$\mathcal{PT}$-symmetric Robin boundary conditions

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Outline

$\mathcal{PT}$-symmetry

$\mathcal{PT}$-symmetric Hamiltonians

- $H = -\dfrac{d^2}{dx^2} + V(x), \; V(x) = \overline{V(-x)}$
  - bounded perturbations: bounded potentials
    $\mathcal{PT}$-symmetric square well: $V(x) = \mathrm{i}Z\,\mathrm{sgn}\,x$

    2001 Znojil, 2001 Znojil and Lévai, ...
  - non-perturbative approach: unbounded potential
    $V(x) = \mathrm{i}x^3$

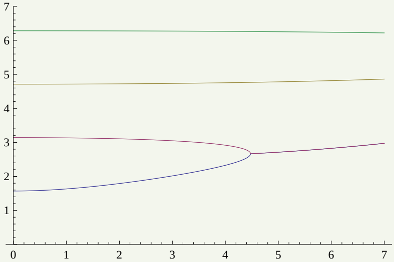
    1998, Bender, Boettcher, 2003 Dorey, Dunning, Tateo, ...
  - relatively (form) bounded perturbations: $\mathcal{PT}$-symmetric point interactions (boundary conditions)
    two δ potentials with complex coupling, $\mathcal{PT}$-symmetric Robin boundary conditions

    2002 Albeverio, Fei, Kurasov, 2005 Albeverio, Kuzhel, 2005 Jakubský, Znojil, 2009 Albeverio, Gunther, Kuzhel, 2006 Krejčiřík, Bíla, Znojil, 2008 Borisov, Krejčiřík, 2010 Krejčiřík, Siegl, ...

Reality of the spectrum and metric operator

Spectrum

- $$\left.\begin{array}{l} \mathcal{PT}-\text{symmetry } (\mathcal{PT})H \subset H(\mathcal{PT}) \\ \text{pseudo}-\text{Hermiticity } H^* = \eta^{-1}H\eta \end{array}\right\} \Rightarrow \lambda \in \sigma(H) \Leftrightarrow \overline{\lambda} \in \sigma(H)$$
- not sufficient for real spectrum, only complex conjugated pairs
- 1D systems: the crossing of real eigenvalues is necessary to produce complex conjugated pair



Metric operator

Metric operator

- $\Theta H = H^* \Theta$
  - $\Theta, \Theta^{-1} \in \mathscr{B}(\mathcal{H})$
  - $\Theta^* = \Theta$
  - $\Theta > 0$
- necessary condition: $\sigma(H) \subset \mathbb{R}$

Existence of metric operator

- H possesses a metric operator Θ
- H is self-adjoint in $\langle \cdot, \Theta \cdot \rangle$
- H is similar to a self-adjoint operator $h = \varrho^{-1} H \varrho = h^*, \quad \Theta = \varrho \varrho^*$
- H possesses a $\mathcal{C}$-symmetry: $\mathcal{C}^2 = I, \ \eta\mathcal{C} > 0, \ \mathcal{C}H = H\mathcal{C}$

Metric operator

Operators with discrete spectrum

- H with discrete spectrum: eigenfunctions $\{\psi_n\}$ form a Riesz basis
- $\Theta = \text{s--}\lim_{N\to\infty} \sum_{j=1}^{N} c_j \langle \phi_j, \cdot \rangle \phi_j$,

  where ϕ_j are eigenfunctions of H^* and $m < c_j < M$

Examples

- existence results: perturbation theory for spectral operators
- few explicit examples:
  - point interactions

    2005 Albeverio, Kuzhel, 2008 Siegl
  - $\mathcal{PT}$-symmetric Robin b.c.

    2006 Krejčiřík, Bíla, Znojil, 2008 Krejčiřík, 2010 Krejčiřík, Siegl, Železný

$\mathcal{PT}$-symmetric Robin boundary conditions

1D model

- $\mathcal{H} = L^2\big((-a,a),\mathrm{d}x\big)$
- $H = -\dfrac{d^2}{dx^2}$
- $\mathrm{Dom}(H) = W^{2,2}\big((-a,a)\big)$ + boundary conditions
- $\psi'(-a) + (\mathrm{i}\alpha - \beta)\psi(-a) = 0, \quad \psi'(a) + (\mathrm{i}\alpha + \beta)\psi(a) = 0$

