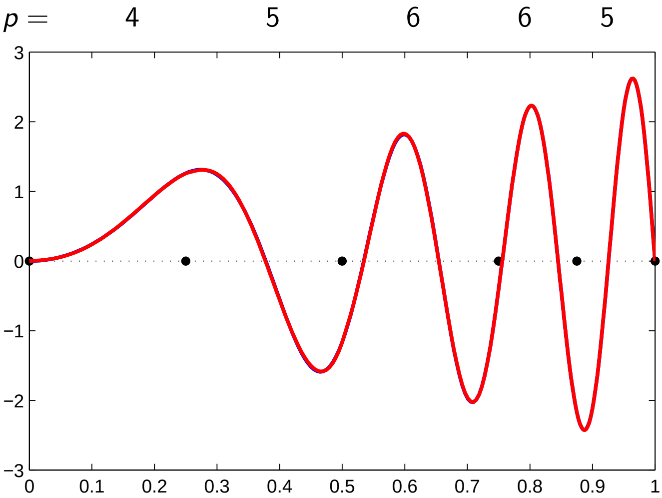
hp-adaptivity



hp-adaptivity

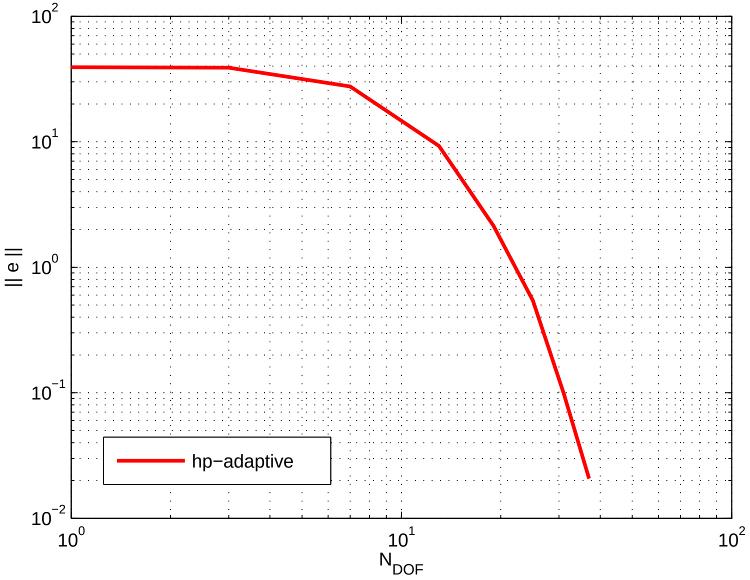


hp-adaptivity

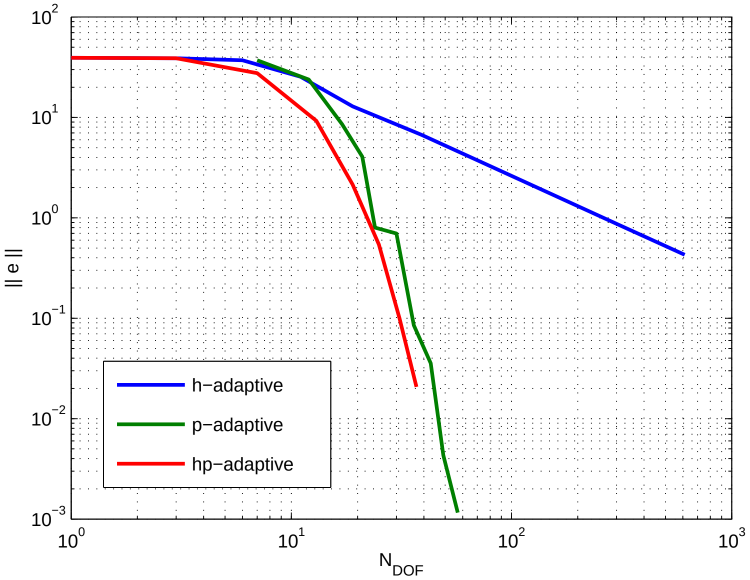


hp-adaptive FEM convergence history

$$\|u - u_{hp}\| \leq C_1 \exp(-C_2\sqrt{N_{\mathrm{DOF}}})$$

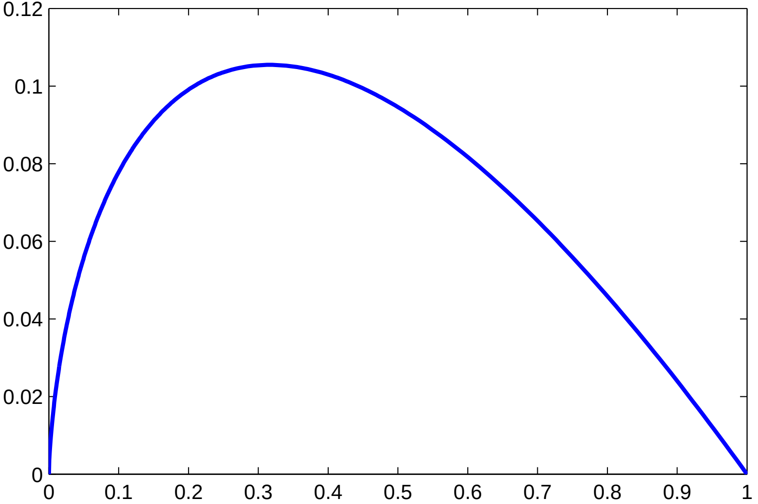


Adaptive FEM convergence comparison

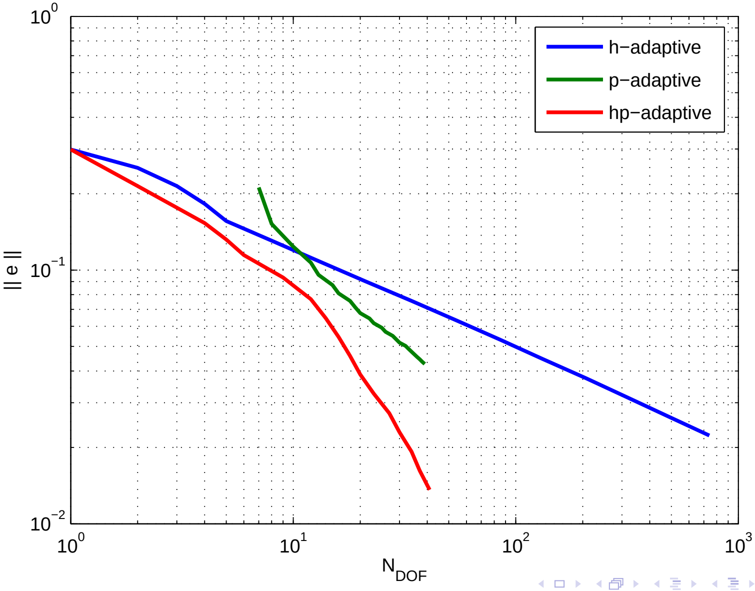


Problem with singularity

$$u(x) = x^{3/4} - x$$



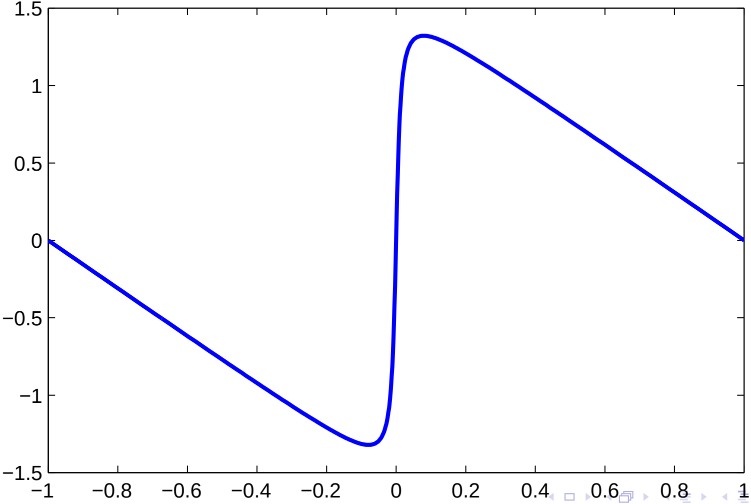
Problem with singularity



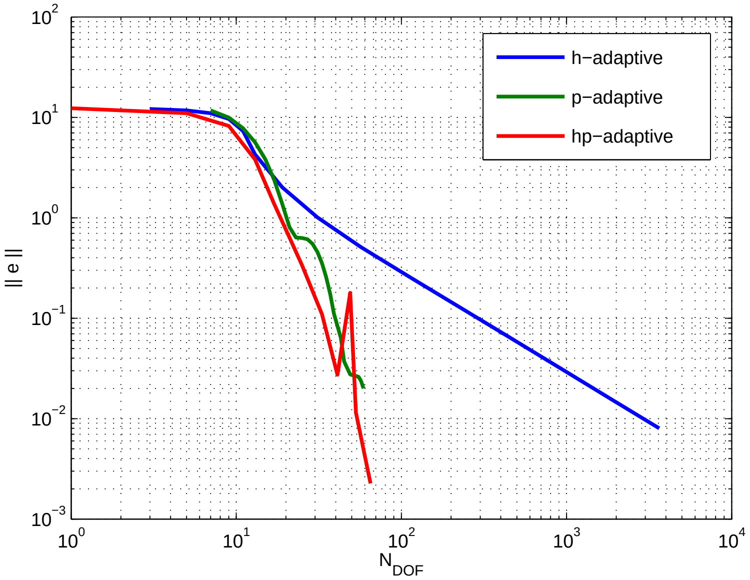
Internal layer

$$u(x) = \arctan(100x) - \ell(x),$$

