Surprising spectra of $\mathcal{PT}$-symmetric point interactions

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$\mathcal{PT}$-symmetric point interactions

Why point interactions?

- solvable models with both continuous and point spectra
- explicit formulas for metric operators

  2005 Albeverio, Kuzhel, 2006 Krejčiřík, Bíla, Znojil, 2008 Siegl
- resolvent criterion for similarity to the self-adjoint operator is applicable in a straightforward way

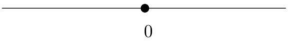
  2005 Albeverio, Kuzhel

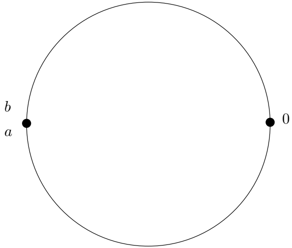
  real spectrum is not sufficient for similarity to the self-adjoint operator (quasi-Hermiticity)

$\mathcal{PT}$-symmetric point interactions

Definitions of operators

- line $L^2(\mathbb{R})$ or finite interval (circle) $L^2(a,b)$
- $H = -\frac{\mathrm{d}^2}{\mathrm{d}x^2}$
- $\mathrm{Dom}(H) = AC^1$ + boundary conditions at $x=0$ or
  at $x=a,b$





$\mathcal{PT}$-symmetric point interactions

$\mathcal{PT}$-symmetric boundary conditions 2002 Albeverio, Fei, Kurasov

$$\begin{pmatrix} \psi(0+) \\ \psi'(0+) \end{pmatrix} = B \begin{pmatrix} \psi(0-) \\ \psi'(0-) \end{pmatrix}$$

$$B = \begin{pmatrix} \sqrt{1+bc}\,e^{\mathrm{i}\phi} & b \\ c & \sqrt{1+bc}\,e^{-\mathrm{i}\phi} \end{pmatrix}, \quad \begin{array}{l} b \geq 0, c \geq -1/b \\ \phi \in (-\pi, \pi] \end{array}$$

System on a line - interaction at $x = 0$

Symmetries

- $\mathcal{PT}$-symmetry: $\mathcal{PT}H\psi = H\mathcal{PT}\psi, \forall\psi \in \mathrm{Dom}(H)$
- $\mathcal{P}$-pseudo-Hermiticity: $H^* = \mathcal{P}H\mathcal{P}$
- $\mathcal{T}$-self-adjointness: $H^* = \mathcal{T}H\mathcal{T}$
- $\mathcal{T}$-complex conjugation, $\mathcal{P}$-parity

Spectrum

- residual part is empty $\sigma_r(H) = \emptyset$ 2008 Borisov, Krejčiřík
- continuous spectrum $\sigma_c(H) = [0, \infty)$
- $b \neq 0, c \neq 0$ point spectrum - at most two eigenvalues real if $bc\sin^2\phi \leq \cos^2\phi$ or $bc\sin^2\phi \geq \cos^2$ and $\cos\phi \geq 0$

  2002 Albeverio, Fei, Kurasov

Special case $b = 0, c = 0$

Definition of operator

- $L^2(\mathbb{R})$
- $H_\phi = -\frac{\mathrm{d}^2}{\mathrm{d}x^2}$
- $\mathrm{Dom}(H_\phi) = AC^1(\mathbb{R})$
- $\psi(0+) = e^{\mathrm{i}\phi}\psi(0-)$
  $\psi'(0+) = e^{-\mathrm{i}\phi}\psi'(0-)$,

Symmetries

- $\mathcal{PT}H_\phi\psi = H_\phi\mathcal{PT}\psi$
- $H_\phi^* = H_{-\phi}$
- $H_\phi^* = \mathcal{P}H_\phi\mathcal{P}$
- $H_\phi^* = \mathcal{T}H_\phi\mathcal{T}$
- H_ϕ is closed

Special case $b = 0, c = 0$

Boundary conditions

$\psi(0+) = e^{i\phi}\psi(0-)$

$\psi'(0+) = e^{-i\phi}\psi'(0-)$

Special cases

- $\phi = 0$ - self-adjoint operator, no interaction
  $\psi(0+) = \psi(0-)$, $\psi'(0+) = \psi'(0-)$
- $\phi \neq \pm\pi/2$ - continuous spectrum $[0, \infty)$, no eigenvalues, quasi-Hermitian
- $\phi = \pm\pi/2$ - SURPRISING CASE
  $\psi(0+) = \pm i\psi(0-)$, $\psi'(0+) = \mp i\psi'(0-)$

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Special case $b = 0, c = 0$, $\phi \neq \pm\pi/2$

