Krein Spaces in de Sitter Quantum Theories

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Motivation

De Sitter solution of Einstein equations

- corresponds to the experimental observation of accelerated expansion of the Universe
- approximates the inflation period in the early Universe
- positive cosmological constant
- maximally symmetric solution
- $SO_0(1,4)$ invariance

Quantum field theory

- quantum elementary systems are associated with unitary irreducible representations of $SO_0(1,4)$
- classification of UIR 1961 Dixmier, 1963 Takahashi
- unsolved problem - quantization of fields for $\Pi_{p,0}$

Structure of the talk

- some de Sitter basics
- origins of "zero-mode problem" and possible ways out
- Gupta-Bleuler like quantization
  - indecomposable representations on Krein space
  - Gupta-Bleuler triplet in the standard form
  - G-R-T construction
  - description of cohomology
- generalization of method to "higher" representations

de Sitter basics

Space-time and coordinates

- hyperboloid embedded in a 4+1-dimensional Minkowski space $\mathbb{M}_5$

$$M_H \equiv \{x \in \mathbb{M}_5;\ x^2 := x \cdot x = \eta_{\alpha\beta}\ x^\alpha x^\beta = -H^{-2}\},$$
$$\alpha, \beta = 0, 1, 2, 3, 4, \quad (\eta_{\alpha\beta}) = \mathrm{diag}(1, -1, -1, -1, -1)\,,$$
$$x := (x^0, \vec{x}, x^4) \ \text{ ambient coordinates}$$

- conformal coordinates

$$x = (H^{-1} \tan\rho, (H\cos\rho)^{-1} u)\,, \quad \rho \in]-\frac{\pi}{2}, \frac{\pi}{2}[, \quad u \in S^3$$

De Sitter group $SO_0(1,4)$

- $SO_0(1,4)$ (or $Sp(2,2)$) - ten parameters
- classification of representations using Casimir operators
- in Dixmier notation, parameters p, q:

$$\mathcal{C}_2 = (-p(p+1) - (q+1)(q-2))\mathbb{I},$$
$$\mathcal{C}_4 = (-p(p+1)q(q-1))\mathbb{I}$$

- p, q represent spin and mass
- our interest - discrete scalar representations $\Pi_{p,0}$ $p \in \mathbb{N}$

Wave equation and modes

- scalar representation $q = 0 \ \Rightarrow \mathcal{C}_4 = 0$
- wave equation $\mathcal{C}_2 = -p(p+1)\mathbb{I}$

  $\mathcal{C}_2$ is proportional to Laplace-Beltrami operator on dS space
- in conformal coordinates:

$$\Box = \frac{1}{\sqrt{g}}\partial_\nu \sqrt{g} g^{\nu\mu}\partial_\mu = H^2 \cos^4\rho \frac{\partial}{\partial\rho}(\cos^{-2}\rho \frac{\partial}{\partial\rho}) - H^2\cos^2\rho\, \Delta_3$$

$$\Delta_3 = \frac{\partial^2}{\partial\alpha^2} + 2\cot\alpha \frac{\partial}{\partial\alpha} + \frac{1}{\sin^2\alpha}\frac{\partial^2}{\partial\theta^2} + \cot\theta \frac{1}{\sin^2\alpha}\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\alpha\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$$

  Δ_3 is Laplace operator on S^3
- solutions of wave equation - carrier space of the representation

Wave equation and modes

- separation of variables

$$\psi(x) = \chi(\rho) D(u), \quad u \in \mathrm{S}^3$$

$$\begin{aligned} [\Delta_3 + L(L+1)] D(u) &= 0, \\ \left(\cos^4 \rho \frac{d}{d\rho} \cos^{-2} \rho \frac{d}{d\rho} + L(L+1) \cos^2 \rho + (p+2)(1-p)\right) \chi(\rho) &= 0. \end{aligned}$$

- $D(u) = \mathrm{Y}_{Llm}(u) = C_{Ll} 2^l l! \,(\sin \alpha)^l \, C_{L-l}^{l+1} (\cos \alpha) \, Y_{lm}(\theta, \phi)$
  for $(L, l, m) \in \mathbb{N}_0 \times \mathbb{N}_0 \times \mathbb{Z}$ with $0 \leq l \leq L$ and $-l \leq m \leq l$.
- $\chi(\rho) = e^{-i(L+1-p)\rho} (\cos \rho)^{1-p} \; {}_2F_1(1-p, L-p+1; L+2; -e^{-2i\rho})$