where $\ell(x)$ is linear such that $u(-1) = u(1) = 0$.



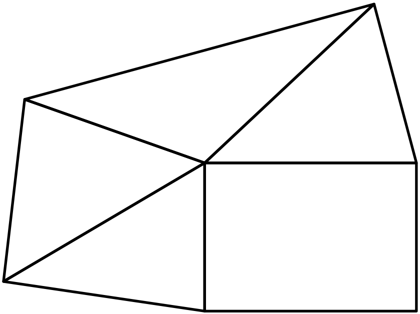
Internal layer



2D

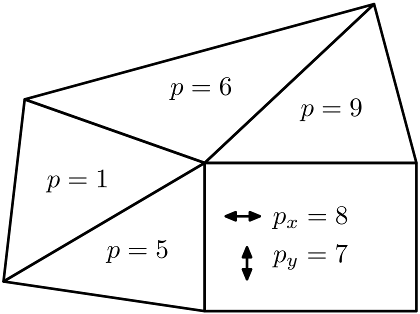
hp-mesh in 2D – minimum rule

$$V_{hp} = \{v_{hp} \in V : v_{hp}|_{K_i} \in P^{p_i}(K_i) \text{ if } K_i \text{ is a triangle}$$
$$v_{hp}|_{K_i} \in Q^{p_{x,i}, p_{y,i}}(K_i) \text{ if } K_i \text{ is a quad}\}$$



hp-mesh in 2D – minimum rule

$$V_{hp} = \{v_{hp} \in V : v_{hp}|_{K_i} \in P^{p_i}(K_i) \text{ if } K_i \text{ is a triangle}$$
$$v_{hp}|_{K_i} \in Q^{p_{x,i}, p_{y,i}}(K_i) \text{ if } K_i \text{ is a quad}\}$$



