Open problems in $\mathcal{PT}$-symmetry

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Outline

$\mathcal{PT}$-symmetry

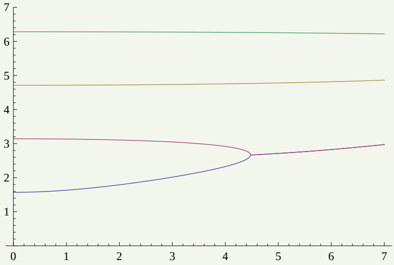
$\mathcal{PT}$-symmetric Hamiltonians

- $H = -\frac{d^2}{dx^2} + V(x), \; V(x) = \overline{V(-x)}$
  - bounded perturbations: bounded potentials
    $\mathcal{PT}$-symmetric square well: $V(x) = \mathrm{i}Z\mathrm{sgn}x$
    2001 Znojil, 2001 Znojil and Lévai, ...
  - non-perturbative approach: unbounded potential
    $V(x) = \mathrm{i}x^3$
    1998, Bender, Boettcher, 2003 Dorey, Dunning, Tateo, ...
  - relatively (form) bounded perturbations: $\mathcal{PT}$-symmetric point interactions (boundary conditions)
    two δ potentials with complex coupling, $\mathcal{PT}$-symmetric Robin boundary conditions
    2002 Albeverio, Fei, Kurasov, 2005 Albeverio, Kuzhel, 2005 Jakubský, Znojil, 2009 Albeverio, Gunther, Kuzhel, 2006 Krejčiřík, Bíla, Znojil, 2008 Borisov, Krejčiřík, 2010 Krejčiřík, Siegl, ...

Reality of the spectrum and metric operator

Spectrum

- $$\left.\begin{array}{l} \mathcal{PT}-\text{symmetry } (\mathcal{PT})H \subset H(\mathcal{PT}) \\ \text{pseudo}-\text{Hermiticity } H^* = \eta^{-1}H\eta \end{array}\right\} \Rightarrow \lambda \in \sigma(H) \Leftrightarrow \overline{\lambda} \in \sigma(H)$$
- pseudo-Hermiticity $\Leftrightarrow$ self-adjointness in Krein space
- not sufficient for real spectrum, only complex conjugated pairs
- 1D systems: the crossing of real eigenvalues is necessary to produce complex conjugated pair



Metric operator

Metric operator

- $\Theta H = H^* \Theta$
  - $\Theta, \Theta^{-1} \in \mathscr{B}(\mathcal{H})$
  - $\Theta^* = \Theta$
  - $\Theta > 0$
- necessary condition: $\sigma(H) \subset \mathbb{R}$

Existence of metric operator

- H possesses a metric operator Θ
- H is self-adjoint in $\langle \cdot, \Theta \cdot \rangle$
- H is similar to a self-adjoint operator $h = \varrho^{-1} H \varrho = h^*, \quad \Theta = \varrho \varrho^*$
- H possesses a $\mathcal{C}$-symmetry: $\mathcal{C}^2 = I,\ \eta\mathcal{C} > 0,\ \mathcal{C}H = H\mathcal{C}$

Metric operator

Operators with discrete spectrum

- H with discrete spectrum: eigenfunctions $\{\psi_n\}$ form a Riesz basis
- $\Theta = \text{s–}\lim_{N\to\infty} \sum_{j=1}^{N} c_j \langle \phi_j, \cdot \rangle \phi_j$,

  where ϕ_j are eigenfunctions of H^* and $m < c_j < M$

Examples

- existence results: perturbation theory for spectral operators
- few explicit examples:
  - point interactions

    2005 Albeverio, Kuzhel, 2008 Siegl
  - $\mathcal{PT}$-symmetric Robin b.c.

    2006 Krejčiřík, Bíla, Znojil, 2008 Krejčiřík, 2010 Krejčiřík, Siegl, Železný

Classes of operators

Antilinear symmetry

- $LH\psi = LH\psi$ for all $\psi \in \mathrm{Dom}(H)$
- L is antilinear bounded operator with bounded inverse
- spectrum: $\lambda \in \sigma_{p,c,r}(H)$ iff $\overline{\lambda} \in \sigma_{p,c,r}(H)$
- example: $L = \mathcal{PT}$, $H = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V(x)$, $\overline{V(-x)} = V(x)$

