Computing guaranteed upper bounds of the approximation error

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Outline

- A posteriori error estimates
- Primal problem: $-\mathcal{L}\mathbf{u} = f \quad \text{in } \Omega \quad + \quad \text{b.c.}$
- Complementary problem: $-\mathcal{L}^*\mathbf{y} = f \quad \text{in } \Omega \quad + \quad \text{b.c.}$
- Error estimate: $|\!|\!|\mathbf{u} - \mathbf{u}_h|\!|\!| \leq \eta(\mathbf{u}_h, \mathbf{y}_h) \quad \forall \mathbf{u}_h \in \mathbf{V} \quad \forall \mathbf{y}_h \in \mathbf{W}$
- Numerical examples
- Conclusions

A posteriori error estimates

- GOAL: Solve the problem with prescribed accuracy.

A posteriori error estimates

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- Adaptive algorithm
  1. Initialize: Construct the initial mesh $\mathcal{T}_h$.
  2. Solve: Find $\mathbf{u}_h$ on $\mathcal{T}_h$.
  3. Estimate error: Compute η_K for all $K \in \mathcal{T}_h$. $\eta^2 = \sum_{K \in \mathcal{T}_h} \eta_K^2$.
  4. Stopping criterion: If $\quad \eta \leq \mathrm{TOL} \quad \Rightarrow \quad$ STOP.
  5. Mark: If $\quad \eta_K \geq \Theta \max_{K \in \mathcal{T}_h} \eta_K \quad \Rightarrow \quad$ mark K. $\qquad 0 < \Theta < 1$
  6. Refine: Refine marked elements and build the new mesh $\mathcal{T}_h$.
  7. GO TO 2.

A posteriori error estimates

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  5. Mark: If $\eta_K \geq \Theta \max_{K \in \mathcal{T}_h} \eta_K \quad \Rightarrow \quad \text{mark } K. \qquad 0 < \Theta < 1$
  6. Refine: Refine marked elements and build the new mesh $\mathcal{T}_h$.
  7. GO TO 2.
- Guaranteed upper bound: $\|\mathbf{u} - \mathbf{u}_h\| \leq \eta \leq \text{TOL}$
- Verification

Primal Problem

$\Omega \subset \mathbb{R}^d$

$$
\begin{aligned}
-\operatorname{div}\left(\mathcal{A}^{11}\nabla u^1\right)\cdots-\operatorname{div}\left(\mathcal{A}^{1N}\nabla u^N\right)+c^{11}u^1\cdots+c^{1N}u^N&=f^1\\
-\operatorname{div}\left(\mathcal{A}^{21}\nabla u^1\right)\cdots-\operatorname{div}\left(\mathcal{A}^{2N}\nabla u^N\right)+c^{21}u^1\cdots+c^{2N}u^N&=f^2\\
&\vdots\\
-\operatorname{div}\left(\mathcal{A}^{N1}\nabla u^1\right)\cdots-\operatorname{div}\left(\mathcal{A}^{NN}\nabla u^N\right)+c^{N1}u^1\cdots+c^{NN}u^N&=f^N
\end{aligned}
$$

$$u^1=u^2=\cdots=u^N=0 \quad \text{on } \partial\Omega$$

Primal Problem

$$-\,\mathbf{div}(\overline{\mathcal{A}}\,\overline{\nabla u}) + C\mathbf{u} = \mathbf{f} \quad \text{in } \Omega$$

$$\mathbf{u} = \mathbf{0} \quad \text{on } \partial\Omega$$

$$\mathbf{u} = \begin{pmatrix} u^1 \\ \vdots \\ u^N \end{pmatrix} \quad \overline{\nabla u} = \begin{pmatrix} \nabla u^1 \\ \vdots \\ \nabla u^N \end{pmatrix} \quad \overline{\mathbf{y}} = \begin{pmatrix} \mathbf{y}^1 \\ \vdots \\ \mathbf{y}^N \end{pmatrix} \quad \mathbf{div}\,\overline{\mathbf{y}} = \begin{pmatrix} \operatorname{div} \mathbf{y}^1 \\ \vdots \\ \operatorname{div} \mathbf{y}^N \end{pmatrix}$$

$$\overline{\mathcal{A}} = \begin{pmatrix} \mathcal{A}^{11} & \cdots & \mathcal{A}^{1N} \\ \vdots & & \vdots \\ \mathcal{A}^{N1} & \cdots & \mathcal{A}^{NN} \end{pmatrix} \quad C = \begin{pmatrix} c^{11} & \cdots & c^{1N} \\ \vdots & & \vdots \\ c^{N1} & \cdots & c^{NN} \end{pmatrix} \quad \mathbf{f} = \begin{pmatrix} f^1 \\ \vdots \\ f^N \end{pmatrix}$$

Assumptions: $\overline{\mathcal{A}} = \overline{\mathcal{A}}^{\frac{1}{2}}\overline{\mathcal{A}}^{\frac{1}{2}}$ is uniformly s.p.d. $\quad C = K^T K$ is s.p.d.

