Perfect transmission scattering as a $\mathcal{PT}$-symmetric spectral problem

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Based on:
1. D. Krejčiřík, P. Siegl, $\mathcal{PT}$-symmetric models in curved manifolds, 2010, Journal of Physics A: Mathematical and Theoretical, 43, 485204.
2. H. Hernandez-Coronado, D. Krejčiřík, P. Siegl, Perfect transmission scattering as a $\mathcal{PT}$-symmetric spectral problem, arXiv:1011.4281.

$\mathcal{PT}$-symmetry

Origins

- Hamiltonian $H = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \mathrm{i}x^3$ has real, positive, discrete spectrum [BeBo98]
- original hypothesis - the reality of spectrum due to $\mathcal{PT}$-symmetry
  - $[\mathcal{PT}, H] = 0$
  - parity $\mathcal{P}$, $(\mathcal{P}\psi)(x) = \psi(-x)$
  - complex conjugation $\mathcal{T}$, $(\mathcal{T}\psi)(x) = \overline{\psi}(x)$

Simple observations

- $\mathcal{PT}$-symmetry is not sufficient for real spectrum
- some $\mathcal{PT}$-symmetric operators are similar to self-adjoint or normal operators $\exists \varrho, \varrho^{-1} \in \mathscr{B}(\mathcal{H})$: $\varrho H \varrho^{-1}$ is self-adjoint or normal

Aims

? spectrum of $\mathcal{PT}$-symmetric operator?

? similarity to self-adjoint or normal operator?

[BeBo98] 1998 Bender and Boettcher, *Physical Review Letters* 80.

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Interpretation of $\mathcal{PT}$-symmetric models

Recent applications in physics

- experimental results in optics [KlGuMo08], [RuMaGaChSeKi10], [Lo10], ...
- superconductivity [RuStMa07], [RuStZu10] , solid state [BeFlKoSh08]
- electromagnetism [RuDeMu05], [Mo09], nuclear physics [ScGeHa92]

Quantum mechanics: similarity to self-adjoint operator

- let $h := \varrho H \varrho^{-1}$, $h^* = h$
- quasi-Hermiticity (equivalent to similarity to s-a operator) $\exists \Theta, \Theta^{-1} \in \mathscr{B}(\mathcal{H}), \Theta > 0 : \Theta H = H^* \Theta$ [Di61]
- $\Theta = \varrho^* \varrho$, H is self-adjoint in $\langle \cdot, \Theta \cdot \rangle$
- for operators with discrete spectrum: equivalent to the Riesz basicity of eigenvectors of H and H^*

[BeFlKoSh08] 2008 Bendix, Fleischmann, Kottos, and Shapiro, *Physical Review Letters* 103,
[Di61] 1961 Dieudonné, *Proceedings Of The International Symposium on Linear Spaces*,
[KlGuMo08] 2008 Klaiman, Günther, and Moiseyev, *Physical Review Letters* 101,
[Lo10] 2010 Longhi, *Physical Review Letters* 105,
[Mo09] 2009 Mostafazadeh, *Physical Review Letters* 102,
[RuStMa07] 2007 Rubinstein, Sternberg, and Ma, *Physical Review Letters* 99,
[RuStZu10] 2010 Rubinstein, Sternberg, and Zumbrun, *Archive for Rational Mechanics and Analysis* 195,
[RuDeMu05] 2005 Ruschhaupt, Delgado, Muga, *Journal of Physics A: Mathematical and General* 38,
[RuMaGaChSeKi10] 2010 Rüter, Makris, El-Ganainy, Christodoulides, Segev, and Kip, *Nature Physics* 6,
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Similarity to self-adjoint operator

Other symmetries

- $\mathcal{PT}$-symmetric operators: often $\mathcal{P}$ and $\mathcal{T}$-self-adjoint
  $H^* = \mathcal{P}H\mathcal{P} \quad H^* = \mathcal{T}H\mathcal{T}$

"Metric" and $\mathcal{C}$ operator

- metric operator Θ: $\Theta H = H^*\Theta$, $\Theta, \Theta^{-1} \in \mathscr{B}(\mathcal{H}), \Theta > 0$.
- operators with discrete spectrum:

$$\Theta = \operatorname*{s-lim}_{N\to+\infty} \sum_{n=1}^{N} c_n \langle \phi_n, \cdot \rangle \phi_n,$$

  with $H^*\phi_n = E_n\phi_n$, $\|\phi_n\| = 1$, $0 < m < c_n < M < +\infty$.
- $\mathcal{C}$ operator: $\mathcal{C} \in \mathscr{B}(\mathcal{H})$, $\mathcal{C}^2 = I$, $\mathcal{PC} > 0$, and $H\mathcal{C} = \mathcal{C}H$
- $\mathcal{PC}$ is a metric operator
- $$\mathcal{C} = \operatorname*{s-lim}_{N\to+\infty} \sum_{n=1}^{N} d_n \langle \phi_n, \cdot \rangle \psi_n,$$

  with $H\psi_n = E_n\psi_n$, $\|\psi_n\| = 1$, d_n restricted by $\mathcal{C}^2 = I$.