Quasi-Hermiticity

- H_ϕ is quasi-Hermitian:
  $\Theta H_\phi^* = H_\phi \Theta, \ \Theta, \Theta^{-1} \in \mathscr{B}(\mathcal{H}), \ \Theta > 0$
- $\Theta = I - \mathrm{i}\sin\phi P_{\mathrm{sign}}\mathcal{P}$
  $(P_{\mathrm{sign}} f)(x) = \mathrm{sign} x f(x)$, $\mathcal{P}$-parity

Metric operator Θ

- spectrum - only two eigenvalues $1 - \sin\phi$, $1 + \sin\phi$
- $\Theta > 0$, $\Theta^{-1} \in \mathscr{B}(\mathcal{H})$
- $\Theta H_\phi^* = H_\phi \Theta$ is valid
- Θ is not invertible if $\phi = \pm\pi/2$!

Special case $b = 0, c = 0, \phi \neq \pm\pi/2$

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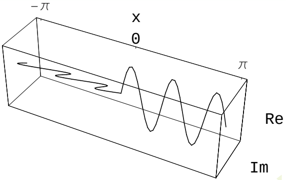
Construction of Θ 2008 Siegl

- finite interval $(-l, l)$ - interaction at $x = 0$, Dirichlet BC at $\pm l$
- discrete spectrum $\lambda_n = \left(\frac{n\pi}{2l}\right)^2,\ n \in \mathbb{N}_0$
- eigenfunctions $$\begin{aligned} \psi_{2n}(x) &= \left(e^{-i\phi}\vartheta(x) + \vartheta(-x)\right)\sin\frac{n\pi}{l}x \\ \psi_{2n+1}(x) &= \left(e^{i\phi}\vartheta(x) - \vartheta(-x)\right)\cos\frac{(2n+1)\pi}{2l}x \end{aligned}$$
- $\Theta = \text{s–}\lim_{N\to\infty}\sum_{j=1}^{N} c_j \langle\phi_j, \cdot\rangle\phi_j = I - i\sin\phi P_{\text{sign}}\mathcal{P}$
- $\phi_n = \mathcal{P}\psi_n$ eigenfunctions of H^*
- limit $l \to \infty$



Surprising case $b = 0, c = 0, \phi = \pi/2$

Properties of $H_{\pi/2}$

- $\psi(0+) = \mathrm{i}\psi(0-)$, $\psi'(0+) = -\mathrm{i}\psi'(0-)$
- $H_{\pi/2}$ is $\mathcal{PT}$-symmetric, $\mathcal{P}$-pseudo-Hermitian, $\mathcal{T}$-self-adjoint
- $H^*_{\pi/2} = H_{-\pi/2}$, $H_{\pi/2}$ is closed
- $\Theta H^*_{\pi/2} = H_{\pi/2}\Theta$
- $\Theta = I - \mathrm{i}P_{\mathrm{sign}}\mathcal{P}$, $\Theta \geq 0$, Θ is not invertible !

Spectrum

- residual spectrum is empty
- continuous spectrum $[0, \infty)$
- point spectrum $\mathbb{C} \setminus [0, \infty)$!

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Eigenfunctions of $H_{\pi/2}$

$$\psi_k(x) = \begin{cases} e^{kx}, & x < 0, \\ \mathrm{i}e^{-kx}, & x > 0, \end{cases} \qquad \varphi_k(x) = \begin{cases} e^{-kx}, & x < 0, \\ \mathrm{i}e^{kx}, & x > 0, \end{cases}$$

$$\zeta_k(x) = \begin{cases} e^{-\mathrm{i}kx}, & x < 0, \\ \mathrm{i}e^{\mathrm{i}kx}, & x > 0. \end{cases}$$

$\psi_k \in L^2(\mathbb{R})$ for $\operatorname{Re} k > 0$, $\varphi_k \in L^2(\mathbb{R})$ for $\operatorname{Re} k < 0$,
$\zeta_k \in L^2(\mathbb{R})$ for $\operatorname{Re} k = 0$ and $\operatorname{Im} k > 0$

Models on finite interval

Models on a finite interval $(-l, l)$

- $L^2(-l,l)$, $H = -\frac{\mathrm{d}^2}{\mathrm{d}x^2}$
- $\mathrm{Dom}(H) = AC^1(-l,l)$
- 2 interactions - at $x = 0$ and $x = \pm l$ - 2 BC
- at $x = 0$ - $\mathcal{PT}$-symmetric interaction $b = 0, c = 0$
- at $x = \pm l$ - both self-adjoint and $\mathcal{PT}$-symmetric interactions

Compact resolvent guaranteed?