Weak formulation

Strong form.:
$$-\mathbf{div}(\overline{\mathcal{A}}\,\overline{\boldsymbol{\nabla} u}) + C\mathbf{u} = \mathbf{f} \quad \text{in } \Omega$$
$$\mathbf{u} = \mathbf{0} \quad \text{on } \partial\Omega$$

Weak form.: $\mathbf{u} \in \mathbf{V}: \quad \mathcal{B}(\mathbf{u},\mathbf{v}) = \mathcal{F}(\mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{V}$

Notation:

- $\mathbf{V} = \left[H_0^1(\Omega)\right]^N$
- $\mathcal{B}(\mathbf{u},\mathbf{v}) = \left(\overline{\mathcal{A}}\,\overline{\boldsymbol{\nabla} u}, \overline{\boldsymbol{\nabla} v}\right) + (C\mathbf{u},\mathbf{v}) \qquad (\overline{\mathbf{p}},\overline{\mathbf{q}}) = \int_\Omega \overline{\mathbf{p}} \cdot \overline{\mathbf{q}}\,\mathrm{d}x$
- $\mathcal{F}(\mathbf{v}) = (\mathbf{f},\mathbf{v})$
- $\|\mathbf{v}\|^2 = \mathcal{B}(\mathbf{v},\mathbf{v})$

Derivation of the estimate

Divergence theorem: $v \in H^1(\Omega) \quad \mathbf{y} \in \mathbf{H}(\mathrm{div}, \Omega)$

$$\int_\Omega v \,\mathrm{div}\, \mathbf{y} \,\mathrm{d}x + \int_\Omega \mathbf{y} \cdot \boldsymbol{\nabla} v \,\mathrm{d}x - \int_{\partial\Omega} v\mathbf{y} \cdot \mathbf{n} \,\mathrm{d}x = 0$$

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$\mathcal{B}(\mathbf{u}, \mathbf{v}) =$

$= (\mathbf{f}, \mathbf{v})$

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$$\mathcal{B}(\mathbf{u} - \mathbf{u}_h, \mathbf{v}) =$$

$$= (\mathbf{f}, \mathbf{v}) - (\overline{\mathcal{A}}\, \overline{\boldsymbol{\nabla} u_h}, \overline{\boldsymbol{\nabla} v}) - (C\mathbf{u}_h, \mathbf{v})$$

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$$= \left([\mathbf{f} - C\mathbf{u}_h + \mathbf{div}\, \overline{\mathbf{y}}]\, , \mathbf{v} \right) + \left(\left[\overline{\mathbf{y}} - \overline{\mathcal{A}}\, \overline{\boldsymbol{\nabla} u_h} \right], \overline{\boldsymbol{\nabla} v} \right)$$

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$$= \left(K^{-T} \left[\mathbf{f} - C\mathbf{u}_h + \mathbf{div}\, \overline{\mathbf{y}}\right], K\mathbf{v}\right) + \left(\overline{\mathcal{A}}^{-\frac{1}{2}} \left[\overline{\mathbf{y}} - \overline{\mathcal{A}}\, \overline{\boldsymbol{\nabla} u_h}\right], \overline{\mathcal{A}}^{\frac{1}{2}} \overline{\boldsymbol{\nabla} v}\right)$$

Derivation of the estimate

Divergence theorem: $v \in H^1(\Omega) \quad \mathbf{y} \in \mathbf{H}(\mathrm{div}, \Omega)$

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$$= \left(K^{-T} \left[\mathbf{f} - C\mathbf{u}_h + \mathbf{div}\, \overline{\mathbf{y}} \right], K\mathbf{v} \right) + \left(\overline{\mathcal{A}}^{-\frac{1}{2}} \left[\overline{\mathbf{y}} - \overline{\mathcal{A}}\, \overline{\boldsymbol{\nabla} u_h} \right], \overline{\mathcal{A}}^{\frac{1}{2}} \overline{\boldsymbol{\nabla} v} \right)$$

$$\leq \left\| K^{-T} \left[\mathbf{f} - C\mathbf{u}_h + \mathbf{div}\, \overline{\mathbf{y}} \right] \right\|_0 \|K\mathbf{v}\|_0 + \left\| \overline{\mathcal{A}}^{-\frac{1}{2}} \left[\overline{\mathbf{y}} - \overline{\mathcal{A}}\, \overline{\boldsymbol{\nabla} u_h} \right] \right\|_0 \left\| \overline{\mathcal{A}}^{\frac{1}{2}} \overline{\boldsymbol{\nabla} v} \right\|_0$$

Derivation of the estimate

Divergence theorem: $v \in H^1(\Omega) \quad \mathbf{y} \in \mathbf{H}(\mathrm{div}, \Omega)$

