$\mathcal{PT}$-symmetry vs. pseudo-Hermiticity

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Antilinear symmetry

Definition

Let $A \in \mathscr{L}(\mathcal{H})$. We say that A possesses an antilinear symmetry if there exists an antilinear bijective operator C and the relation

$$AC\psi = CA\psi$$

holds for all $\psi \in \mathrm{Dom}(A)$.

Proposition

Let $A \in \mathscr{L}(\mathcal{H})$ be a closed operator having an antilinear symmetry C. Then $\lambda \in \mathbb{C}$ is in the spectrum of A if and only if $\overline{\lambda}$ is in the spectrum of A. Moreover, this equivalence is valid also for the disjoint parts of spectrum, i.e. $\lambda \in \sigma_{p,c,r}(A) \Longleftrightarrow \overline{\lambda} \in \sigma_{p,c,r}(A)$.

Pseudo-Hermiticity

Definition

Let $A \in \mathscr{L}(\mathcal{H})$ be densely defined. A is called weakly pseudo-Hermitian, if there exists an operator η with properties

(i) $\eta, \eta^{-1} \in \mathscr{B}(\mathcal{H})$,

(ii) $A = \eta^{-1} A^* \eta$.

If η is self-adjoint then A is called pseudo-Hermitian.

Proposition

Let $A \in \mathscr{L}(\mathcal{H})$ be a weakly pseudo-Hermitian operator. Then A is closed.

Pseudo-Hermiticity

Proposition

Let $A \in \mathscr{L}(\mathcal{H})$ be a weakly-pseudo-Hermitian operator. Then point, continuous and residual spectrum $\sigma_{p,c,r}(A)$ of A and $\sigma_{p,c,r}(A^*)$ of A^* are equal.

Remark

Operators with antilinear symmetry are not closed in general, thus a trivial example of operator with antilinear symmetry without pseudo-Hermiticity is non-closed or even non-closable operator with antilinear symmetry.

J-self adjoint operators

Definition

Let $A \in \mathscr{L}(\mathcal{H})$ be densely defined. Let J be an antilinear isometric involution, i.e. $J^2 = I$ and $\langle Jx, Jy \rangle = \langle y, x \rangle$ for all $x, y \in \mathcal{H}$. A is called J-symmetric if $A \subset JA^*J$. A is called J-self-adjoint if $A = JA^*J$.

Lemma

Let A be a J-self-adjoint operator. Then
(i) $\dim(\mathrm{Ker}(A - \lambda)) = \dim(\mathrm{Ker}(A^* - \overline{\lambda}))$,
(ii) $\sigma_r(A) = \emptyset$.

Finite dimension

Lemma

Every $A \in \mathscr{L}(V_n)$ is similar to the J-self-adjoint operator, i.e. there exists invertible $X \in \mathscr{L}(V_n)$ such that XAX^{-1} is J-self-adjoint.

Proposition

Let $A \in \mathscr{L}(V_n)$. Then A is pseudo-Hermitian if and only if it possesses an antilinear symmetry.

Proof.

- Let A be pseudo-Hermitian, $A = \eta^{-1} A^* \eta$
  - $XAX^{-1} = A_J = JA_J^* J = J(X^{-1})^* A^* X^* J = J(X^{-1})^* \eta A \eta^{-1} X^* J$
  - $CA = AC$, $C := \eta^{-1} X^* J X$ antilinear symmetry

Proof.

- Let A be pseudo-Hermitian, $A = \eta^{-1} A^* \eta$
  - $XAX^{-1} = A_J = JA_J^* J = J(X^{-1})^* A^* X^* J = J(X^{-1})^* \eta A \eta^{-1} X^* J$
  - $CA = AC$, $C := \eta^{-1} X^* JX$ antilinear symmetry
- Let C be the antilinear symmetry of A
  - $A = C^{-1} X^{-1} J(X^{-1})^* A^* X^* JXC$
  - A is η-weakly-pseudo-Hermitian, $\eta := X^* JXC$
  - weak pseudo-Hermiticity and pseudo-Hermiticity are equivalent properties on V_n