  $\alpha, \beta \in \mathbb{R}$

1D model - $\mathcal{PT}$-symmetry

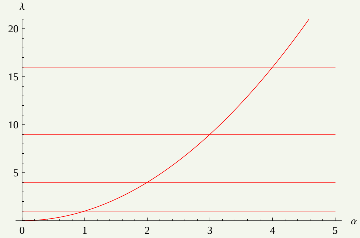
- $\forall \psi \in \mathrm{Dom}(H), \quad \psi \in \mathrm{Dom}(H) \Leftrightarrow \mathcal{PT}\psi \in \mathrm{Dom}(H)$
- $\forall \psi \in \mathrm{Dom}(H), \quad H\mathcal{PT}\psi = \mathcal{PT}H\psi$
- $H(\alpha,\beta)^* = H(-\alpha,\beta)$
- $H = \mathcal{P}H^*\mathcal{P}$, $H = \mathcal{T}H^*\mathcal{T}$

$\mathcal{PT}$-symmetric Robin boundary conditions

Spectrum of 1D model: I. $\beta = 0$

- $\psi'(-a) + i\alpha\psi(-a) = 0, \quad \psi'(a) + i\alpha\psi(a) = 0$
- $\sigma(H) = \sigma_d(H) \subset \mathbb{R}$, crossings

Spectrum of 1D model: I. $\beta = 0$



$$\sigma(H) = \{\alpha^2\} \cup \left\{\left(\tfrac{n\pi}{2a}\right)^2\right\}_{n\in\mathbb{N}}$$

$$\psi_0(x) = e^{-i\alpha x}$$

$$\psi_n(x) = \cos\left(\tfrac{n\pi}{2a}x\right) - i\alpha\tfrac{2a}{n\pi}\sin\left(\tfrac{n\pi}{2a}x\right)$$

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Spectrum of 1D model: I. $\beta = 0$, metric operator

- $\Theta H = H^* \Theta$
- $\Theta = I + \phi_0 \langle \phi_0, \cdot \rangle + \Theta_0 + \mathrm{i}\alpha\Theta_1 + \alpha^2 \Theta_2$,

  where

  - $\phi_0 = \sqrt{\frac{1}{2a} \exp(\mathrm{i}\alpha x)}$,
  - $(\Theta_0 \psi)(x) = -\frac{1}{2a} (J\psi)(2a)$,
  - $(\Theta_1 \psi)(x) = 2(J\psi)(x) - \frac{x}{2a}(J\psi)(2a) - \frac{1}{d}(J^2\psi)(2a)$,
  - $(\Theta_2 \psi)(x) = -(J^2\psi)(x) + \frac{x}{2a}(J^2\psi)(2a)$,
  - with $(J\psi)(x) = \int_{-a}^{x} \psi$. 2006 Krejčiřík, Bíla, Znojil

- $\Theta = I + K$,

  where K is an integral operator with kernel

  $K(x, y) =$

  $$\frac{e^{\mathrm{i}\alpha(x-y)} - 1}{2a} + \mathrm{i}\frac{\alpha(y-x)}{2a} - \alpha^2 \frac{xy}{2a} + \begin{cases} -\mathrm{i}\alpha + \alpha^2 x, & x < y \\ \mathrm{i}\alpha + \alpha^2 y, & y < x \end{cases}$$

  2010 Krejčiřík, Siegl, Železný

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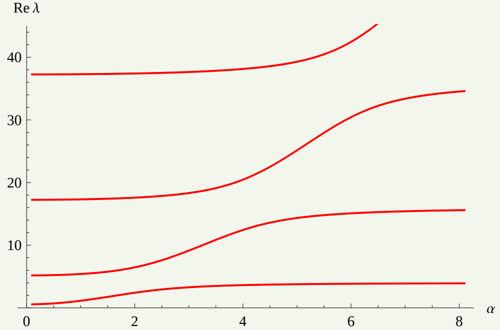
Spectrum of 1D model: II. $\beta > 0$

- $\psi'(-a) + (\mathrm{i}\alpha - \beta)\psi(-a) = 0, \quad \psi'(a) + (\mathrm{i}\alpha + \beta)\psi(a) = 0$
- $\sigma(H) = \sigma_d(H) \subset \mathbb{R}, \quad (k^2 - \alpha^2 - \beta^2)\sin 2ka - 2\beta k \cos 2ka = 0$
- metric operator exists for every α, β, no crossings

