Local error indicators and guaranteed upper bounds

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Motivation – adaptive algorithm

1. Initialize: Construct the initial mesh $\mathcal{T}_h$.
2. Solve: Find u_h on $\mathcal{T}_h$.
3. Error indicators: Compute η_K for all $K \in \mathcal{T}_h$.
4. Error estimator: $\eta^2 = \sum_{K \in \mathcal{T}_h} \eta_K^2$.
5. Stopping criterion: If $\quad \eta \leq \text{TOL} \quad \Rightarrow \quad$ STOP.
6. Mark: If $\quad \eta_K \geq \Theta \max_{K \in \mathcal{T}_h} \eta_K \quad \Rightarrow \quad$ mark K. $\qquad 0 < \Theta < 1$
7. Refine: Refine marked elements and build the new mesh $\mathcal{T}_h$.
8. GO TO 2.

Remark: Guaranteed upper bound: $\|u - u_h\| \leq \eta \leq \text{TOL}$

Primal Problem

Strong form.:

$$-\Delta u = f \quad \text{in } \Omega \subset \mathbb{R}^d$$
$$u = 0 \quad \text{on } \partial\Omega$$

Weak form.: $u \in V: \quad \mathcal{B}(u,v) = \mathcal{F}(v) \quad \forall v \in V$

Notation:

- $V = H_0^1(\Omega)$
- $\mathcal{B}(u,v) = (\nabla u, \nabla v) \qquad (\mathbf{p},\mathbf{q}) = \int_\Omega \mathbf{p}\cdot\mathbf{q}\,\mathrm{d}x$
- $\mathcal{F}(v) = (f,v) \qquad (f,v) = \int_\Omega fv\,\mathrm{d}x$
- $\|v\|^2 = \mathcal{B}(v,v)$

Derivation of the estimate

Divergence theorem: $v \in H^1(\Omega) \quad \mathbf{y} \in \mathbf{H}(\mathrm{div}, \Omega)$

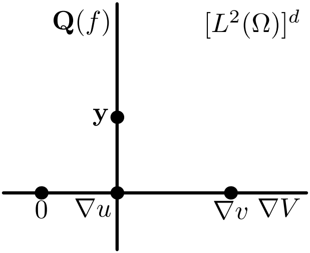
$$\int_\Omega v \,\mathrm{div}\, \mathbf{y} \,\mathrm{d}x + \int_\Omega \mathbf{y} \cdot \nabla v \,\mathrm{d}x - \int_{\partial\Omega} v \mathbf{y} \cdot \mathbf{n} \,\mathrm{d}x = 0$$

$$\mathbf{Q}(f) = \{\mathbf{y} \in \mathbf{H}(\mathrm{div}, \Omega) : (\mathbf{y}, \nabla v) = (f, v) \quad \forall v \in V\}$$

Orthogonality:

$$\int_\Omega (\nabla u - \mathbf{y}) \cdot \nabla v \,\mathrm{d}x = 0$$

$$\forall v \in V, \ \forall \mathbf{y} \in \mathbf{Q}(f)$$



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$$v \in V, \quad \mathbf{y} \in \mathbf{Q}(f) = \{\mathbf{y} \in \mathbf{H}(\mathrm{div}, \Omega) : (\mathbf{y}, \boldsymbol{\nabla} v) = (f, v) \quad \forall v \in V\}$$

$$\begin{aligned}
\mathcal{B}(u - u_h, v) &= (f, v) - (\boldsymbol{\nabla} u_h, \boldsymbol{\nabla} v) + (v, \mathrm{div}\, \mathbf{y}) + (\mathbf{y}, \boldsymbol{\nabla} v) \\
&= (f + \mathrm{div}\, \mathbf{y}, v) + (\mathbf{y} - \boldsymbol{\nabla} u_h, \boldsymbol{\nabla} v) \\
&= (\mathbf{y} - \boldsymbol{\nabla} u_h, \boldsymbol{\nabla} v) \\
&\leq \|\mathbf{y} - \boldsymbol{\nabla} u_h\|_0 \, \|\boldsymbol{\nabla} v\|_0
\end{aligned}$$

$$|\!|\!| u - u_h |\!|\!| \leq \|\mathbf{y} - \boldsymbol{\nabla} u_h\|_0 \qquad \forall u_h \in V$$