Pseudo-Hermiticity

- weak pseudo-Hermiticity:
  - $H = \eta^{-1} H^* \eta$
  - $\eta, \eta^{-1} \in \mathscr{B}(\mathcal{H})$
- pseudo-Hermiticity: $\eta = \eta^*$
- pseudo-Hermiticity $\Leftrightarrow$ self-adjointness in Krein space
- spectrum: $\sigma_{p,c,r}(H) = \sigma_{p,c,r}(H^*)$

C-symmetric operators

Definition

Let $A \in \mathscr{L}(\mathcal{H})$ be densely defined. Let C be an antilinear isometric involution, i.e. $C^2 = I$ and $\langle Cx, Cy \rangle = \langle y, x \rangle$ for all $x, y \in \mathcal{H}$. A is called C-symmetric if $A \subset CA^*C$. A is called C-self-adjoint if $A = CA^*C$.

Lemma

Let A be a C-self-adjoint operator. Then

(i) $\dim(\mathrm{Ker}(A - \lambda)) = \dim(\mathrm{Ker}(A^* - \overline{\lambda}))$,

(ii) $\sigma_r(A) = \emptyset$.

Finite dimension

Lemma

Every $A \in \mathscr{L}(V_n)$ is similar to the C-self-adjoint operator, i.e. there exists invertible $X \in \mathscr{L}(V_n)$ such that XAX^{-1} is C-self-adjoint.

Proposition

Let $A \in \mathscr{L}(V_n)$. Then A is pseudo-Hermitian if and only if it possesses an antilinear symmetry.

Antilinear symmetry without pseudo-Hermiticity

Example

- $\{e_n\}_{n=1}^{\infty}$ standard orthonormal basis of $\mathcal{H} = l_2(\mathbb{N})$, $e_n(m) = \delta_{mn}$
- $Te_n := e_{n-1}, \quad n \in \mathbb{N}, \quad e_0 := 0$
- $T^* e_n := e_{n+1}, \quad n \in \mathbb{N}$
- $T = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & 0 & \\ 0 & 0 & 0 & 1 & 0 & \\ 0 & 0 & 0 & 0 & 1 & \\ \vdots & & & & \ddots & \ddots \end{pmatrix}$

Antilinear symmetry without pseudo-Hermiticity

Example

- antilinear symmetry $\mathcal{T}$
- every $|\lambda| < 1$ is in the point spectrum $\sigma_p(T)$, $x_\lambda = \sum_{n=1}^{\infty} \lambda^{n-1} e_n$
- $\sigma_p(T^*) = \emptyset$, point spectrum of T^* is empty
- $\{\lambda \in \mathbb{C} | |\lambda| < 1\} \subset \sigma_r(T^*)$, residual spectrum is non-empty
- T is not pseudo-Hermitian, $\sigma_p(T) \neq \sigma_p(T^*)$

Pseudo-Hermiticity without antilinear symmetry

Example

- $\{e_i\}_{-\infty}^{\infty}$ orthonormal basis of $\mathcal{H} = l^2(\mathbb{Z})$, $e_n(m) = \delta_{mn}$
- $$Te_i := \begin{cases} \lambda_0 e_i + e_{i+1}, & i \geq 1, \\ 0, & i = 0, \\ \overline{\lambda}_0 e_{-1}, & i = -1, \\ \overline{\lambda}_0 e_i + e_{i+1}, & i < -1, \end{cases}$$
- $\lambda_0 \in \mathbb{C}, \ \operatorname{Im} \lambda_0 > \frac{1}{2}$

Pseudo-Hermiticity without antilinear symmetry

Example

- $$T = \begin{pmatrix} \ddots & & & & & & \\ \ddots & \overline{\lambda_0} & 0 & 0 & 0 & 0 & \\ & 1 & \overline{\lambda_0} & 0 & 0 & 0 & \\ & 0 & 0 & 0 & 0 & 0 & \\ & 0 & 0 & 0 & \lambda_0 & 0 & \\ & 0 & 0 & 0 & 1 & \lambda_0 & \\ & & & & & \ddots & \ddots \end{pmatrix}$$

Pseudo-Hermiticity without antilinear symmetry

Example

- T is pseudo-Hermitian, $\mathcal{P}e_i := e_{-i}$, $T = \mathcal{P}T^*\mathcal{P}$
- $\overline{\lambda}_0 \in \sigma_p(T) = \sigma_p(T^*)$
- $\lambda_0 \in \sigma_r(T) = \sigma_r(T^*)$
- T has not any antilinear symmetry, $\lambda \in \sigma_p(T) \Leftrightarrow \overline{\lambda} \in \sigma_p(T^*)$