Theorem (Kato)

Let $T_1, T_2 \in \mathscr{C}(\mathcal{H})$ have non-empty resolvent sets. Let T_1, T_2 be extensions of a common operator T_0, with order of extension for T_1 being finite. Then T_1 has compact resolvent if and only if T_2 has compact resolvent.

$\mathcal{PT}$-symmetric and symmetric interaction

Symmetric point interaction at $x = \pm l$

$$(U - I)\Psi(l) + \mathrm{i}\, L_0 (U + I)\Psi'(l) = 0$$

$$\Psi(l) = \begin{pmatrix} \psi(l) \\ \psi(-l) \end{pmatrix} \qquad \Psi'(l) = \begin{pmatrix} \psi'(l) \\ -\psi'(-l) \end{pmatrix}$$

U is unitary matrix

$\mathcal{PT}$-symmetric point interaction at $x = 0$

$\psi(0+) = e^{\mathrm{i}\phi}\psi(0-)$

$\psi'(0+) = e^{-\mathrm{i}\phi}\psi'(0-)$

$\mathcal{PT}$-symmetric and symmetric interaction

Spectrum

- discrete ($\lambda = k^2$) if $\phi \neq \pm\pi/2$

$$\cos\phi\Big(P_1(U) - 2\mathrm{i}kL_0P_2(U)\cos 2kl + k^2L_0^2P_3(U)\sin 2kl\Big) + \\ +2\mathrm{i}kL_0\Big(u_{12} + u_{21} + \mathrm{i}(u_{11} - u_{22})\sin\phi\Big) = 0$$

- $\phi = \pm\pi/2$
  - empty if $u_{12} + u_{21} \pm \mathrm{i}(u_{11} - u_{22}) \neq 0$
  - entire $\mathbb{C}$ if $u_{12} + u_{21} \pm \mathrm{i}(u_{11} - u_{22}) = 0$
    Dirichlet $U = -I$, Neumann $U = I$, Robin $U = \alpha I, \alpha \in \mathbb{R}$

Two $\mathcal{PT}$-symmetric interactions

$\mathcal{PT}$-symmetric interactions

$$\psi(0+) = e^{\mathrm{i}\phi_1}\psi(0-)$$

$$\psi'(0+) = e^{-\mathrm{i}\phi_1}\psi'(0-)$$

$$\begin{pmatrix} \psi(l) \\ \psi'(l) \end{pmatrix} = B \begin{pmatrix} \psi(-l) \\ \psi'(-l) \end{pmatrix}$$

$$B = \begin{pmatrix} \sqrt{1+b_2c_2}\,e^{\mathrm{i}\phi_2} & b_2 \\ c_2 & \sqrt{1+b_2c_2}\,e^{-\mathrm{i}\phi_2} \end{pmatrix}$$

Two $\mathcal{PT}$-symmetric interactions

Spectrum

- discrete ($\lambda = k^2$) if
  - $\phi_1 \neq \pm\pi/2, \phi_2 \neq \pm\pi/2$
  - $\phi_1 \neq \pm\pi/2, \phi_2 = \pm\pi/2$ and $b_2 \neq 0$ or $c_2 \neq 0$

$$\cos\phi_1 \left(\left(b_2 k^2 - c_2\right) \sin 2kl + 2k\sqrt{1 + b_2 c_2} \cos\phi_2 \cos 2kl \right) + \\ + 2k \left(\sqrt{1 + b_2 c_2} \sin\phi_1 \sin\phi_2 - 1 \right) = 0.$$

- empty if $\phi_1 = \pm\pi/2$ and $\sqrt{1 + b_2 c_2} \sin\phi_2 - 1 \neq 0$
- entire $\mathbb{C}$ if $\phi_1 = \pm\pi/2$ and $\sqrt{1 + b_2 c_2} \sin\phi_2 - 1 = 0$
- $b_2 = c_2 = 0$
  - empty if $\phi_1 = \pm\pi/2$ and $\phi_2 \neq \pm\pi/2$
  - entire $\mathbb{C}$ if $\phi_1 = \phi_2 = \pm\pi/2$

Conclusions

Conclusions

- Spectral properties of $\mathcal{PT}$-symmetric operators can be very rich
- $\mathcal{PT}$-symmetry, pseudo-Hermiticity, J-self-adjointness do not guarantee non-empty spectrum, countable point spectrum, spectral decomposition
- Examples of $\mathcal{PT}$-symmetric point interactions
  - line $\mathbb{R}$ - $\sigma = \mathbb{C}$, $\sigma_c = [0, \infty)$, $\sigma_p = \mathbb{C} \setminus [0, \infty)$
  - finite interval $(-l, l)$ - $\sigma = \emptyset$ versus $\sigma = \sigma_p = \mathbb{C}$