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$$V_{hp} = \{v_{hp} \in V : v_{hp}|_{K_i} \in P^{p_i}(K_i) \text{ if } K_i \text{ is a triangle}$$
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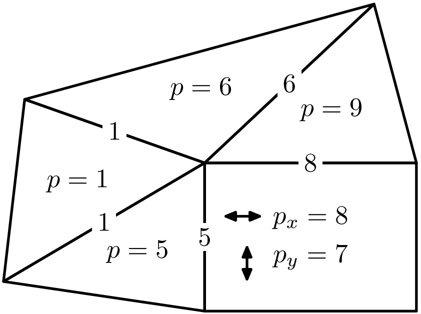
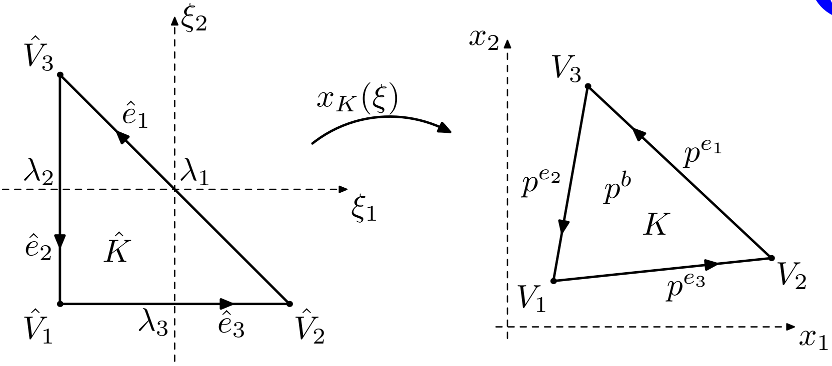


Basis functions



Shape functions
$\hat{\varphi}(\xi)$ on $\hat{K}$
$P^{p^b}(\hat{K})$

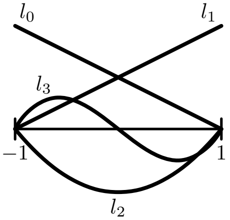
$x = \mathbf{x}_K(\xi)$
$\longmapsto$

Basis functions
$\varphi(x) := \hat{\varphi}\left(\mathbf{x}_K^{-1}(x)\right)$ on K
$P^{p^b}(K)$

$$\mathbf{x}_K(\xi) = \sum_{i=1}^{3} V_i \lambda_i(\xi)$$

Lobatto polynomials and kernel functions

$$l_0(\xi) = (1-\xi)/2 \quad \xi \in \hat{K} = [-1,1]$$
$$l_1(\xi) = (1+\xi)/2$$
$$l_j(\xi) = \sqrt{\frac{2j-1}{2}} \int_{-1}^{\xi} P_{j-1}(x)\,\mathrm{d}x$$
$$\int_{-1}^{1} l_i'(\xi) l_j'(\xi)\,\mathrm{d}\xi = \delta_{ij} \quad i,j = 2,3,\ldots$$



$$l_j(\xi) = l_0(\xi) l_1(\xi) \kappa_{j-2}(\xi)$$

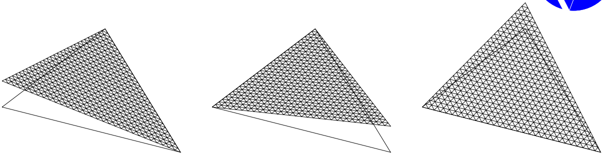
$$\kappa_k(\xi) = -\sqrt{\frac{2k+3}{2}} \frac{4}{(k+2)(k+1)} P_{k+1}'(\xi) \quad k = 0,1,\ldots$$

$$\int_{-1}^{1} \frac{(1-\xi^2)}{4} \kappa_\ell(\xi) \kappa_k(\xi)\,\mathrm{d}\xi = \begin{cases} 0, & \ell \neq k \\ \frac{4}{(k+2)(k+1)}, & \ell = k \end{cases}$$

Shape functions

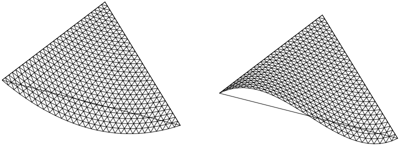
Vertex, 3:

$\hat{\varphi}^{v_1}(\xi) = \lambda_1(\xi)$
$\hat{\varphi}^{v_2}(\xi) = \lambda_2(\xi)$
$\hat{\varphi}^{v_3}(\xi) = \lambda_3(\xi)$



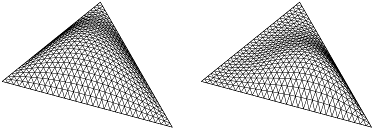
Edge, $p^{e_i} - 1$:

$\hat{\varphi}_k^{e_1} = \lambda_2 \lambda_3 \kappa_{k-2}(\lambda_3 - \lambda_2)$
$\hat{\varphi}_k^{e_2} = \lambda_3 \lambda_1 \kappa_{k-2}(\lambda_1 - \lambda_3)$
$\hat{\varphi}_k^{e_3} = \lambda_1 \lambda_2 \kappa_{k-2}(\lambda_2 - \lambda_1)$
$k = 2, 3, \ldots, p^{e_i}$



Bubble, $(p^b - 1)(p^b - 2)/2$:

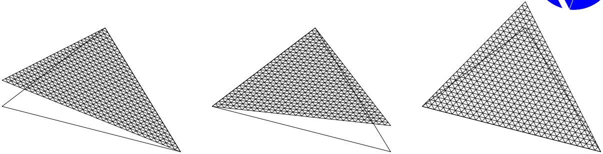
$\hat{\varphi}_{n,m}^{b,l} = \lambda_1 \lambda_2^n \lambda_3^m$
$m + n + 1 \leq p^b;\ m, n \geq 1$



Shape functions

Vertex, 3:

$$\hat{\varphi}^{v_1}(\xi) = \lambda_1(\xi)$$
$$\hat{\varphi}^{v_2}(\xi) = \lambda_2(\xi)$$
$$\hat{\varphi}^{v_3}(\xi) = \lambda_3(\xi)$$



Edge, $p^{e_i} - 1$:

$$\hat{\varphi}_k^{e_1} = \lambda_2\lambda_3\kappa_{k-2}(\lambda_3 - \lambda_2)$$
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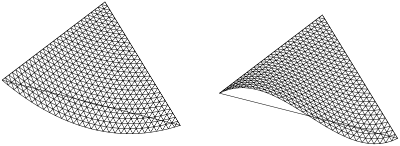
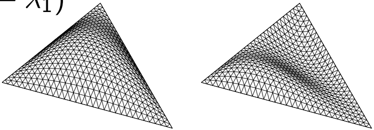
$$\hat{\varphi}_k^{e_3} = \lambda_1\lambda_2\kappa_{k-2}(\lambda_2 - \lambda_1)$$
$$k = 2, 3, \ldots, p^{e_i}$$



Bubble, $(p^b - 1)(p^b - 2)/2$:

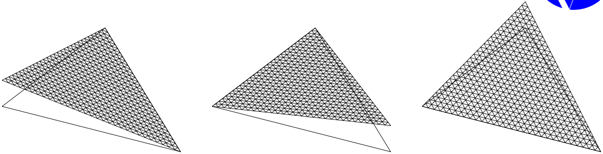
$$\hat{\varphi}_{n,m}^{b,II} = \lambda_1\lambda_2\lambda_3\kappa_{n-1}(\lambda_3 - \lambda_2)\kappa_{m-1}(\lambda_2 - \lambda_1)$$
$$m + n + 1 \le p^b;\ m, n \ge 1$$



Shape functions

Vertex, 3:

$$\hat{\varphi}^{v_1}(\xi) = \lambda_1(\xi)$$
$$\hat{\varphi}^{v_2}(\xi) = \lambda_2(\xi)$$
$$\hat{\varphi}^{v_3}(\xi) = \lambda_3(\xi)$$



Edge, $p^{e_i} - 1$:

$$\hat{\varphi}_k^{e_1} = \lambda_2\lambda_3\kappa_{k-2}(\lambda_3 - \lambda_2)$$
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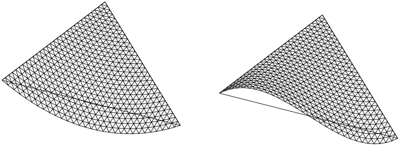
$$k = 2, 3, \dots, p^{e_i}$$



Bubble, $(p^b - 1)(p^b - 2)/2$:

$\hat{\varphi}_{n,m}^{b,III} = \dots$ orthonormal by Gram-Schmidt process

$$m + n + 1 \leq p^b;\ m, n \geq 1$$

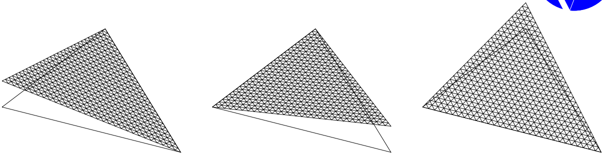
Shape functions

Vertex, 3:

$\hat{\varphi}^{v_1}(\xi) = \lambda_1(\xi)$

$\hat{\varphi}^{v_2}(\xi) = \lambda_2(\xi)$

$\hat{\varphi}^{v_3}(\xi) = \lambda_3(\xi)$



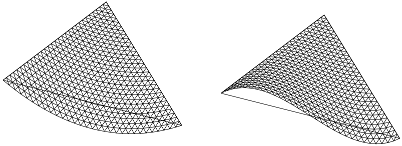
Edge, $p^{e_i} - 1$:

$\hat{\varphi}_k^{e_1} = \lambda_2\lambda_3\kappa_{k-2}(\lambda_3 - \lambda_2)$

$\hat{\varphi}_k^{e_2} = \lambda_3\lambda_1\kappa_{k-2}(\lambda_1 - \lambda_3)$

$\hat{\varphi}_k^{e_3} = \lambda_1\lambda_2\kappa_{k-2}(\lambda_2 - \lambda_1)$

$k = 2, 3, \ldots, p^{e_i}$

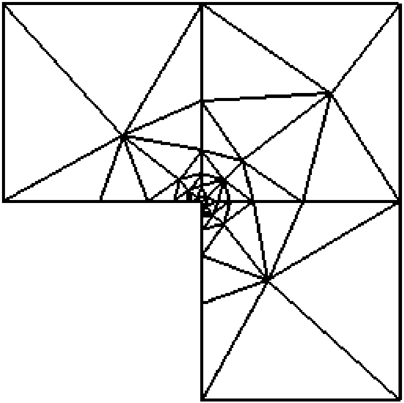


Bubble, $(p^b - 1)(p^b - 2)/2$:

$\hat{\varphi}_{n,m}^{b,IV} = \ldots$ generalized eigenfunctions of discrete Laplacian

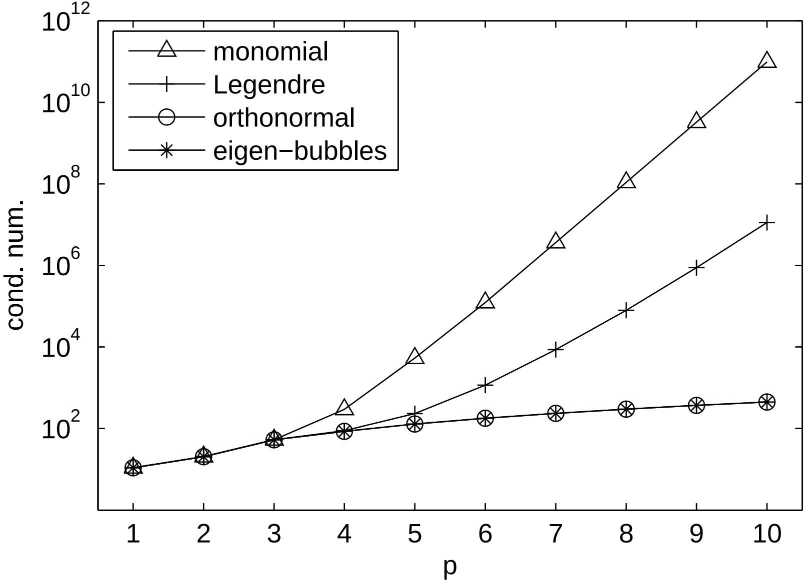
$m + n + 1 \leq p^b;\ m, n \geq 1$

Stiffness matrix conditioning

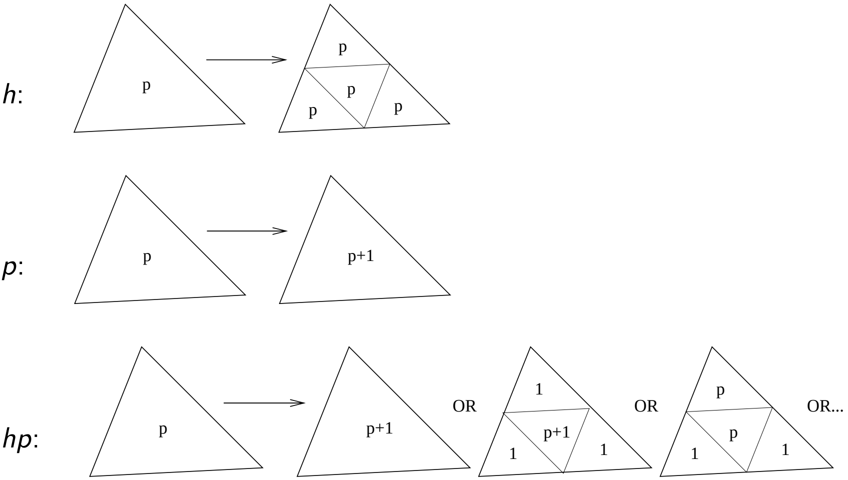


$$-\Delta u = f \quad \text{in } \Omega$$
$$u = 0 \quad \text{on } \partial\Omega$$