Irregular $\mathcal{PT}$-symmetric boundary conditions7

$\mathcal{PT}$-symmetric irregular boundary conditions

- parametrization of $\mathcal{PT}$-symmetric b.c. 2002 Albeverio, Fei, Kurasov
- $\mathcal{PT}$-symmetric b.c. are strongly regular except one case (irregular)
- $H = -\dfrac{\mathrm{d}^2}{\mathrm{d}x^2}$ on $L_2(-1,1)$
  $\psi(-1) = \psi(1) = 0, \quad \psi(0+) = e^{\mathrm{i}\tau}\psi(0-)$ and $\psi'(0+) = e^{-\mathrm{i}\tau}\psi'(0-)$
- irregular for $\tau = \pm\pi/2$: $\sigma(H) = \mathbb{C}$ 2005 Albeverio, Kuzhel
- for $\tau \neq \pm\pi/2$: $\sigma(H) = \left\{\left(\frac{n\pi}{2}\right)^2\right\}$
  - $\Theta = I - \mathrm{i}\sin\tau P_{sgn}\mathcal{P}$ 2009 Siegl
  - $\dim \mathrm{Ker}(\Theta) = \infty$ for $\tau = \pm\pi/2$
- can we approximate irregular case with regular ones?
  - resolvent does not exists
  - strong graph limit: $H_{\pi/2} = \mathrm{str.\,gr.}\lim H_n$
  - str. gr. limit preserves $\mathcal{PT}$-symmetry and $\mathcal{P}$-self-adjointness

$\mathcal{PT}$-symmetric Robin boundary conditions

1D model

2006 Krejčiřík, Bíla, Znojil, 2010 Krejčiřík, Siegl

- $\mathcal{H} = L^2\big((-a,a), \mathrm{d}x\big)$
- $H = -\dfrac{d^2}{dx^2}$
- $\mathrm{Dom}(H) = W^{2,2}\big((-a,a)\big)$ + boundary conditions
- $\psi'(-a) + (\mathrm{i}\alpha - \beta)\psi(-a) = 0, \quad \psi'(a) + (\mathrm{i}\alpha + \beta)\psi(a) = 0$

  $\alpha, \beta \in \mathbb{R}$

1D model - $\mathcal{PT}$-symmetry

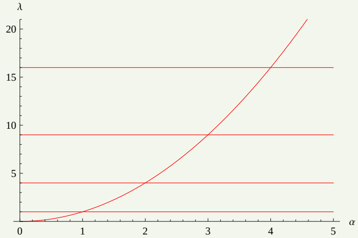
- $\forall \psi \in \mathrm{Dom}(H), \quad \psi \in \mathrm{Dom}(H) \Leftrightarrow \mathcal{PT}\psi \in \mathrm{Dom}(H)$
- $\forall \psi \in \mathrm{Dom}(H), \quad H\mathcal{PT}\psi = \mathcal{PT}H\psi$
- $H(\alpha,\beta)^* = H(-\alpha,\beta)$
- $H = \mathcal{P}H^*\mathcal{P}, \, H = \mathcal{T}H^*\mathcal{T}$

$\mathcal{PT}$-symmetric Robin boundary conditions

Spectrum of 1D model: I. $\beta = 0$

- $\psi'(-a) + \mathrm{i}\alpha\psi(-a) = 0, \quad \psi'(a) + \mathrm{i}\alpha\psi(a) = 0$
- $\sigma(H) = \sigma_d(H) \subset \mathbb{R}$, crossings

Spectrum of 1D model: I. $\beta = 0$



$$\sigma(H) = \{\alpha^2\} \cup \left\{\left(\tfrac{n\pi}{2a}\right)^2\right\}_{n\in\mathbb{N}}$$

$$\psi_0(x) = e^{-\mathrm{i}\alpha x}$$
$$\psi_n(x) = \cos\left(\tfrac{n\pi}{2a}x\right) - \mathrm{i}\alpha\tfrac{2a}{n\pi}\sin\left(\tfrac{n\pi}{2a}x\right)$$