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$$\begin{aligned}
&\mathcal{B}(\mathbf{u} - \mathbf{u}_h, \mathbf{v}) = \\
&= (\mathbf{f}, \mathbf{v}) - (\overline{\mathcal{A}}\, \overline{\boldsymbol{\nabla} u_h}, \overline{\boldsymbol{\nabla} v}) - (C\mathbf{u}_h, \mathbf{v}) + (\mathbf{div}\, \overline{\mathbf{y}}, \mathbf{v}) + (\overline{\mathbf{y}}, \overline{\boldsymbol{\nabla} v}) \\
&= \left(K^{-T} \left[\mathbf{f} - C\mathbf{u}_h + \mathbf{div}\, \overline{\mathbf{y}} \right], K\mathbf{v} \right) + \left(\overline{\mathcal{A}}^{-\frac{1}{2}} \left[\overline{\mathbf{y}} - \overline{\mathcal{A}}\, \overline{\boldsymbol{\nabla} u_h} \right], \overline{\mathcal{A}}^{\frac{1}{2}} \overline{\boldsymbol{\nabla} v} \right) \\
&\leq \left\| K^{-T} \left[\mathbf{f} - C\mathbf{u}_h + \mathbf{div}\, \overline{\mathbf{y}} \right] \right\|_0 \| K\mathbf{v} \|_0 + \left\| \overline{\mathcal{A}}^{-\frac{1}{2}} \left[\overline{\mathbf{y}} - \overline{\mathcal{A}}\, \overline{\boldsymbol{\nabla} u_h} \right] \right\|_0 \left\| \overline{\mathcal{A}}^{\frac{1}{2}} \overline{\boldsymbol{\nabla} v} \right\|_0 \\
&\leq \underbrace{\left(\| \mathbf{f} - C\mathbf{u}_h + \mathbf{div}\, \overline{\mathbf{y}} \|^2_{C^{-1}} + \| \overline{\mathbf{y}} - \overline{\mathcal{A}}\, \overline{\boldsymbol{\nabla} u_h} \|^2_{\overline{\mathcal{A}}^{-1}} \right)^{\frac{1}{2}}}_{\eta(\mathbf{u}_h, \overline{\mathbf{y}})} |\!|\!| \mathbf{v} |\!|\!|
\end{aligned}$$

Derivation of the estimate

Divergence theorem: $v \in H^1(\Omega) \quad \mathbf{y} \in \mathbf{H}(\mathrm{div}, \Omega)$

$$\int_\Omega v \,\mathrm{div}\,\mathbf{y}\,\mathrm{d}x + \int_\Omega \mathbf{y} \cdot \boldsymbol{\nabla} v \,\mathrm{d}x - \int_{\partial\Omega} v\mathbf{y} \cdot \mathbf{n}\,\mathrm{d}x = 0$$

$$\mathcal{B}(\mathbf{u} - \mathbf{u}_h, \mathbf{v}) =$$

$$= (\mathbf{f}, \mathbf{v}) - (\overline{\mathcal{A}}\,\overline{\boldsymbol{\nabla} u_h}, \overline{\boldsymbol{\nabla} v}) - (C\mathbf{u}_h, \mathbf{v}) + (\mathbf{div}\,\overline{\mathbf{y}}, \mathbf{v}) + (\overline{\mathbf{y}}, \overline{\boldsymbol{\nabla} v})$$

$$= \left(K^{-T}\left[\mathbf{f} - C\mathbf{u}_h + \mathbf{div}\,\overline{\mathbf{y}}\right], K\mathbf{v}\right) + \left(\overline{\mathcal{A}}^{-\frac{1}{2}}\left[\overline{\mathbf{y}} - \overline{\mathcal{A}}\,\overline{\boldsymbol{\nabla} u_h}\right], \overline{\mathcal{A}}^{\frac{1}{2}}\overline{\boldsymbol{\nabla} v}\right)$$

$$\le \left\| K^{-T}\left[\mathbf{f} - C\mathbf{u}_h + \mathbf{div}\,\overline{\mathbf{y}}\right]\right\|_0 \|K\mathbf{v}\|_0 + \left\|\overline{\mathcal{A}}^{-\frac{1}{2}}\left[\overline{\mathbf{y}} - \overline{\mathcal{A}}\,\overline{\boldsymbol{\nabla} u_h}\right]\right\|_0 \left\|\overline{\mathcal{A}}^{\frac{1}{2}}\overline{\boldsymbol{\nabla} v}\right\|_0$$

$$\le \underbrace{\left(\|\mathbf{f} - C\mathbf{u}_h + \mathbf{div}\,\overline{\mathbf{y}}\|^2_{C^{-1}} + \|\overline{\mathbf{y}} - \overline{\mathcal{A}}\,\overline{\boldsymbol{\nabla} u_h}\|^2_{\overline{\mathcal{A}}^{-1}}\right)^{\frac{1}{2}}}_{\eta(\mathbf{u}_h, \overline{\mathbf{y}})} \|\!|\mathbf{v}|\!\|$$

$$\|\!|\mathbf{u} - \mathbf{u}_h|\!\| \le \eta(\mathbf{u}_h, \overline{\mathbf{y}}) \qquad \forall \mathbf{u}_h \in \mathbf{V} \quad \forall \overline{\mathbf{y}} \in \mathbf{W} = [\mathbf{H}(\mathrm{div}, \Omega)]^N$$