The estimator

Definition: $\eta(u_h, \mathbf{y}_h) = \|\mathbf{y}_h - \boldsymbol{\nabla} u_h\|_0$

Theorem: $|\!|\!| u - u_h |\!|\!| \leq \eta(u_h, \mathbf{y}_h) \qquad \forall u_h \in V \quad \forall \mathbf{y}_h \in \mathbf{Q}(f)$

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Complementary problem:

(A) Find $\mathbf{y} \in \mathbf{Q}(f) : \eta(u_h, \mathbf{y}) \leq \eta(u_h, \mathbf{w}) \quad \forall \mathbf{w} \in \mathbf{Q}(f)$

(B) Find $\mathbf{y} \in \mathbf{Q}(f) : \frac{1}{2}\|\mathbf{y}\|_0^2 \leq \frac{1}{2}\|\mathbf{w}\|_0^2 \quad \forall \mathbf{w} \in \mathbf{Q}(f)$

(C) Find $\mathbf{y} \in \mathbf{Q}(f) : (\mathbf{y}, \mathbf{w}^0) = 0 \quad \forall \mathbf{w}^0 \in \mathbf{Q}(0)$

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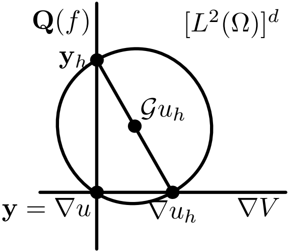
Lemma 1: (A) $\Leftrightarrow$ (B) $\Leftrightarrow$ (C)

Lemma 2: $\mathbf{y} = \nabla u \in \mathbf{Q}(f)$ is the unique solution of (A)–(C)

Lemma 3: $\eta^2(u, \mathbf{y}_h) + \eta^2(u_h, \mathbf{y}) = \eta^2(u_h, \mathbf{y}_h) \qquad \forall u_h \in V,\ \mathbf{y}_h \in \mathbf{Q}(f)$

$$\|\mathbf{y}_h - \mathbf{y}\|_0^2 + \|\nabla u - \nabla u_h\|_0^2 = \|\mathbf{y}_h - \nabla u_h\|_0^2$$

Method of hypercircle



Theorem: If

- $u \in V$ is primal solution
- $u_h \in V$, $\mathbf{y}_h \in \mathbf{Q}(f)$ arbitrary
- $\mathcal{G}u_h = (\mathbf{y}_h + \nabla u_h)/2$

Then

$$\|\nabla u - \mathcal{G}u_h\|_0 = \frac{1}{2}\eta(u_h, \mathbf{y}_h).$$

Proof:

$$\begin{aligned} 4\,\|\nabla u - \mathcal{G}u_h\|_0^2 &= \|\nabla u - \mathbf{y}_h + \nabla u - \nabla u_h\|_0^2 \\ &= \|\nabla u - \mathbf{y}_h\|_0^2 + \|\nabla u - \nabla u_h\|_0^2 = \|\nabla u_h - \mathbf{y}_h\|_0^2 \end{aligned}$$

Handelling $\mathbf{Q}(f)$, $d = 2$, Ω simply connected

$$\overline{\mathbf{q}}(x_1, x_2) = -\left(\int_0^{x_1} f(s, x_2)\,\mathrm{d}s, 0\right)^T \quad \Rightarrow \quad -\operatorname{div}\overline{\mathbf{q}} = f$$

$$\mathbf{Q}(0) = \mathbf{curl}\, H^1(\Omega), \qquad \mathbf{curl} = (\partial_2, -\partial_1)^\top$$

$$\mathbf{Q}(f) = \overline{\mathbf{q}} + \mathbf{Q}(0) = \overline{\mathbf{q}} + \mathbf{curl}\, H^1(\Omega)$$

(Comp) $\quad \mathbf{y} = \overline{\mathbf{q}} + \mathbf{curl}\, z \in \mathbf{Q}(f) : (\mathbf{y}, \mathbf{w}^0) = 0 \quad \forall \mathbf{w}^0 \in \mathbf{Q}(0)$