Spectrum of 1D model: II. $\beta > 0$



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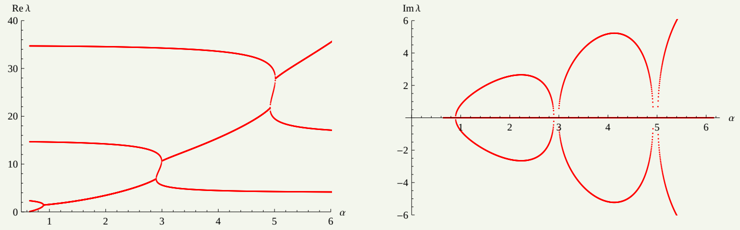
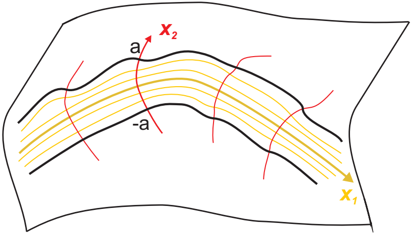
Spectrum of 1D model: III. $\beta < 0$

- $\psi'(-a) + (\mathrm{i}\alpha - \beta)\psi(-a) = 0, \quad \psi'(a) + (\mathrm{i}\alpha + \beta)\psi(a) = 0$
- $\sigma(H) = \sigma_d(H)$, $(k^2 - \alpha^2 - \beta^2)\sin 2ka - 2\beta k \cos 2ka = 0$
- either one or any complex conjugated pair
- known localization: $\Re\lambda$ in neighborhood of $\alpha^2 + \beta^2$
- metric operator exists if $\sigma(H) \subset \mathbb{R}$

Spectrum of 1D model: III. $\beta < 0$



$\mathcal{PT}$-symmetric models in curved manifolds



Metric tensor g

$$g = \begin{pmatrix} f(x_1, x_2) & 0 \\ 0 & 1 \end{pmatrix}$$

$|g| = det(g)$

Jacobi equation

$\partial_2^2 f + K f = 0$

$f(\cdot, 0) = 1, \; \partial_2 f(\cdot, 0) = k$

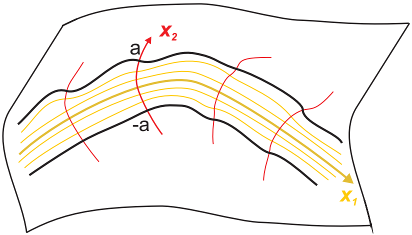
Laplace-Beltrami operator

$H = -|g|^{-1/2} \partial_i |g|^{1/2} g^{ij} \partial_j$ in $L^2\big((-\pi, \pi) \times (-a, a), \mathrm{d}\Omega\big)$

$\mathrm{Dom}(H) = W^{2,2}$ + boundary conditions

$\mathrm{d}\Omega = |g|^{1/2} \mathrm{d}x_1 \mathrm{d}x_2$

$\mathcal{PT}$-symmetric models in curved manifolds



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$$\mathrm{Dom}(H) = W^{2,2} + \text{boundary conditions}$$

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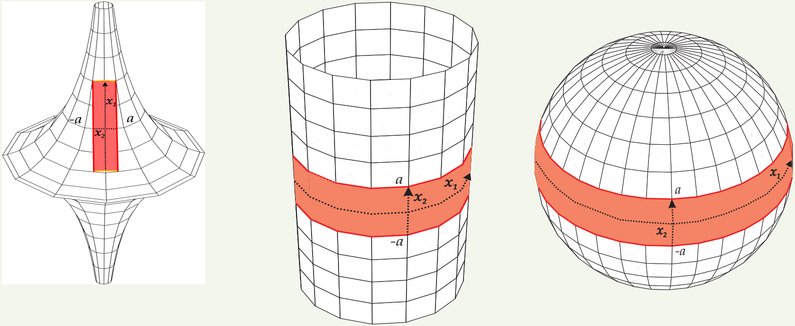
$\mathcal{PT}$-symmetric models in curved manifolds