The estimator

Definition: $\eta^2(\mathbf{u}_h, \overline{\mathbf{y}}_h) = \|\mathbf{f} - C\mathbf{u}_h + \mathbf{div}\,\overline{\mathbf{y}}\|^2_{C^{-1}} + \left\|\overline{\mathbf{y}} - \overline{\mathcal{A}}\,\overline{\boldsymbol{\nabla} u_h}\right\|^2_{\overline{\mathcal{A}}^{-1}}$

Theorem: $\|\mathbf{u} - \mathbf{u}_h\| \leq \eta(\mathbf{u}_h, \overline{\mathbf{y}}_h) \qquad \forall \mathbf{u}_h \in \mathbf{V} \quad \forall \overline{\mathbf{y}}_h \in \mathbf{W}$

Complementary problem:

(A) Find $\overline{\mathbf{y}} \in \mathbf{W} : \eta(\mathbf{u}_h, \overline{\mathbf{y}}) = \min_{\overline{\mathbf{w}} \in \mathbf{W}} \eta(\mathbf{u}_h, \overline{\mathbf{w}})$

(B) Find $\overline{\mathbf{y}} \in \mathbf{W} : \mathcal{B}^*(\overline{\mathbf{y}}, \overline{\mathbf{w}}) = \mathcal{F}^*(\overline{\mathbf{w}}) \quad \forall \overline{\mathbf{w}} \in \mathbf{W}$

- $\mathbf{W} = [\mathbf{H}(\mathrm{div}, \Omega)]^N$
- $\mathcal{B}^*(\overline{\mathbf{y}}, \overline{\mathbf{w}}) = \left(C^{-1}\,\mathbf{div}\,\overline{\mathbf{y}}, \mathbf{div}\,\overline{\mathbf{w}}\right) + \left(\overline{\mathcal{A}}^{-1}\overline{\mathbf{y}}, \overline{\mathbf{w}}\right)$
- $\mathcal{F}^*(\overline{\mathbf{w}}) = -\left(C^{-1}\mathbf{f}, \mathbf{div}\,\overline{\mathbf{w}}\right)$
- $\|\overline{\mathbf{w}}\|^2_* = \mathcal{B}^*(\overline{\mathbf{w}}, \overline{\mathbf{w}})$

The estimator

Definition: $\eta^2(\mathbf{u}_h, \overline{\mathbf{y}}_h) = \|\mathbf{f} - C\mathbf{u}_h + \mathbf{div}\,\overline{\mathbf{y}}\|^2_{C^{-1}} + \left\|\overline{\mathbf{y}} - \overline{\mathcal{A}}\,\overline{\boldsymbol{\nabla} u_h}\right\|^2_{\overline{\mathcal{A}}^{-1}}$

Theorem: $\|\mathbf{u} - \mathbf{u}_h\| \leq \eta(\mathbf{u}_h, \overline{\mathbf{y}}_h) \qquad \forall \mathbf{u}_h \in \mathbf{V} \quad \forall \overline{\mathbf{y}}_h \in \mathbf{W}$

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(A) Find $\overline{\mathbf{y}} \in \mathbf{W} : \eta(\mathbf{u}_h, \overline{\mathbf{y}}) = \min_{\overline{\mathbf{w}} \in \mathbf{W}} \eta(\mathbf{u}_h, \overline{\mathbf{w}})$

(B) Find $\overline{\mathbf{y}} \in \mathbf{W} : \mathcal{B}^*(\overline{\mathbf{y}}, \overline{\mathbf{w}}) = \mathcal{F}^*(\overline{\mathbf{w}}) \quad \forall \overline{\mathbf{w}} \in \mathbf{W}$

Lemma 1: (A) $\Leftrightarrow$ (B)

Lemma 2: $\overline{\mathbf{y}} = \overline{\mathcal{A}}\,\overline{\boldsymbol{\nabla}\mathbf{u}} \in \mathbf{W}$ is the unique solution of (A)–(B)

Lemma 3: $\eta^2(\mathbf{u}_h, \overline{\mathbf{y}}) + \eta^2(\mathbf{u}, \overline{\mathbf{y}}_h) = \eta^2(\mathbf{u}_h, \overline{\mathbf{y}}_h) \qquad \forall \mathbf{u}_h \in \mathbf{V},\ \overline{\mathbf{y}}_h \in \mathbf{W}$

$$\|\mathbf{u} - \mathbf{u}_h\|^2 + \|\overline{\mathbf{y}}_h - \overline{\mathbf{y}}\|^2_* = \eta^2(\mathbf{u}_h, \overline{\mathbf{y}}_h)$$

Method of hypercircle

Theorem: If

- $\mathbf{u} \in \mathbf{V}$ is primal solution
- $\mathbf{u}_h \in \mathbf{V}$, $\overline{\mathbf{y}}_h \in \mathbf{W}$ arbitrary
- $\widetilde{\mathbf{u}}_h = [C^{-1}(\mathbf{f} + \mathbf{div}\,\overline{\mathbf{y}}_h) + \mathbf{u}_h]/2$
- $\mathcal{G}\widetilde{\mathbf{u}}_h = (\overline{\mathbf{y}}_h + \overline{\mathcal{A}}\,\overline{\boldsymbol{\nabla} u_h})/2$