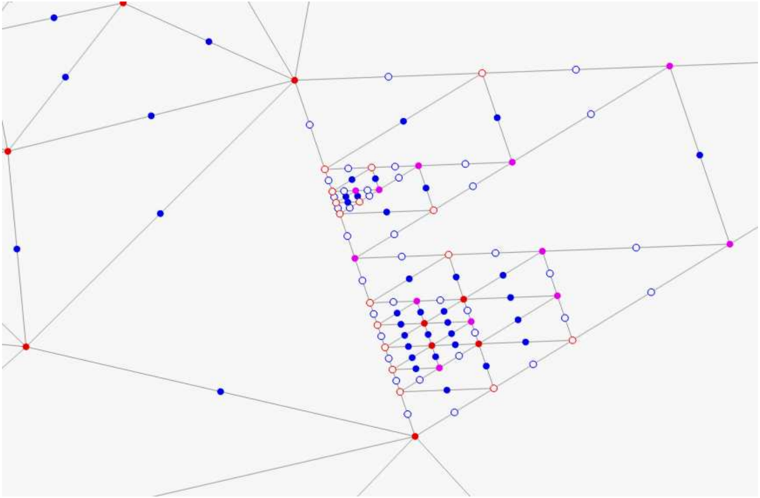
Stiffness matrix conditioning



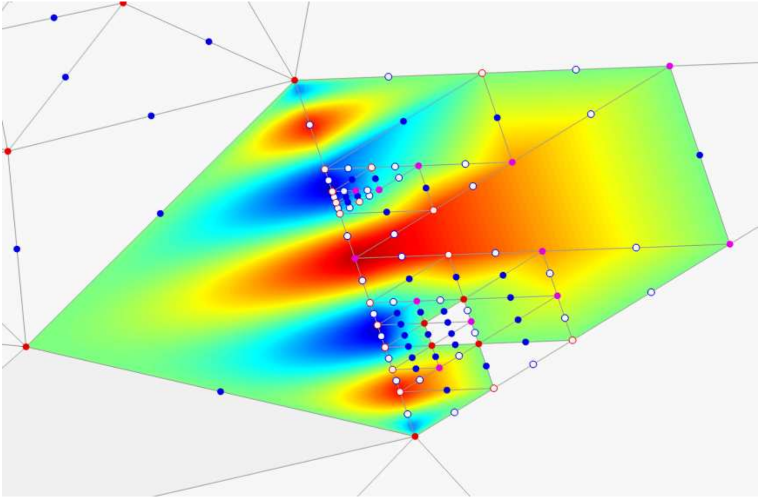
hp-adaptivity in 2D



Arbitrary level hanging nodes



Arbitrary level hanging nodes



Static condensation of internal DOFs

Static condensation of internal DOFs

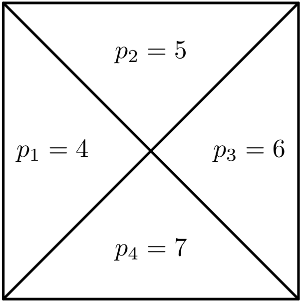
Basis: $$\{\underbrace{\varphi_1^{bub}, \ldots, \varphi_M^{bub}}_{\text{bubbles}}, \underbrace{\varphi_{M+1}^{non}, \ldots, \varphi_N^{non}}_{\text{non-bubbles}}\}$$

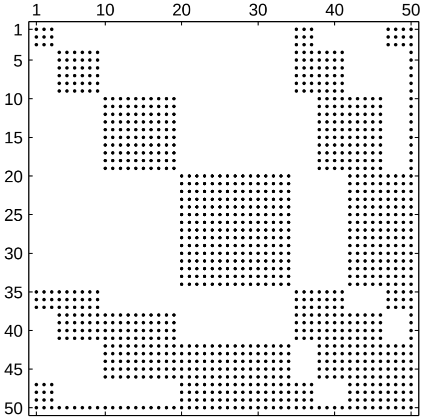
$$u_{hp} = \sum_{j=1}^{M} x_j \varphi_j^{bub} + \sum_{j=1}^{M-N} y_j \varphi_{M+j}^{non}$$

$$\begin{pmatrix} A & B^T \\ B & D \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} = \begin{pmatrix} F \\ G \end{pmatrix} \quad \Leftrightarrow \quad \begin{pmatrix} I & 0 \\ 0 & S \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} = \begin{pmatrix} \widetilde{F} \\ \widetilde{G} \end{pmatrix}$$

$$S = D - BA^{-1}B^T$$
$$\widetilde{F} = A^{-1}(F - B^T\mathbf{y})$$
$$\widetilde{G} = G - BA^{-1}F$$

Structure of the stiffness matrix





Static condensation – results

refin. steps	static condensation M	N_{iter}	CPU time	ILU-PCG N	N_{iter}	CPU time
0	34	3	0.004 s	50	3	0.005 s
1	136	5	0.012 s	225	5	0.017 s
2	544	7	0.049 s	953	7	0.130 s
3	2176	11	0.389 s	3921	11	1.665 s
4	8704	21	4.697 s	15905	21	26.10 s
5	34816	40	71.10 s	64065	40	415.5 s

Time harmonic Maxwell's equations

H(curl) conforming elements (edge elements)

Time harmonic Maxwell's equations in 2D

$$\mathbf{curl}\left(\mu_r^{-1}\,\mathrm{curl}\,\mathbf{E}\right) - \kappa^2\epsilon_r\mathbf{E} = \mathbf{F} \quad \text{in } \Omega$$

$$\mathbf{E}\cdot\tau = 0 \quad \text{on } \Gamma_P \quad \text{(Perfect conductor)}$$

$$\mu_r^{-1}\,\mathrm{curl}\,\mathbf{E} - \mathrm{i}\kappa\lambda\mathbf{E}\cdot\tau = \mathbf{g}\cdot\tau \quad \text{on } \Gamma_I \quad \text{(Impedance)}$$

- $\mathbf{curl} = (\partial/\partial x_2, -\partial/\partial x_1)^\top$
- $\mathrm{curl}\,\mathbf{E} = \partial E_2/\partial x_1 - \partial E_1/\partial x_2$
- $\Omega \subset \mathbb{R}^2$
- $\mu_r = \mu_r(x) \in \mathbb{R}$ relative permeability
- $\epsilon_r = \epsilon_r(x) \in \mathbb{C}^{2\times 2}$ relative permittivity
- $\mathbf{E} = \mathbf{E}(x) \in \mathbb{C}^2$ phaser of the electric field intensity
- $\mathbf{F} = \mathbf{F}(x) \in \mathbb{C}^2$
- $\kappa \in \mathbb{R}$ the wave number
- $\tau = (-\nu_2, \nu_1)^\top$ positively oriented unit tangent vector
- $\lambda = \lambda(x) > 0$ impedance
- $\mathbf{g} = \mathbf{g}(x) \in \mathbb{C}^2$