(Comp) $\quad z \in H^1(\Omega) : \underbrace{(\mathbf{curl}\, z, \mathbf{curl}\, v)}_{(\nabla z, \nabla v)} = -(\overline{\mathbf{q}}, \mathbf{curl}\, v) \quad \forall v \in H^1(\Omega)$

Numerical example

Primal problem

$$-u'' + \kappa^2 u = f$$
$$u(0) = u(1) = 0$$

Complementary problem

$$-y'' + \kappa^2 y = f' \quad \text{in } (0,1)$$
$$y'(0) = -f(0) \quad y'(1) = -f(1)$$

$$f = 1, \qquad \kappa = \text{const.}$$

$$u = \frac{1}{\kappa^2}\left(1 - \frac{\sinh \kappa(1-x) + \sinh \kappa x}{\sinh \kappa}\right), \qquad y = u'$$

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FEM mesh

$$0 = x_0 < x_1 < \cdots < x_M = 1 \qquad K_i = [x_{i-1}, x_i], \quad i = 1, 2, \ldots, M$$

Primal FEM

$$V_h = \{v_h \in H_0^1(0,1) : v_h|_{K_i} \in P^1(K_i)\ \forall K_i\}$$

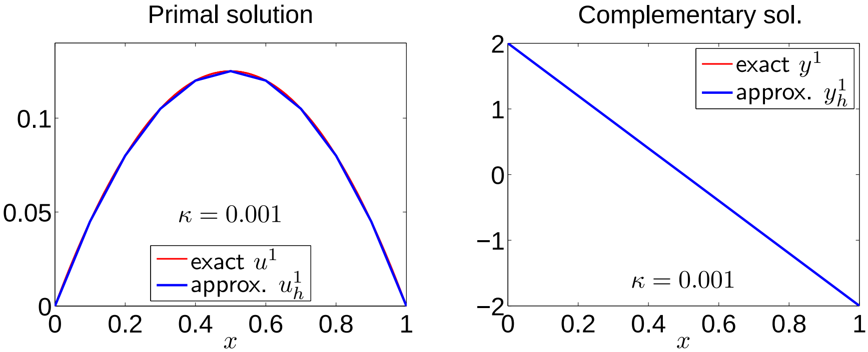
$$u_h \in V_h : \quad (u_h', v_h') + (\kappa^2 u_h, v_h) = (f, v_h) \quad \forall v_h \in V_h$$

Complementary FEM

$$W_h = \{w_h \in H^1(0,1) : w_h|_{K_i} \in P^1(K_i)\ \forall K_i\}$$

$$y_h \in W_h : \quad (y_h', w_h') + (\kappa^2 y_h, w_h) = -(f, w_h') \quad \forall w_h \in W_h$$

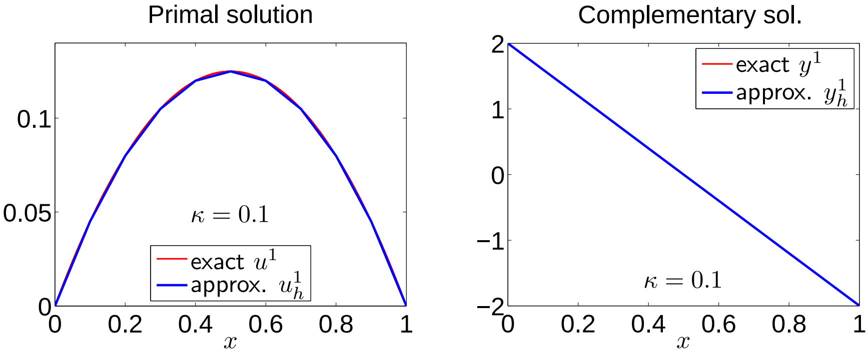
Primal and complementary solutions



$$\frac{\|u-u_h\|}{\|u_h\|} \le \frac{\eta}{\|u_h\|} = 9.486\,\%$$

$$\frac{\|u-u_h\|}{\|u\|} = 9.440\,\% \qquad \frac{\eta}{\|u\|} = 9.440\,\% \qquad I_{eff} = 1.000$$