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Perfect transmission

Scattering, perfect transmission

- scattering on real line, potential V with support in $(-a, a)$
- solutions of Schrödinger equation

$$\psi(x) = \begin{cases} e^{\mathrm{i}kx} + Re^{-\mathrm{i}kx}, & x < -a \\ Te^{\mathrm{i}kx}, & x > a \end{cases}$$

- we look for such k that $R = 0$, *i.e.* perfect transmission
- inside $(-a, a)$

$$\begin{cases} -\psi''(x) + V(x)\psi(x) = k^2\psi(x), & x \in (-a, a) \\ \psi'(\pm a) - \mathrm{i}\, k\, \psi(\pm a) = 0 \end{cases}$$

- nonlinear problem $\to$ spectral problem

$$\begin{cases} -\psi''(x) + V(x)\psi(x) = \mu(\alpha)\psi(x), & x \in (-a, a) \\ \psi'(\pm a) - \mathrm{i}\, \alpha\, \psi(\pm a) = 0 \end{cases}$$

- perfect transmission energies (PTEs) $\mu(\alpha_*)$

$$\mu(\alpha_*) = \alpha_*^2$$

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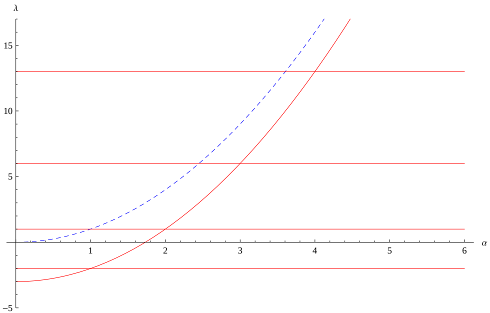
Perfect transmission - examples

Square well

- $V(x) = V_0\,\chi_{[-a,a]}(x)$,
- eigenvalues of associated operator: $\begin{cases} \mu_0(\alpha) & = \alpha^2 + V_0, \\ \mu_n(\alpha) & = (n\pi/2a)^2 + V_0 \end{cases}$
- PTEs: $k_*^2 = (n\pi/2a)^2 - V_0$
- if $V_0 = 0$, then PTEs: $k_*^2 \in \mathbb{R}^+$



Perfect transmission - examples

Two δ potentials

- $V(x) = \beta(\delta(x+a) + \delta(x-a))$
- definition of Hamiltonian via boundary conditions:

$$\begin{cases} \psi(\pm a-) = \psi(\pm a+) = \psi(\pm a) \\ \psi'(\pm a+) - \psi'(\pm a-) = \beta\psi(\pm a) \end{cases}$$

- can be approximated by

$$V_\varepsilon(x) = \frac{\beta}{\varepsilon}\left(\chi_{[-a,-a+\varepsilon]}(x) + \chi_{[a,a-\varepsilon]}(x)\right)$$

- associated spectral problem:

$$\begin{cases} -\psi''(x) = \mu(\alpha)\psi(x), \quad x \in (-a,a) \\ \psi'(\pm a) - (\mathrm{i}\,\alpha \pm \beta)\,\psi(\pm a) = 0 \end{cases}$$

$$\mu(\alpha_*) = \alpha_*^2$$

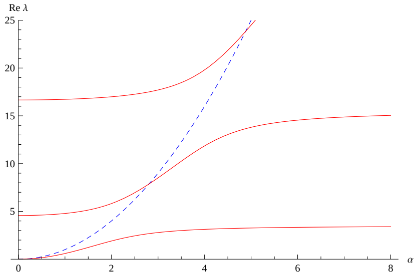
Two δ potentials

$\beta > 0$

- Eigenvalue equation

$$(k^2 - \alpha^2 - \beta^2)\sin 2ka - 2\beta k \cos 2ka = 0$$

- all eigenvalues are real [KrSi10]



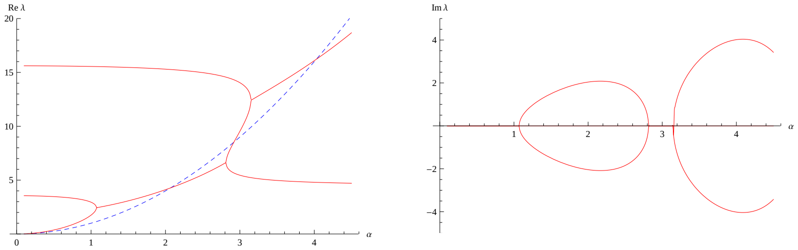
[KrSi10] 2010 Krejčiřík, Siegl, *Journal of Physics A: Mathematical and General* 43

$\beta < 0$

- Eigenvalue equation

$$(k^2 - \alpha^2 - \beta^2)\sin 2ka - 2\beta k \cos 2ka = 0$$

- one or zero pair of complex conjugated eigenvalues [KrSi10]