Strips in curved manifolds



Boundary conditions

$$\partial_2 \Psi(x_1, a) + (\mathrm{i}\alpha + \beta)\Psi(x_1, a) = 0$$

$$\partial_2 \Psi(x_1, -a) + (\mathrm{i}\alpha - \beta)\Psi(x_1, -a) = 0$$

2010 Krejčiřík, Siegl

$\mathcal{PT}$-symmetric models in curved manifolds

Strips in curved manifolds: non-constant curvature and interaction

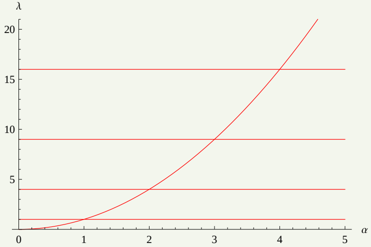
- H: m-sectorial operator (under suitable assumptions on α, f)
- $\sigma(H) = \sigma_d(H)$
- H: $\mathcal{PT}$-symmetric, $\mathcal{P}$-self-adjoint if $f(x_1, x_2) = f(x_1, -x_2)$
- $(\mathcal{P}\Psi)(x_1, x_2) := \Psi(x_1, -x_2)$

Strips in curved manifolds: constant curvature and interaction

- solvable models $H_{(K)} = \bigoplus_{m \in \mathbb{Z}} H^m_{(K)}$
- $H^m_{(0)} = -\dfrac{d^2}{dx^2} + m^2, \qquad \psi'(\pm a) + \mathrm{i}\alpha\psi(\pm a) = 0$
- $H^m_{(+1)} = -\dfrac{d^2}{dx^2} + V^m_{(+1)}, \quad \psi'(\pm a) + (\mathrm{i}\alpha \pm \beta)\psi(\pm a) = 0, \ \beta > 0$
- $H^m_{(-1)} = -\dfrac{d^2}{dx^2} + V^m_{(-1)}, \quad \psi'(\pm a) + (\mathrm{i}\alpha \pm \beta)\psi(\pm a) = 0, \ \beta < 0$
- eigenfunctions form Riesz basis

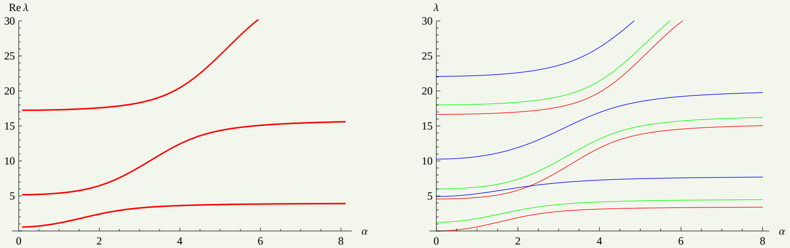
$\mathcal{PT}$-symmetric models in curved manifolds

Cylinder, zero curvature



$\mathcal{PT}$-symmetric models in curved manifolds

Sphere, positive curvature

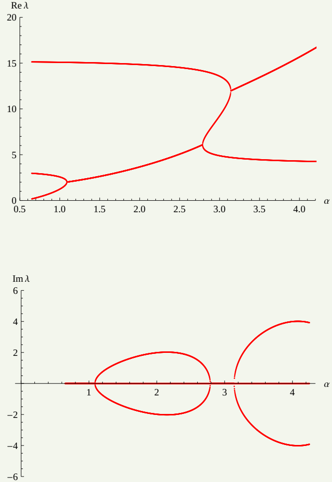


Sphere, positive curvature, spectrum

- for every m: all eigenvalues with $\Re\lambda > \Lambda_m$ are real
- conjecture (numerics): all eigenvalues are real

$\mathcal{PT}$-symmetric models in curved manifolds

Sphere, negative curvature



Physical interpretation

Alternative interpretation

- usual interpretation based on $\langle \cdot, \Theta \cdot \rangle$, ϱ
- can we find alternative interpretation?
- perfect transmission energies for scattering

Scattering, perfect transmission

- scattering on real line, potential V with support in $(-a, a)$
- solutions of Schrödinger equation

$$\psi(x) = \begin{cases} e^{ikx} + Re^{-ikx}, & x < -a \\ Te^{ikx}, & x > a \end{cases}$$

- we look for such k that $R = 0$, *i.e.* perfect transmission

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Perfect transmission

Scattering, perfect transmission

- inside $(-a, a)$

$$\begin{cases} -\psi''(x) + V(x)\psi(x) = k^2\psi(x), & x \in (-a, a) \\ \psi'(\pm a) - \mathrm{i}\, k\, \psi(\pm a) = 0 \end{cases}$$

-

$$\begin{cases} -\psi''(x) + V(x)\psi(x) = \mu(\alpha)\psi(x), & x \in (-a, a) \\ \psi'(\pm a) - \mathrm{i}\, \alpha\, \psi(\pm a) = 0 \end{cases}$$