$\mathcal{PT}$-symmetric Robin boundary conditions

Spectrum of 1D model: I. $\beta = 0$, metric operator

- $\Theta H = H^* \Theta$
- $\Theta = I + \phi_0 \langle \phi_0, \cdot \rangle + \Theta_0 + \mathrm{i}\alpha\Theta_1 + \alpha^2 \Theta_2$,

  where
  - $\phi_0 = \sqrt{\frac{1}{2a} \exp(\mathrm{i}\alpha x)}$,
  - $(\Theta_0 \psi)(x) = -\frac{1}{2a}(J\psi)(2a)$,
  - $(\Theta_1 \psi)(x) = 2(J\psi)(x) - \frac{x}{2a}(J\psi)(2a) - \frac{1}{d}(J^2\psi)(2a)$,
  - $(\Theta_2 \psi)(x) = -(J^2\psi)(x) + \frac{x}{2a}(J^2\psi)(2a)$,
  - with $(J\psi)(x) = \int_{-a}^{x} \psi$. 2006 Krejčiřík, Bíla, Znojil
- $\Theta = I + K$,

  where K is an integral operator with kernel

  $K(x, y) =$

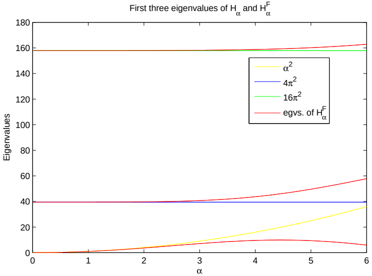
  $$\frac{e^{\mathrm{i}\alpha(x-y)} - 1}{2a} + \mathrm{i}\,\frac{\alpha(y-x)}{2a} - \alpha^2 \frac{xy}{2a} + \begin{cases} -\mathrm{i}\alpha + \alpha^2 x, & x < y \\ \mathrm{i}\alpha + \alpha^2 y, & y < x \end{cases}$$

  2010 Krejčiřík, Siegl, Železný

$\mathcal{PT}$-symmetric Robin boundary conditions

Equivalent self-adjoint Hamiltonian

- $\beta = 0$ and α small, notation $H_\alpha \equiv H(\alpha, 0)$
- approximative formula for $\rho \approx \sqrt{\Theta}$
- equivalent self-adjoint Hamiltonian $H_\alpha^F = \rho H_\alpha \rho^{-1}$
- $h\psi = -\psi'' + \frac{1}{4}\alpha^2(\psi(-a) + \psi(a)) + O(\alpha^3)$
- $\mathrm{Dom}(H_\alpha^F) = \left\{\psi \in W^{2,2}((-a,a)) \middle| \psi'(a) = -\psi'(-a) = \frac{1}{4}\alpha^2 \int_{-a}^{a} \psi(y)\mathrm{d}y\right\}$

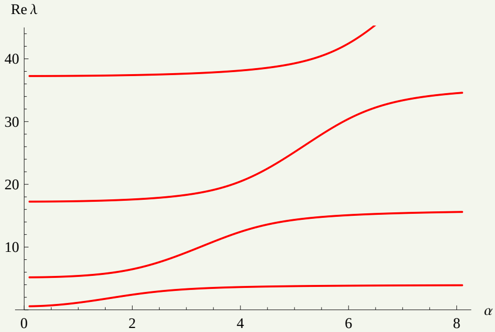


$\mathcal{PT}$-symmetric Robin boundary conditions

Spectrum of 1D model: II. $\beta > 0$

- $\psi'(-a) + (\mathrm{i}\alpha - \beta)\psi(-a) = 0, \quad \psi'(a) + (\mathrm{i}\alpha + \beta)\psi(a) = 0$
- $\sigma(H) = \sigma_d(H) \subset \mathbb{R}, \quad (k^2 - \alpha^2 - \beta^2)\sin 2ka - 2\beta k \cos 2ka = 0$
- metric operator exists for every α, β, no crossings

Spectrum of 1D model: II. $\beta > 0$



$\mathcal{PT}$-symmetric Robin boundary conditions

Spectrum of 1D model: III. $\beta < 0$

- $\psi'(-a) + (\mathrm{i}\alpha - \beta)\psi(-a) = 0, \quad \psi'(a) + (\mathrm{i}\alpha + \beta)\psi(a) = 0$
- $\sigma(H) = \sigma_d(H)$, $(k^2 - \alpha^2 - \beta^2)\sin 2ka - 2\beta k \cos 2ka = 0$
- either one or any complex conjugated pair
- known localization: $\Re\lambda$ in neighborhood of $\alpha^2 + \beta^2$
- metric operator exists if $\sigma(H) \subset \mathbb{R}$

Spectrum of 1D model: III. $\beta < 0$