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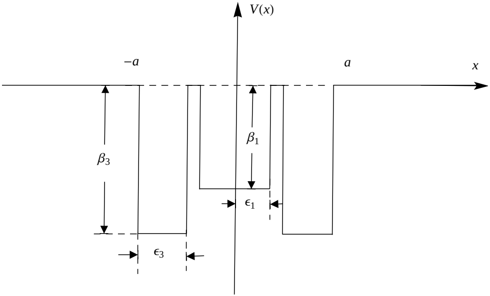
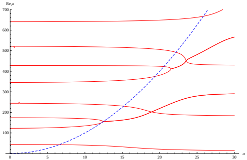
Interpretation of complex EVs

Three steps potential



Spectral picture

$a = \pi/4,\ \varepsilon_1 = 0.2,\ \varepsilon_3 = 0.5,\ \beta_1 = -90,\ \beta_3 = -100$



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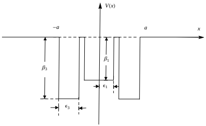

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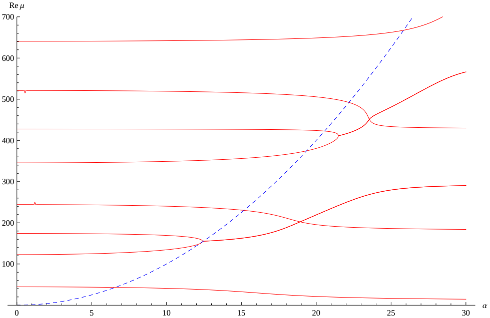
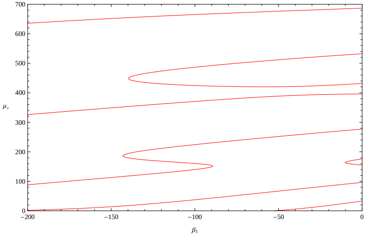


Interpretation of complex EVs

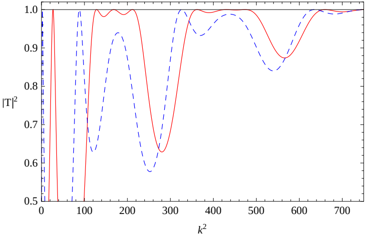
PTE as a function of the potential

$a = \pi/4$, $\varepsilon_1 = 0.2$, $\varepsilon_3 = 0.5$, $\beta_3 = -100$, and $\beta_2 = 0$

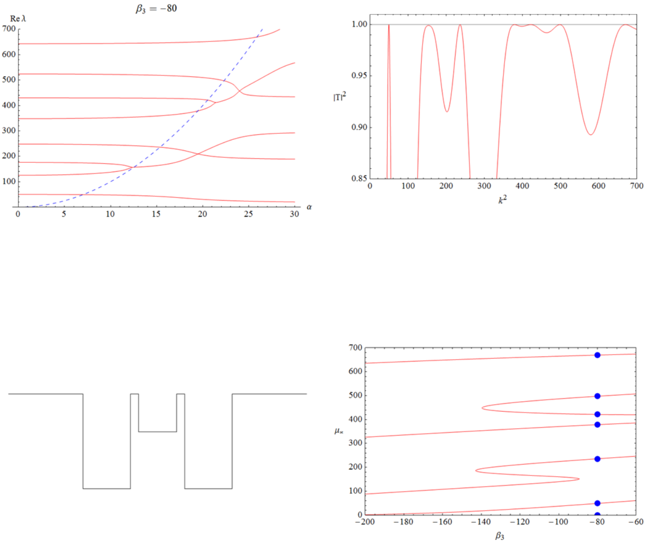


Transmission coefficient

$\beta_1 = -120$ (cont. red), and $\beta_1 = -200$ (dashed blue)



Three steps potential - animations



Inverse problem

Addition of constant potential

- what can we say about the spectrum of

$$\begin{cases} -\psi''(x) + V(x)\psi(x) = \mu(\alpha)\psi(x), & x \in (-a, a) \\ \psi'(\pm a) - \mathrm{i}\,\alpha\,\psi(\pm a) = 0 \end{cases} \qquad ?$$

addition of constant potential V_0

- $$\begin{cases} -\psi''(x) + (V(x) + V_0)\psi(x) = \mu(V_0, \alpha)\psi(x), & x \in (-a, a) \\ \psi'(\pm a) - \mathrm{i}\,\alpha\,\psi(\pm a) = 0 \end{cases}$$

$$\mu(V_0, \alpha_*) = \alpha_*^2$$

- V_0 shifts the eigenvalues up and down, the parabola determining perfect transmission energies is not shifted
- appearance of complex conjugated pair can be traced *e.g.* as a loss of two close perfect transmission energies

Conclusions

Summary

- introduction to $\mathcal{PT}$-symmetry
- metric and $\mathcal{C}$ operators - more details in talk of J. Železný
- perfect transmission scattering - $\mathcal{PT}$-symmetric spectral problem
- interpretation of complex eigenvalues
- inverse problem
- framework for arbitrary V: perturbation theory for spectral operators