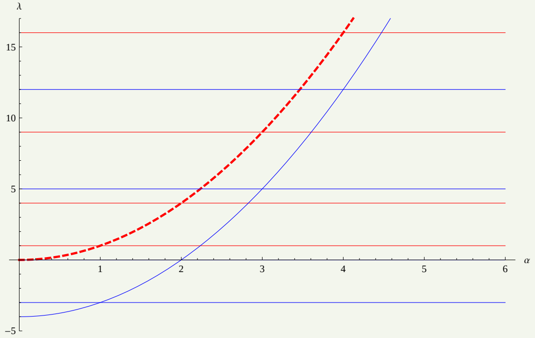
- perfect transmission energies $\mu(\alpha_*)$

$$\mu(\alpha_*) = \alpha_*^2$$

Perfect transmission - examples

Square well

- $V(x) = -V_0\,\chi_{[-a,a]}(x)$, with $V_0 > 0$
- $k_*^2 = \left(\frac{n\pi}{2a}\right)^2 - V_0$
- if $V_0 = 0$, then $k_*^2 \in \mathbb{R}^+$

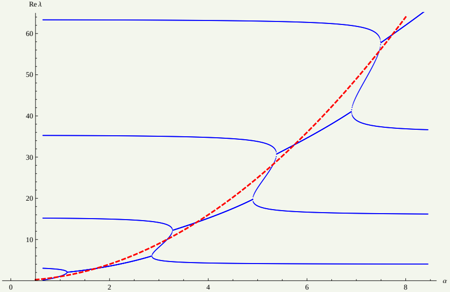


Perfect transmission - examples

Two steps potential

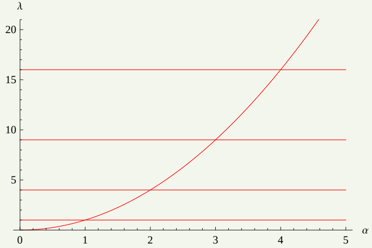
- $V(x) = \dfrac{\beta}{\varepsilon}\ \left(\chi_{[-a,-a+\varepsilon]}(x) + \chi_{[a,a-\varepsilon]}(x)\right)$

$\beta = -0.1, \varepsilon = 0.1, a = \pi/4$



Exceptional points

$\mathcal{PT}$-symmetric Robin boundary conditions



- crossings - $\dim \operatorname{Ker}(\Theta) = 1$
- 2x2 Jordan block of H

Exceptional points

$\mathcal{PT}$-symmetric irregular boundary conditions

- parametrization of $\mathcal{PT}$-symmetric b.c. 2002 Albeverio, Fei, Kurasov
- $\mathcal{PT}$-symmetric b.c. are strongly regular except one case (irregular)
- $H = -\dfrac{\mathrm{d}^2}{\mathrm{d}x^2}$ on $L_2(-1,1)$
  $\psi(-1) = \psi(1) = 0, \quad \psi(0+) = e^{\mathrm{i}\tau}\psi(0-)$ and $\psi'(0+) = e^{-\mathrm{i}\tau}\psi'(0-)$
- irregular for $\tau = \pm\pi/2$: $\sigma(H) = \mathbb{C}$ 2005 Albeverio, Kuzhel
- for $\tau \neq \pm\pi/2$: $\sigma(H) = \left\{\left(\frac{n\pi}{2}\right)^2\right\}$
  - $\Theta = I - \mathrm{i}\sin\tau P_{sgn}\mathcal{P}$
  - $\dim \operatorname{Ker}(\Theta) = \infty$ for $\tau = \pm\pi/2$
- can we approximate irregular case with regular ones?
  - resolvent does not exists
  - strong graph limit: $H_{\pi/2} = \text{str. gr.} \lim H_n$
  - str. gr. limit preserves $\mathcal{PT}$-symmetry and $\mathcal{P}$-self-adjointness

Conclusions

- $\mathcal{PT}$-symmetric Robin boundary conditions - 1D models, strips in curved manifolds, $\mathcal{PT}$-symmetric waveguide

  2008 Borisov, Krejčiřík, 2008 Krejčiřík, Tater

- rich properties: degeneracies in the spectrum, complex conjugated pairs, explicit formula for metric operator, existence of metric operator
- alternative physical interpretation: perfect transmission energies
- exceptional points: irregular cases can be approximated with strong graph limit