Klein-Gordon inner product

- solutions of wave equation $\phi^p_{Llm}(x) = \chi^p_L(\rho) Y_{Llm}(u)$
- Klein-Gordon inner product

$$\langle \Phi_1, \Phi_2 \rangle = \frac{i}{H^2} \int_{\rho=0} \overline{\Phi_1(\rho,u)} \overleftrightarrow{\partial}_\rho \Phi_2(\rho,u) \, du \,,$$

  where $du = \sin^2\alpha \, \sin\theta \, d\alpha \, d\theta \, d\phi$ is the invariant measure on S^3

- Klein-Gordon inner product is dS invariant $\langle \pi(g)\cdot, \pi(g)\cdot \rangle = \langle \cdot, \cdot \rangle$
- we need $\langle \phi^p_{Llm}, \phi^p_{L'l'm'} \rangle = \delta_{LL'} \delta_{ll'} \delta_{mm'}$
- orthogonality is satisfied, normalization?

Normalization

- $\|\phi^{p}_{Llm}\|^{2} = \frac{2^{3}}{H^{2}} \frac{\Gamma(L+p+2)}{(\Gamma(p+2))^{2}\Gamma(L-p-1)}$
- for $p=0$ and $L=0$: $\|\phi^{0}_{000}\| = 0$
- for $p>1$ we have $p(p+1)(2p+1)/6$ zero norm solutions
- origin of so-called "zero-mode" problem
- no-go result by Allen, 1985
- we need non-degenerate, $SO_0(1,4)$-invariant set of modes

Set of modes

- for $p = 0$: $\phi_{000} \equiv \psi_g = const.$
- $\{\phi_{Llm}\}_{L>0}$ is not dS invariant
- $\{\phi_{Llm}\}_{L\geq 0}$ is degenerate (for K-G product)
- possible ways out:
  - only $O(4)$-invariance Allen 1985
  - Gupta-Bleuler like quantization Gazeau, Renaud, Takook 2000

Construction of carrier space

- to obtain zero mode with non-zero K-G norm - add the second solution of wave equation ($L = 0$)

$$\psi_g = \frac{H}{2\pi}$$
$$\psi_s = -i\frac{H}{2\pi}\left(\rho + \frac{1}{2}\sin 2\rho\right)$$

- $\phi_0 := \psi_g + \psi_s/2$, $\langle \psi_g, \psi_s \rangle = 1 \;\Rightarrow \|\phi_0\| = 1$
- $\{\phi_{Llm}\}_{L>0} \cup \{\phi_0\}$ is non-degenerate and orthonormal
- $\{\phi_{Llm}\}_{L>0} \cup \{\phi_0\}$ is not dS invariant!
- it is necessary to include $\{\overline{\phi_{Llm}}\}_{L>0} \cup \{\overline{\phi_0}\}$
- $\|\overline{\phi_{Llm}}\| < 0$, $\|\overline{\phi_0}\| < 0$
- we change the notation $\{\phi_{Llm}\}_{L\geq 0} \;\rightarrow\; \{\psi_n\}_{n\in\mathbb{N}_0}$

Construction of carrier space

- dS invariant, non-degenerate carrier space $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$

$$\mathcal{H}_+ := \Big\{ \sum_{n\in\mathbb{N}_0} c_n \psi_n \Big| \sum_{n\in\mathbb{N}_0} |c_n|^2 < \infty \Big\}$$

$$\mathcal{H}_- := \Big\{ \sum_{n\in\mathbb{N}_0} d_n \overline{\psi_n} \Big| \sum_{n\in\mathbb{N}_0} |d_n|^2 < \infty \Big\}$$

- subspaces

$$\mathcal{N} := \mathbb{C}\psi_g, \quad \psi_g = 1/2(\psi_0 + \overline{\psi_0}), \qquad \|\psi_g\| = 0,$$

$$\mathcal{S} := \mathbb{C}\psi_s, \quad \psi_s = 1/(2i)\,(\psi_0 - \overline{\psi_0}), \qquad \|\psi_s\| = 0,$$

$$\mathcal{K}_+ := \Big\{ \sum_{n\in\mathbb{N}} c_n \psi_n \Big| \sum_{n\in\mathbb{N}} |c_n|^2 < \infty \Big\}, \qquad \|\psi_n\| > 0, \ \ n \in \mathbb{N},$$

$$\mathcal{K}_- := \Big\{ \sum_{n\in\mathbb{N}} d_n \overline{\psi_n} \Big| \sum_{n\in\mathbb{N}} |d_n|^2 < \infty \Big\}, \qquad \|\psi_n\| < 0, \ \ n \in \mathbb{N}$$