Then

$$\left\| \overline{\boldsymbol{\nabla} u} - \overline{\mathcal{A}}^{-1} \mathcal{G}\widetilde{\mathbf{u}}_h \right\|_{\overline{\mathcal{A}}}^2 + \|\mathbf{u} - \widetilde{\mathbf{u}}_h\|_C^2 = \frac{1}{4}\eta^2(\mathbf{u}_h, \overline{\mathbf{y}}_h).$$

Other variants (good for C singular)

(1) $\eta^2(\mathbf{u}_h, \overline{\mathbf{y}}_h) = \|\mathbf{f} - C\mathbf{u}_h + \mathbf{div}\,\overline{\mathbf{y}}_h\|^2_{C^{-1}} + \left\|\overline{\mathbf{y}}_h - \overline{\mathcal{A}}\,\overline{\boldsymbol{\nabla} u_h}\right\|^2_{\overline{\mathcal{A}}^{-1}}$

(2) $\widetilde{\eta}(\mathbf{u}_h, \widetilde{\mathbf{y}}_h) = \left\|\widetilde{\mathbf{y}}_h - \overline{\mathcal{A}}\,\overline{\boldsymbol{\nabla} u_h}\right\|_{\overline{\mathcal{A}}^{-1}} \qquad \widetilde{\mathbf{y}}_h \in \mathbf{Q}(\mathbf{f}, \mathbf{u_h})$

$$\mathbf{Q}(\mathbf{f}, \mathbf{u_h}) = \{\widetilde{\mathbf{y}} \in \mathbf{W} : \mathbf{f} - C\mathbf{u}_h + \mathbf{div}\,\widetilde{\mathbf{y}} = 0\}$$

(3) $\widehat{\eta}(\mathbf{u}_h, \widehat{\mathbf{y}}_h) = \dfrac{\gamma_F}{\lambda}\|\mathbf{f} - C\mathbf{u}_h + \mathbf{div}\,\widehat{\mathbf{y}}_h\|_0 + \left\|\widehat{\mathbf{y}}_h - \overline{\mathcal{A}}\,\overline{\boldsymbol{\nabla} u_h}\right\|_{\overline{\mathcal{A}}^{-1}}$

Friedrichs ineq.: $\|v\|_0 \le \gamma_F \|\boldsymbol{\nabla} v\|_0 \quad \forall v \in H^1_0(\Omega)$

Ellipticity: $(\overline{\mathcal{A}}\,\overline{\mathbf{w}}, \overline{\mathbf{w}}) \ge \lambda^2 \|\overline{\mathbf{w}}\|^2_0 \quad \forall \mathbf{w} \in \mathbf{W}$

Lemma: $|\!|\!|\mathbf{u} - \mathbf{u}_h|\!|\!| \le \min\{\eta(\mathbf{u}_h, \overline{\mathbf{y}}_h),\ \widetilde{\eta}(\mathbf{u}_h, \widetilde{\mathbf{y}}_h),\ \widehat{\eta}(\mathbf{u}_h, \widehat{\mathbf{y}}_h)\}$

$$\forall \mathbf{u}_h \in \mathbf{V} \quad \forall \overline{\mathbf{y}}_h \in \mathbf{W} \quad \forall \widetilde{\mathbf{y}}_h \in \mathbf{Q}(\mathbf{f}, \mathbf{u}_h) \quad \forall \widehat{\mathbf{y}}_h \in \mathbf{W}$$

Numerical example

Primal problem

$$
\begin{aligned}
-\left(\overline{\mathcal{A}}\mathbf{u}'\right)' + C\mathbf{u} &= \mathbf{f} \\
\mathbf{u}(0) = \mathbf{u}(1) &= \mathbf{0}
\end{aligned}
$$

$$
\overline{\mathcal{A}} = \begin{pmatrix} 4 & 2 & -2 \\ 2 & 2 & 0 \\ -2 & 0 & 6 \end{pmatrix} \quad C = \kappa^2 \begin{pmatrix} 4 & 1 & -1 \\ 1 & 2 & 1 \\ -1 & 1 & 4 \end{pmatrix} \quad \mathbf{f} = \begin{pmatrix} 4 \\ 4 \\ 4 \end{pmatrix}
$$

$$
\mathbf{u} = \begin{pmatrix} u^1 \\ u^2 \\ u^3 \end{pmatrix} = \frac{1}{\kappa^2}\left(1 - \frac{\sinh \kappa(1-x) + \sinh \kappa x}{\sinh \kappa}\right)\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}
$$

Numerical example

Primal problem

$$-\left(\overline{\mathcal{A}}\mathbf{u}'\right)' + C\mathbf{u} = \mathbf{f}$$
$$\mathbf{u}(0) = \mathbf{u}(1) = \mathbf{0}$$