Weak and *hp*-FEM formulations

- $$V = \{\mathbf{E} \in \mathbf{H}(\mathrm{curl}, \Omega) : \mathbf{E} \cdot \tau = 0 \text{ on } \Gamma_P\}$$
  $$\mathbf{E} \in V : \quad \boxed{a(\mathbf{E}, \mathbf{\Phi}) = \mathcal{F}(\mathbf{\Phi})} \quad \forall \mathbf{\Phi} \in V$$
- $$a(\mathbf{E}, \mathbf{\Phi}) = \left(\mu_{\mathrm{r}}^{-1} \,\mathrm{curl}\, \mathbf{E}, \mathrm{curl}\, \mathbf{\Phi}\right) - \kappa^2 \left(\epsilon_{\mathrm{r}} \mathbf{E}, \mathbf{\Phi}\right) - \mathrm{i}\kappa \left\langle \lambda \mathbf{E} \cdot \tau, \mathbf{\Phi} \cdot \tau \right\rangle$$
  $$\mathcal{F}(\mathbf{\Phi}) = (\mathbf{F}, \mathbf{\Phi}) + \langle \mathbf{g} \cdot \tau, \mathbf{\Phi} \cdot \tau \rangle$$

Weak and *hp*-FEM formulations

- $$V = \{\mathbf{E} \in \mathbf{H}(\mathrm{curl}, \Omega) : \mathbf{E} \cdot \tau = 0 \text{ on } \Gamma_P\}$$
  $$\mathbf{E} \in V : \quad \boxed{a(\mathbf{E}, \mathbf{\Phi}) = \mathcal{F}(\mathbf{\Phi})} \quad \forall \mathbf{\Phi} \in V$$
- $$a(\mathbf{E}, \mathbf{\Phi}) = \left(\mu_{\mathrm{r}}^{-1} \,\mathrm{curl}\, \mathbf{E}, \mathrm{curl}\, \mathbf{\Phi}\right) - \kappa^2 \left(\epsilon_{\mathrm{r}} \mathbf{E}, \mathbf{\Phi}\right) - \mathrm{i}\kappa \left\langle \lambda \mathbf{E} \cdot \tau, \mathbf{\Phi} \cdot \tau \right\rangle$$
  $$\mathcal{F}(\mathbf{\Phi}) = (\mathbf{F}, \mathbf{\Phi}) + \langle \mathbf{g} \cdot \tau, \mathbf{\Phi} \cdot \tau \rangle$$
- $$V_h = \left\{ \mathbf{E}_h \in V : \mathbf{E}_h|_{K_j} \in \mathbf{P}^{p_j}(K_j) \text{ and } \mathbf{E}_h \cdot \tau_k \text{ is continuous on each edge } e_k \right\}$$
  $$\mathbf{E}_h \in V_h : \quad \boxed{a(\mathbf{E}_h, \mathbf{\Phi}_h) = \mathcal{F}(\mathbf{\Phi}_h)} \quad \forall \mathbf{\Phi}_h \in V_h$$

Weak and *hp*-FEM formulations

- $$V = \{\mathbf{E} \in \mathbf{H}(\mathrm{curl}, \Omega) : \mathbf{E} \cdot \tau = 0 \text{ on } \Gamma_P\}$$
  $$\mathbf{E} \in V : \quad \boxed{a(\mathbf{E}, \mathbf{\Phi}) = \mathcal{F}(\mathbf{\Phi})} \quad \forall \mathbf{\Phi} \in V$$
- $$a(\mathbf{E}, \mathbf{\Phi}) = \left(\mu_{\mathrm{r}}^{-1}\,\mathrm{curl}\,\mathbf{E}, \mathrm{curl}\,\mathbf{\Phi}\right) - \kappa^2\left(\epsilon_{\mathrm{r}}\mathbf{E}, \mathbf{\Phi}\right) - \mathrm{i}\kappa\,\langle \lambda \mathbf{E} \cdot \tau, \mathbf{\Phi} \cdot \tau \rangle$$
  $$\mathcal{F}(\mathbf{\Phi}) = (\mathbf{F}, \mathbf{\Phi}) + \langle \mathbf{g} \cdot \tau, \mathbf{\Phi} \cdot \tau \rangle$$
- $$V_h = \big\{\mathbf{E}_h \in V : \mathbf{E}_h|_{K_j} \in \mathbf{P}^{p_j}(K_j) \text{ and } \mathbf{E}_h \cdot \tau_k \text{ is continuous on each edge } e_k\big\}$$
  $$\mathbf{E}_h \in V_h : \quad \boxed{a(\mathbf{E}_h, \mathbf{\Phi}_h) = \mathcal{F}(\mathbf{\Phi}_h)} \quad \forall \mathbf{\Phi}_h \in V_h$$
- $$\mathbf{E}_h = \sum_j^N \underbrace{c_j}_{\in \mathbb{C}} \psi_j; \qquad \psi_j \ \dots \ \text{hierarchic basis}$$

Shape functions

Whitney functions:

$$\hat{\psi}_0^{e_1} = \frac{\lambda_3 \mathbf{n}_2}{\mathbf{n}_2 \cdot \mathbf{t}_1} + \frac{\lambda_2 \mathbf{n}_3}{\mathbf{n}_3 \cdot \mathbf{t}_1}$$

$$\hat{\psi}_0^{e_2} = \frac{\lambda_1 \mathbf{n}_3}{\mathbf{n}_3 \cdot \mathbf{t}_2} + \frac{\lambda_3 \mathbf{n}_1}{\mathbf{n}_1 \cdot \mathbf{t}_2}$$