Construction of carrier space

- $\mathcal{H}$ is a Krein space

$$J\psi_n := \psi_n, \quad n \in \mathbb{N}, \qquad J\psi_g := \psi_s$$
$$J\overline{\psi_n} := -\overline{\psi_n}, \quad n \in \mathbb{N}, \qquad J\psi_s := \psi_g$$

- $J^2 = I$, $\forall \psi, \phi \in \mathcal{H}$, $\langle \psi, J\phi \rangle = \langle J\psi, \phi \rangle$
- $(\cdot, \cdot) := \langle \cdot, J\cdot \rangle$ is a positive inner product on $\mathcal{H}$
- $\mathcal{H} = \mathcal{N} \oplus_J \mathcal{S} \oplus_J \mathcal{K}_+ \oplus_J \mathcal{K}_- = (\mathcal{N} \dotplus \mathcal{S}) \oplus \mathcal{K}_+ \oplus \mathcal{K}_-$
- n.b. $\langle \psi_g, \psi_s \rangle = 1$

Gupta-Bleuler triplet

- Gupta-Bleuler triplet

$$\left.\begin{array}{l} \pi_3 \to \pi_2 \to \pi_1 \\ \mathcal{H}_3 \supset \mathcal{H}_2 \supset \mathcal{H}_1 \end{array}\right\}$$

- Indecomposable representation

$$\pi(g) = \begin{pmatrix} \pi_1(g) & c_{12}(g) & c_{13}(g) \\ 0 & \pi_2(g) & c_{23}(g) \\ 0 & 0 & \pi_3(g) \end{pmatrix}$$

- π_j is a representation on $\mathcal{R}_j := \mathcal{H}_{j+1}/\mathcal{H}_j$

  $\mathcal{H}_3 := \mathcal{H}$ and $\mathcal{R}_1 := \mathcal{H}_1$

G-B triplet in the standard form

- definition by Araki (1985)

$$\left.\begin{array}{l} \pi_1^{\#} \to \pi_2 \to \pi_1 \\ \mathcal{H}_3 \supset \mathcal{H}_2 \supset \mathcal{H}_1 \end{array}\right\}$$

- π_3 is a conjugate of π_1

$$\forall g \in G,\ \phi \in \mathcal{H}_1^{\#},\ \psi \in \mathcal{H}_1,\ \langle \pi_1^{\#}(g^{-1})\phi, \psi\rangle = \langle \phi, \pi_1(g)\psi\rangle$$

- for mmcf

$$\mathcal{H}_1 = \mathcal{N}, \quad \mathcal{H}_2 := \mathcal{H}_1^{\perp} = \mathcal{N} \oplus \mathcal{K}_- \oplus \mathcal{K}_+,\ \mathcal{H} \equiv \mathcal{H}_3 = (\mathcal{N} \dotplus \mathcal{S}) \oplus \mathcal{K}_- \oplus \mathcal{K}_+$$
$$\mathcal{R}_1 = \mathcal{N}, \quad \mathcal{R}_2 = \mathcal{H}_2/\mathcal{H}_1 \simeq \mathcal{K}_- \oplus \mathcal{K}_+,\ \mathcal{R}_3 = \mathcal{H}_3/\mathcal{H}_2 \simeq \mathcal{S}$$

- 'one particle sector' ($\mathcal{R}_2$) contains also modes with negative norm

Modified G-B triplet

- structure of the G-R-T construction

$$\left.\begin{array}{l} \pi_3 \to \pi_2 \to \pi_1 \\ \mathcal{H}_3 \supset \mathcal{H}_2 \supset \mathcal{H}_1 \end{array}\right\}$$

- π_3 is not a conjugate of π_1
- $\mathcal{N} \oplus \mathcal{K}_+$ is an invariant subspace

$$\begin{array}{lll} \mathcal{H}_1 = \mathcal{N}, & \mathcal{H}_2 := \mathcal{N} \oplus \mathcal{K}_+ \subset \mathcal{H}_1^\perp, & \mathcal{H} \equiv \mathcal{H}_3 = (\mathcal{N} \dotplus \mathcal{S}) \oplus \mathcal{K}_- \oplus \mathcal{K}_+ \\ \mathcal{R}_1 = \mathcal{N}, & \mathcal{R}_2 = \mathcal{H}_2/\mathcal{H}_1 \simeq \mathcal{K}_+, & \mathcal{R}_3 = \mathcal{H}_3/\mathcal{H}_2 \simeq \mathcal{S} \oplus \mathcal{K}_- \end{array}$$