Complementary problem

$$-\left(C^{-1}\mathbf{y}'\right)' + \overline{\mathcal{A}}^{-1}\mathbf{y} = \left(C^{-1}\mathbf{f}\right)' \quad \text{in } (0,1)$$
$$\mathbf{y}'(0) = -\mathbf{f}(0) \quad \mathbf{y}'(1) = -\mathbf{f}(1)$$

$$\overline{\mathcal{A}} = \begin{pmatrix} 4 & 2 & -2 \\ 2 & 2 & 0 \\ -2 & 0 & 6 \end{pmatrix} \quad C = \kappa^2 \begin{pmatrix} 4 & 1 & -1 \\ 1 & 2 & 1 \\ -1 & 1 & 4 \end{pmatrix} \quad \mathbf{f} = \begin{pmatrix} 4 \\ 4 \\ 4 \end{pmatrix}$$

$$\mathbf{u} = \begin{pmatrix} u^1 \\ u^2 \\ u^3 \end{pmatrix} = \frac{1}{\kappa^2}\left(1 - \frac{\sinh \kappa(1-x) + \sinh \kappa x}{\sinh \kappa}\right)\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\mathbf{y} = \overline{\mathcal{A}}\mathbf{u}'$$

Numerical example

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$$\mathbf{u}(0) = \mathbf{u}(1) = \mathbf{0}$$

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$$-\left(C^{-1}\mathbf{y}'\right)' + \overline{\mathcal{A}}^{-1}\mathbf{y} = \left(C^{-1}\mathbf{f}\right)' \quad \text{in } (0,1)$$
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FEM mesh

$$0 = x_0 < x_1 < \cdots < x_M = 1 \qquad K_i = [x_{i-1}, x_i], \quad i = 1, 2, \ldots, M$$

Primal FEM

$$\mathbf{V}_h = \{v_h \in H_0^1(0,1) : v_h|_{K_i} \in P^1(K_i)\ \forall K_i\}^3$$

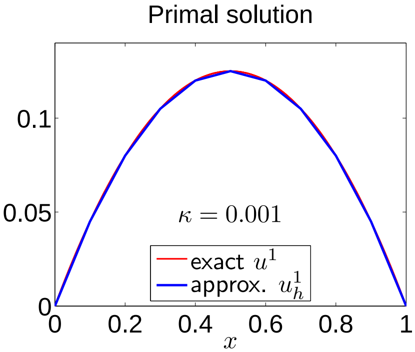
$$\mathbf{u}_h \in \mathbf{V}_h : \quad (\overline{\mathcal{A}}\mathbf{u}_h', \mathbf{v}_h') + (C\mathbf{u}_h, \mathbf{v}_h) = (\mathbf{f}, \mathbf{v}_h) \quad \forall \mathbf{v}_h \in \mathbf{V}_h$$

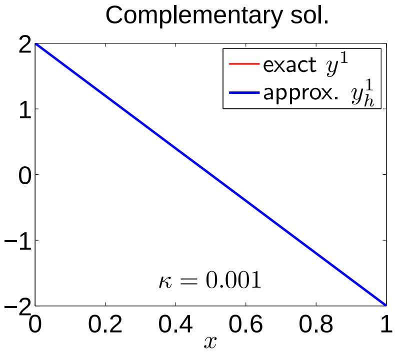
Complementary FEM

$$\mathbf{W}_h = \{w_h \in H^1(0,1) : w_h|_{K_i} \in P^1(K_i)\ \forall K_i\}^3$$

$$\mathbf{y}_h \in \mathbf{W}_h : \quad (C^{-1}\mathbf{y}_h', \mathbf{w}_h') + (\overline{\mathcal{A}}^{-1}\mathbf{y}_h, \mathbf{w}_h) = -(C^{-1}\mathbf{f}, \mathbf{w}_h') \quad \forall \mathbf{w}_h \in \mathbf{W}_h$$

Primal and complementary solutions

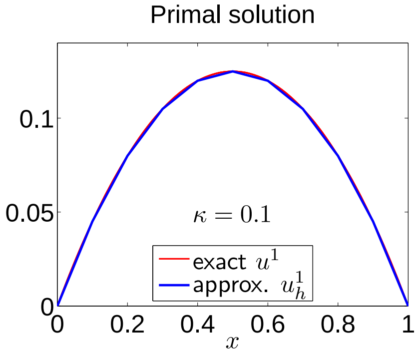


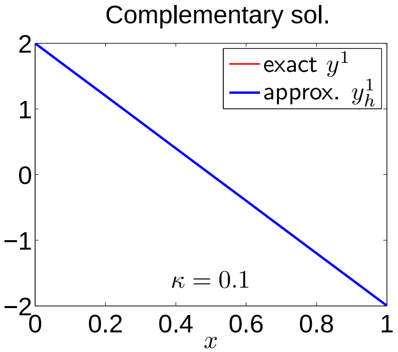


$$\frac{\|\mathbf{u}-\mathbf{u}_h\|}{\|\mathbf{u}_h\|} \leq \frac{\eta}{\|\mathbf{u}_h\|} = 9.486\,\%$$

$$\frac{\|\mathbf{u}-\mathbf{u}_h\|}{\|\mathbf{u}\|} = 9.440\,\% \qquad \frac{\eta}{\|\mathbf{u}\|} = 9.440\,\% \qquad I_{eff} = 1.000$$