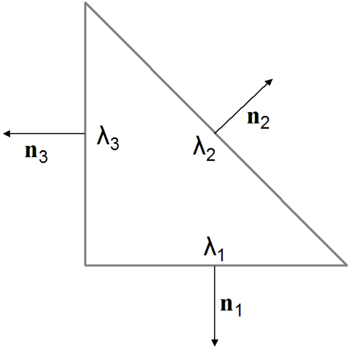
$$\hat{\psi}_0^{e_3} = \frac{\lambda_2 \mathbf{n}_1}{\mathbf{n}_1 \cdot \mathbf{t}_3} + \frac{\lambda_1 \mathbf{n}_2}{\mathbf{n}_2 \cdot \mathbf{t}_3}$$

First order functions:

$$\hat{\psi}_1^{e_1} = \frac{\lambda_3 \mathbf{n}_2}{\mathbf{n}_2 \cdot \mathbf{t}_1} - \frac{\lambda_2 \mathbf{n}_3}{\mathbf{n}_3 \cdot \mathbf{t}_1}$$

$$\hat{\psi}_1^{e_2} = \frac{\lambda_1 \mathbf{n}_3}{\mathbf{n}_3 \cdot \mathbf{t}_2} - \frac{\lambda_3 \mathbf{n}_1}{\mathbf{n}_1 \cdot \mathbf{t}_2}$$

$$\hat{\psi}_1^{e_3} = \frac{\lambda_2 \mathbf{n}_1}{\mathbf{n}_1 \cdot \mathbf{t}_3} - \frac{\lambda_1 \mathbf{n}_2}{\mathbf{n}_2 \cdot \mathbf{t}_3}$$



$$\mathbf{t}_i = \begin{bmatrix} -\mathbf{n}_{i,2} \\ \mathbf{n}_{i,1} \end{bmatrix}$$

Shape functions

Edge functions:

$$\hat{\psi}_k^{e_1} = \frac{2k-1}{k} L_{k-1}(\lambda_3 - \lambda_2)\hat{\psi}_1^{e_1} - \frac{k-1}{k} L_{k-2}(\lambda_3 - \lambda_2)\hat{\psi}_0^{e_1},$$

$$\hat{\psi}_k^{e_2} = \frac{2k-1}{k} L_{k-1}(\lambda_1 - \lambda_3)\hat{\psi}_1^{e_2} - \frac{k-1}{k} L_{k-2}(\lambda_1 - \lambda_3)\hat{\psi}_0^{e_2},$$

$$\hat{\psi}_k^{e_3} = \frac{2k-1}{k} L_{k-1}(\lambda_2 - \lambda_1)\hat{\psi}_1^{e_3} - \frac{k-1}{k} L_{k-2}(\lambda_2 - \lambda_1)\hat{\psi}_0^{e_3}, \quad k = 2, 3, \ldots$$

Edge based and genuine bubblea:

$$\hat{\psi}_k^{b,e_1} = \lambda_3\lambda_2 L_{k-2}(\lambda_3 - \lambda_2)\mathbf{n}_1,$$

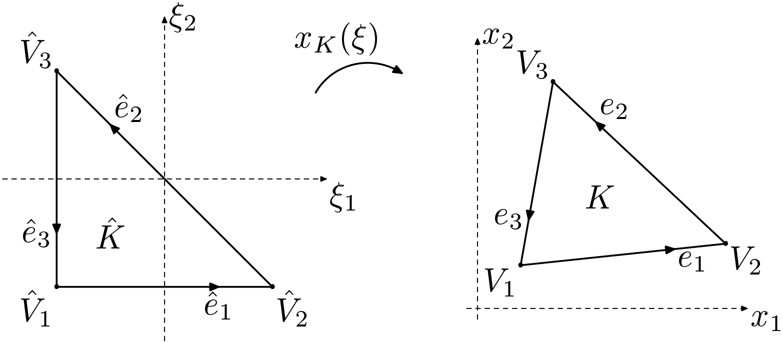
$$\hat{\psi}_k^{b,e_2} = \lambda_1\lambda_3 L_{k-2}(\lambda_1 - \lambda_3)\mathbf{n}_2,$$

$$\hat{\psi}_k^{b,e_3} = \lambda_2\lambda_1 L_{k-2}(\lambda_2 - \lambda_1)\mathbf{n}_3, \quad k = 2, 3, \ldots$$

$$\hat{\psi}_{n_1,n_2}^{b,1} = \lambda_1\lambda_2\lambda_3 L_{n_1-1}(\lambda_3 - \lambda_2) L_{n_2-1}(\lambda_2 - \lambda_1) \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

$$\hat{\psi}_{n_1,n_2}^{b,2} = \lambda_1\lambda_2\lambda_3 L_{n_1-1}(\lambda_3 - \lambda_2) L_{n_2-1}(\lambda_2 - \lambda_1) \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad 1 \le n_1, n_2$$

Affine transformation



Bubble functions: $\psi_K^b\left(x_K(\xi)\right) = \left(\frac{\mathrm{D}x_K}{\mathrm{D}\xi}\right)^{-\top} \hat{\psi}^b(\xi)$

Edge functions: $\psi_{K,k}^{e_i}\left(x_K(\xi)\right)/\|e_i\| = \pm\left(\frac{\mathrm{D}x_K}{\mathrm{D}\xi}\right)^{-\top} \hat{\psi}_k^{\hat{e}_i}(\xi)/\|\hat{e}_i\|$

Examples

See “sem MU Hermes.pdf”.

Thank you for your attention

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