- only modes with positive norm in 'one particle sector' ($\mathcal{R}_2$)

Cohomology - definitions

Definition

A representation π on $\mathcal{H}$ is called irreducible if there is no non-trivial closed invariant subspace.

$$\mathcal{W} \subset \mathcal{H}, \quad \pi(G)\mathcal{W} \subset \mathcal{W} \;\Rightarrow\; \mathcal{W} = \mathcal{H} \text{ or } \mathcal{W} = 0.$$

π is called topologically indecomposable if there are no non-zero closed invariant subspaces $\mathcal{U}$ and $\mathcal{V}$ of $\mathcal{H}$ such that $((\mathcal{U}+\mathcal{V})^{\perp})^{\perp} = \mathcal{H}$ and $\mathcal{U} \cap \mathcal{V} = 0$

$$\begin{pmatrix} \pi_1 & c_{12} \\ 0 & \pi_2 \end{pmatrix}$$

Cohomology - definitions

Definition

Let π_1, π_2 be representations on $\mathcal{H}_1, \mathcal{H}_2$. A function $c(g_1, ..., g_n)$ of $g_k \in G$ with values in $\mathscr{L}(\mathcal{H}_1, \mathcal{H}_2)$, *i.e.* in the set of everywhere-defined linear mappings from $\mathcal{H}_1$ into $\mathcal{H}_2$, is called n-cochain. The set of all n-cochains is denoted by $C^n(\pi_1, \pi_2)$.

Definition

The coboundary operation δ (satisfying $\delta^2 = 0$)

$$(\delta c_n)(g_1, ..., g_{n+1}) := \pi_2(g_1) c_n(g_2, ..., g_n) + \sum_{i=1}^{n} (-1)^n c_n(g_1, ..., g_i g_{i+1}, ..., g_{n+1}) + (-1)^{n+1} c_n(g_1, ..., g_n) \pi_1(g_{n+1}),$$

where $c_n \in C^n(\pi_1, \pi_2), g_1, ..., g_n \in G$.

Cohomology - definitions

Definition

Cocycle:
$Z^n(\pi_1, \pi_2) := \{c_n \in C^n \,|\, \delta c_n = 0\}$
Coboundary:
$B^n(\pi_1, \pi_2) := \delta C^{n-1}(\pi_1, \pi_2), \quad B^0(\pi_1, \pi_2) := 0$
Cohomology:
$H^n(\pi_1, \pi_2) := Z^n(\pi_1, \pi_2)/B^n(\pi_1, \pi_2)$

Definition

Let $c_1 \in C^m(\pi_2, \pi_3)$ and $c_2 \in C^n(\pi_1, \pi_2)$ we define $\times$ operation as

$$(c_1 \times c_2)(g_1, ..., g_{m+n}) := c_1(g_1, ..., g_m)c_2(g_{m+1}, ..., g_{m+n})$$

Cohomology and G-B triplet

$$\pi(g) = \begin{pmatrix} \pi_1(g) & c_{12}(g) & c_{13}(g) \\ 0 & \pi_2(g) & c_{23}(g) \\ 0 & 0 & \pi_3(g) \end{pmatrix}$$

π is a representation

- $\delta c_{12} = \delta c_{23} = 0$
- $\delta c_{13} = -c_{12} \times c_{23}$

Invariant complements

- $\mathcal{H}_1$ does not have any invariant complement iff $c_{12} \notin B^1(\pi_2, \pi_1)$.
- the existence of G-B triplet implies $H^1(\pi_2, \pi_1) \neq 0$, $H^1(\pi_3, \pi_2) \neq 0$

Summary

Results

- Zero mode problem can be solved by Gupta-Bleuler like quantization
- G-R-T construction is not Gupta-Bleuler triplet in the standard form
- the existence of Gupta-Bleuler triplet can be formulated in cohomological conditions

Open questions

- the second solution (construction of ϕ_0)
- relation of G-R-T construction and Gupta-Bleuler triplet in the standard form
- can be the method for $p = 0$ extended to higher representations? $\dim(\mathcal{N}) > 1$

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