$\square$

Antilinear symmetry without pseudo-Hermiticity

Example

- $\{e_n\}_{n=1}^{\infty}$ standard orthonormal basis of $\mathcal{H} = l_2(\mathbb{N})$, $e_n(m) = \delta_{mn}$
- $Te_n := e_{n-1}, \quad n \in \mathbb{N}, \quad e_0 := 0$
- $T^* e_n := e_{n+1}, \quad n \in \mathbb{N}$
- $T = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & 0 & \\ 0 & 0 & 0 & 1 & 0 & \\ 0 & 0 & 0 & 0 & 1 & \\ \vdots & & & & \ddots & \ddots \end{pmatrix}$

Antilinear symmetry without pseudo-Hermiticity

Example

- antilinear symmetry $\mathcal{T}$
- every $|\lambda| < 1$ is in the point spectrum $\sigma_p(T)$, $x_\lambda = \sum_{n=1}^{\infty} \lambda^{n-1} e_n$
- $\sigma_p(T^*) = \emptyset$, point spectrum of T^* is empty
- $\{\lambda \in \mathbb{C} | |\lambda| < 1\} \subset \sigma_r(T^*)$, residual spectrum is non-empty
- T is not pseudo-Hermitian, $\sigma_p(T) \neq \sigma_p(T^*)$

Pseudo-Hermiticity without antilinear symmetry

Example

- $\{e_i\}_{-\infty}^{\infty}$ orthonormal basis of $\mathcal{H} = l^2(\mathbb{Z})$, $e_n(m) = \delta_{mn}$
- $$Te_i := \begin{cases} \lambda_0 e_i + e_{i+1}, & i \geq 1, \\ 0, & i = 0, \\ \overline{\lambda}_0 e_{-1}, & i = -1, \\ \overline{\lambda}_0 e_i + e_{i+1}, & i < -1, \end{cases}$$
- $\lambda_0 \in \mathbb{C}, \ \operatorname{Im} \lambda_0 > \frac{1}{2}$

Pseudo-Hermiticity without antilinear symmetry

Example

- $$T = \begin{pmatrix} \ddots & & & & & & & \\ \ddots & \overline{\lambda_0} & 0 & 0 & 0 & 0 & & \\ & 1 & \overline{\lambda_0} & 0 & 0 & 0 & & \\ & 0 & 0 & 0 & 0 & 0 & & \\ & 0 & 0 & 0 & \lambda_0 & 0 & & \\ & 0 & 0 & 0 & 1 & \lambda_0 & & \\ & & & & & \ddots & \ddots & \end{pmatrix}$$

Pseudo-Hermiticity without antilinear symmetry

Example

- T is pseudo-Hermitian, $\mathcal{P}e_i := e_{-i}$, $T = \mathcal{P}T^*\mathcal{P}$
- $\overline{\lambda}_0 \in \sigma_p(T) = \sigma_p(T^*)$
- $\lambda_0 \in \sigma_r(T) = \sigma_r(T^*)$
- T has not any antilinear symmetry, $\lambda \in \sigma_p(T) \Leftrightarrow \overline{\lambda} \in \sigma_p(T^*)$

Conclusions and applications

- antilinear symmetry and pseudo-Hermiticity are not equivalent properties even for bounded operators
- the counterexamples are not spectral operators - point and residual spectra are not countable
- the equivalence can be shown for spectral operators of scalar type
  - proof based on J-self adjoint operators
  - generalization of previous works
    2002 Mostafazadeh JMP 43, 2002 Solombrino quant-ph/0203101,
    2002 Solombrino and Scolarici quant-ph/0211161
  - problem for general spectral operator is still open

Conclusions and applications

- non-empty residual spectrum!
  - physical meaning of the residual spectrum ?
  - antilinear symmetry together with pseudo-Hermiticity exclude non-empty residual spectrum
  - real numbers cannot be in the residual spectrum
- applications of equivalence
  - problem of boundedness of the inverse of metric Θ
  - quasi-Hermitian operators in Dieudonné's sense - Θ^{-1} is not bounded, complex values in the spectrum 1961 Quasi-Hermitian operators, Proceedings Of The International Symposium on Linear Spaces
  - boundedness of Θ^{-1} is guaranteed for pseudo-Hermitian operators with antilinear symmetry, if Θ exists + technical assumptions