Primal and complementary solutions

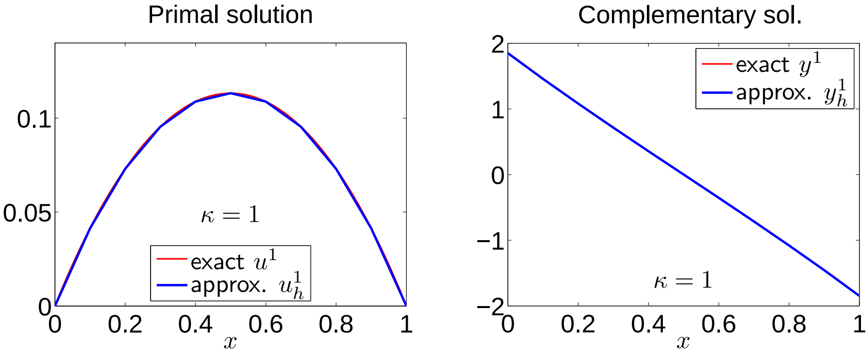




$$\frac{\|\mathbf{u}-\mathbf{u}_h\|}{\|\mathbf{u}_h\|} \leq \frac{\eta}{\|\mathbf{u}_h\|} = 9.486\,\%$$

$$\frac{\|\mathbf{u}-\mathbf{u}_h\|}{\|\mathbf{u}\|} = 9.436\,\% \qquad \frac{\eta}{\|\mathbf{u}\|} = 9.439\,\% \qquad I_{eff} = 1.0003$$

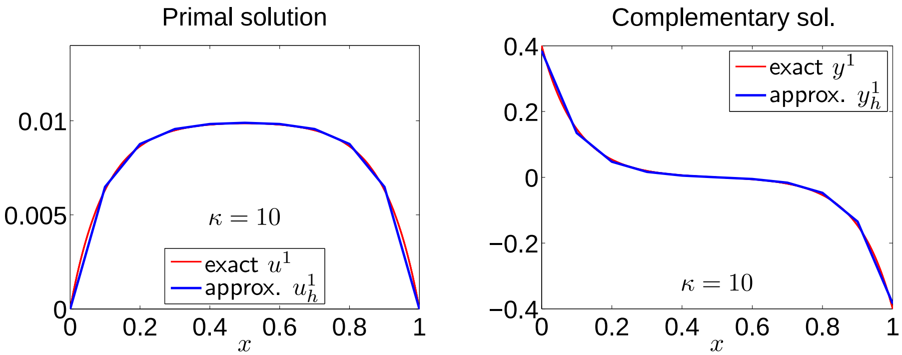
Primal and complementary solutions



$$\frac{\|\mathbf{u} - \mathbf{u}_h\|}{\|\mathbf{u}_h\|} \leq \frac{\eta}{\|\mathbf{u}_h\|} = 9.530\,\%$$

$$\frac{\|\mathbf{u} - \mathbf{u}_h\|}{\|\mathbf{u}\|} = 9.158\,\% \qquad \frac{\eta}{\|\mathbf{u}\|} = 9.487\,\% \qquad I_{eff} = 1.036$$

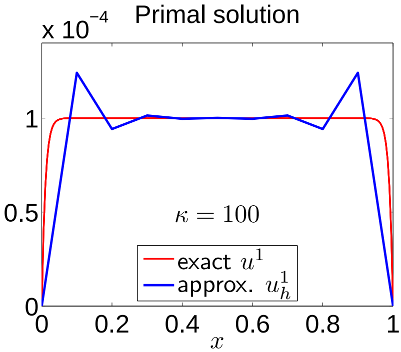
Primal and complementary solutions

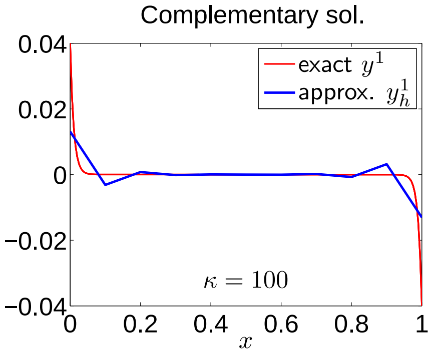


$$\frac{\|\mathbf{u} - \mathbf{u}_h\|}{\|\mathbf{u}_h\|} \le \frac{\eta}{\|\mathbf{u}_h\|} = 13.45\,\%$$

$$\frac{\|\mathbf{u} - \mathbf{u}_h\|}{\|\mathbf{u}\|} = 9.617\,\% \qquad \frac{\eta}{\|\mathbf{u}\|} = 13.39\,\% \qquad I_{eff} = 1.392$$