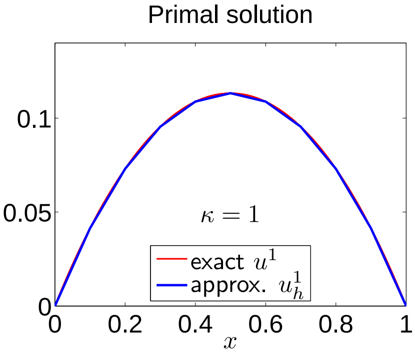
Primal and complementary solutions

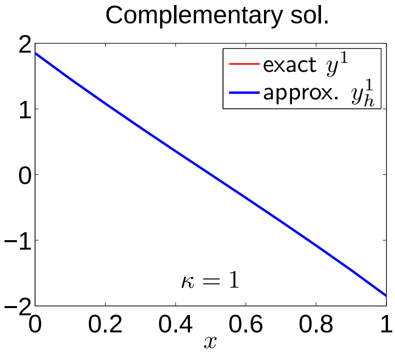


$$\frac{|||u - u_h|||}{|||u_h|||} \leq \frac{\eta}{|||u_h|||} = 9.486\,\%$$

$$\frac{|||u - u_h|||}{|||u|||} = 9.436\,\% \qquad \frac{\eta}{|||u|||} = 9.439\,\% \qquad I_{eff} = 1.0003$$

Primal and complementary solutions

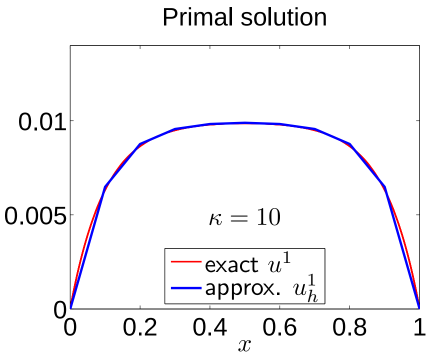


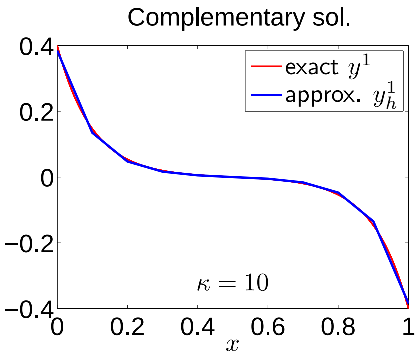


$$\frac{\|u - u_h\|}{\|u_h\|} \leq \frac{\eta}{\|u_h\|} = 9.530\,\%$$

$$\frac{\|u - u_h\|}{\|u\|} = 9.158\,\% \qquad \frac{\eta}{\|u\|} = 9.487\,\% \qquad I_{eff} = 1.036$$

Primal and complementary solutions

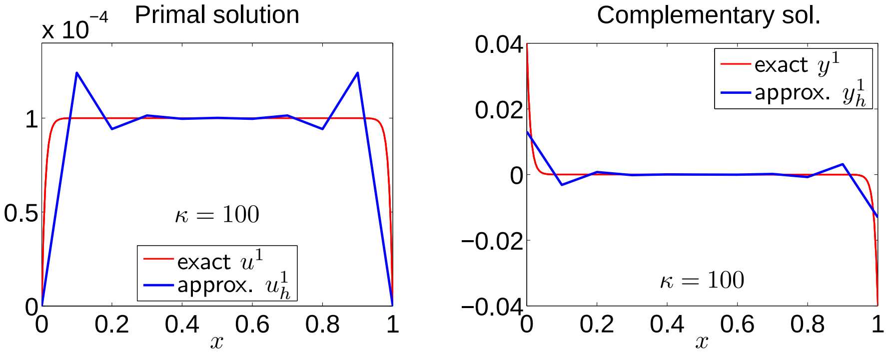




$$\frac{\|u - u_h\|}{\|u_h\|} \leq \frac{\eta}{\|u_h\|} = 13.45\,\%$$

$$\frac{\|u - u_h\|}{\|u\|} = 9.617\,\% \qquad \frac{\eta}{\|u\|} = 13.39\,\% \qquad I_{eff} = 1.392$$

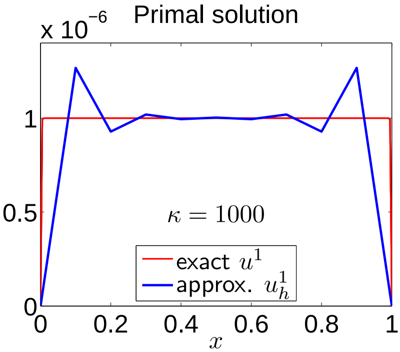
Primal and complementary solutions

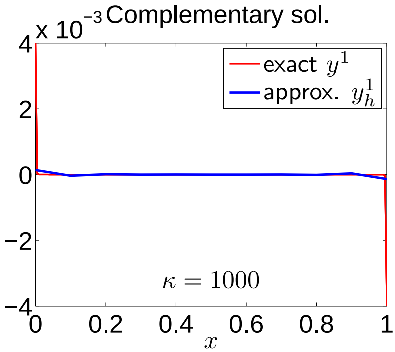


$$\frac{|||u - u_h|||}{|||u_h|||} \leq \frac{\eta}{|||u_h|||} = 22.26\,\%$$

$$\frac{|||u - u_h|||}{|||u|||} = 15.48\,\% \qquad \frac{\eta}{|||u|||} = 21.99\,\% \qquad I_{eff} = 1.420$$

Primal and complementary solutions

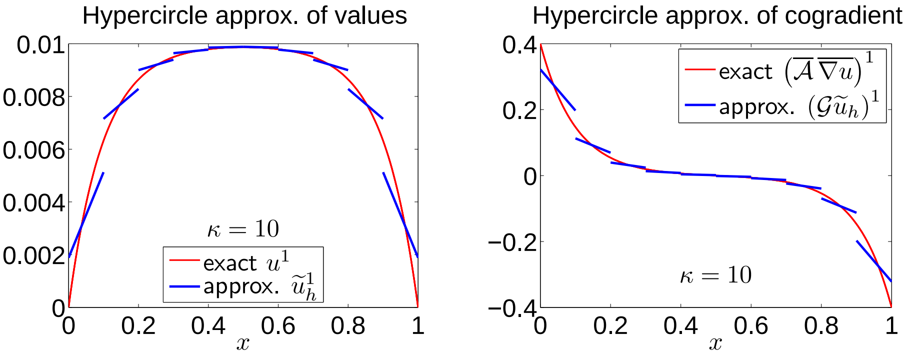




$$\frac{\| u - u_h \|}{\| u_h \|} \leq \frac{\eta}{\| u_h \|} = 22.23\,\%$$

$$\frac{\| u - u_h \|}{\| u \|} = 19.86\,\% \qquad \frac{\eta}{\| u \|} = 21.90\,\% \qquad I_{eff} = 1.103$$

Method of hypercircle



$$\frac{\|[u - \widetilde{u}_h, \overline{\nabla} u - \mathcal{G}\widetilde{u}_h]\|_{hc}}{\|u_h\|} = \frac{1}{2}\frac{\eta}{\|u_h\|} = 6.72\,\%$$

Generalizations

- General elliptic problem (nonsymmetric)
- Elasticity
- Stokes problem (incompressible viscous fluids)
- Variational inequalities
- Nonlinear problems (special)
- Differential equations of higher order
- Equations with curl
- Linear evolutionary problems
- Optimal control

Conclusions

$$|||u - u_h||| \leq \eta(u_h, \mathbf{y}_h)$$

- Guaranteed upper bounds
- Optimal $\mathbf{y}$ solves a complementary problem
- Postprocessing of ∇u_h
  $\Rightarrow$ fast algorithms for $\mathbf{y}_h$ (many open problems)
- $u_h \in V$ arbitrary
  $\Rightarrow$ including algebraic errors, quadrature errors, human errors
  $\Rightarrow$ any conforming numerical method
  including Splines and IsoGeometric Analysis

Thank you for your attention

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