Primal and complementary solutions

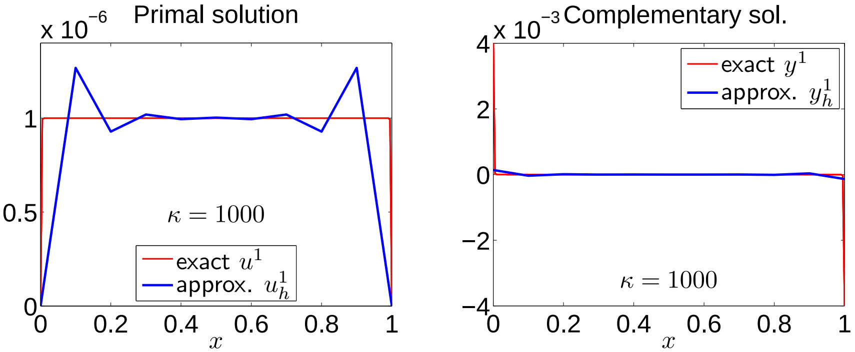




$$\frac{\|\mathbf{u}-\mathbf{u}_h\|}{\|\mathbf{u}_h\|} \leq \frac{\eta}{\|\mathbf{u}_h\|} = 22.26\,\%$$

$$\frac{\|\mathbf{u}-\mathbf{u}_h\|}{\|\mathbf{u}\|} = 15.48\,\% \qquad \frac{\eta}{\|\mathbf{u}\|} = 21.99\,\% \qquad I_{eff} = 1.420$$

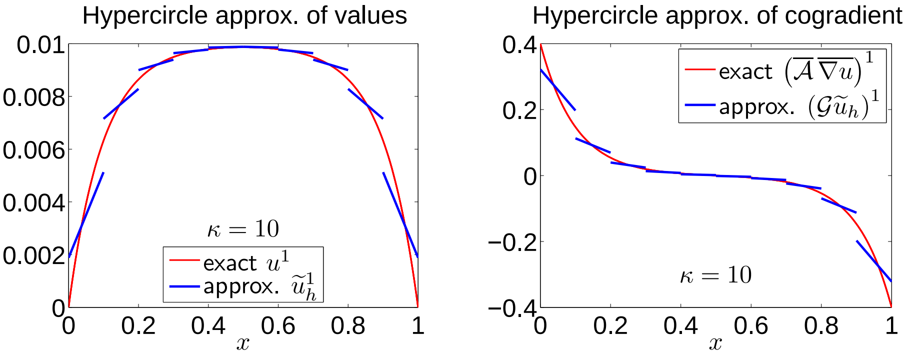
Primal and complementary solutions



$$\frac{\|\mathbf{u}-\mathbf{u}_h\|}{\|\mathbf{u}_h\|} \leq \frac{\eta}{\|\mathbf{u}_h\|} = 22.23\,\%$$

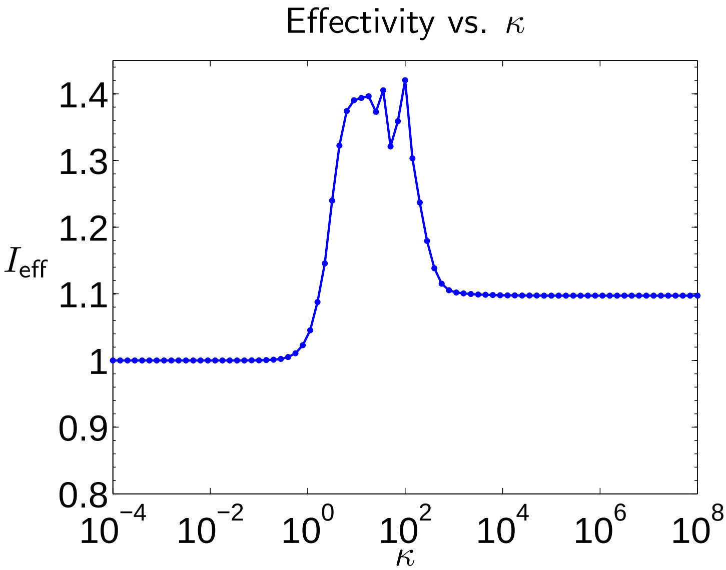
$$\frac{\|\mathbf{u}-\mathbf{u}_h\|}{\|\mathbf{u}\|} = 19.86\,\% \qquad \frac{\eta}{\|\mathbf{u}\|} = 21.90\,\% \qquad I_{eff} = 1.103$$

Method of hypercircle



$$\frac{\|[\mathbf{u}-\widetilde{\mathbf{u}}_h, \overline{\boldsymbol{\nabla}} u - \mathcal{G}\widetilde{\mathbf{u}}_h]\|_{hc}}{\|\mathbf{u}_h\|} = \frac{1}{2}\frac{\eta}{\|\mathbf{u}_h\|} = 6.72\,\%$$

Effectivity index



Conclusions

$$\|\mathbf{u} - \mathbf{u}_h\| \leq \eta(\mathbf{u}_h, \overline{\mathbf{y}}_h)$$

- Computable guaranteed upper bounds
- Optimal $\overline{\mathbf{y}}$ solves a complementary problem
- Postprocessing of $\overline{\mathcal{A}}\,\overline{\boldsymbol{\nabla} u_h}$
  $\Rightarrow$ fast algorithms for $\overline{\mathbf{y}}_h$ (many open problems)
- $\mathbf{u}_h \in \mathbf{V}$ arbitrary
  $\Rightarrow$ including algebraic errors, human errors
- Exist generalizations to other classes of problems

Thank you for your attention

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