$\mathcal{PT}$-symmetric models in curved manifolds

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Based on:

1. D. Krejčiřík, P. Siegl, $\mathcal{PT}$-symmetric models in curved manifolds, Journal of Physics A: Mathematical and Theoretical, 2010, to appear,
2. D. Krejčiřík, P. Siegl, and J. Železný, work in progress.

$\mathcal{PT}$-symmetry

Origins

- Hamiltonian $H = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \mathrm{i}x^3$ has real, positive, discrete spectrum [BeBo98]
- original hypothesis - the reality of spectrum due to $\mathcal{PT}$-symmetry
  - $[\mathcal{PT}, H] = 0$
  - parity $\mathcal{P}$, $(\mathcal{P}\psi)(x) = \psi(-x)$
  - complex conjugation $\mathcal{T}$, $(\mathcal{T}\psi)(x) = \overline{\psi}(x)$

Simple observations

- $\mathcal{PT}$-symmetry is not sufficient for real spectrum
- some $\mathcal{PT}$-symmetric operators are similar to self-adjoint or normal operators
  $\exists \varrho, \varrho^{-1} \in \mathscr{B}(\mathcal{H})$: $\varrho H \varrho^{-1}$ is self-adjoint or normal

Aims

? spectrum of $\mathcal{PT}$-symmetric operator?

? similarity to self-adjoint or normal operator?

[BeBo98] 1998 Bender and Boettcher, *Physical Review Letters* 80.

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Interpretation of $\mathcal{PT}$-symmetric models

Recent applications in physics

- experimental results in optics [KlGuMo08], [RuMaGaChSeKi10], [Lo10], ...
- superconductivity [RuStMa07], [RuStZu10] , solid state [BeFlKoSh08]
- electromagnetism [RuDeMu05], [Mo09], nuclear physics [ScGeHa92]

Quantum mechanics: similarity to self-adjoint operator

- let $h := \varrho H \varrho^{-1}$, $h^* = h$
- quasi-Hermiticity (equivalent to similarity to s-a operator) $\exists \Theta, \Theta^{-1} \in \mathscr{B}(\mathcal{H}), \Theta > 0 : \Theta H = H^* \Theta$ [Di61]
- $\Theta = \varrho^* \varrho$, H is self-adjoint in $\langle \cdot, \Theta \cdot \rangle$
- for operators with discrete spectrum: equivalent to the Riesz basicity of eigenvectors of H and H^*

[BeFlKoSh08] 2008 Bendix, Fleischmann, Kottos, and Shapiro, *Physical Review Letters* 103,
[Di61] 1961 Dieudonné, *Proceedings Of The International Symposium on Linear Spaces*,
[KlGuMo08] 2008 Klaiman, Günther, and Moiseyev, *Physical Review Letters* 101,
[Lo10] 2010 Longhi, *Physical Review Letters* 105,
[Mo09] 2009 Mostafazadeh, *Physical Review Letters* 102,
[RuStMa07] 2007 Rubinstein, Sternberg, and Ma, *Physical Review Letters* 99,
[RuStZu10] 2010 Rubinstein, Sternberg, and Zumbrun, *Archive for Rational Mechanics and Analysis* 195,
[RuDeMu05] 2005 Ruschhaupt, Delgado, Muga, *Journal of Physics A: Mathematical and General* 38,
[RuMaGaChSeKi10] 2010 Rüter, Makris, El-Ganainy, Christodoulides, Segev, and Kip, *Nature Physics* 6,
[ScGeHa92] 1992 Scholtz, Geyer, and Hahne, *Annals of Physics* 213.

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Similarity to self-adjoint operator

Other symmetries

- $\mathcal{PT}$-symmetric operators: often $\mathcal{P}$ and $\mathcal{T}$-self-adjoint
  $H^* = \mathcal{P}H\mathcal{P} \quad H^* = \mathcal{T}H\mathcal{T}$

"Metric" and $\mathcal{C}$ operator

- metric operator Θ: $\Theta H = H^*\Theta$, $\Theta, \Theta^{-1} \in \mathscr{B}(\mathcal{H}), \Theta > 0$.
- operators with discrete spectrum:

$$\Theta = \operatorname*{s-lim}_{N\to+\infty} \sum_{n=1}^{N} c_n \langle \phi_n, \cdot \rangle \phi_n,$$

  with $H^*\phi_n = E_n\phi_n$, $\|\phi_n\| = 1$, $0 < m < c_n < M < +\infty$.
- $\mathcal{C}$ operator: $\mathcal{C} \in \mathscr{B}(\mathcal{H})$, $\mathcal{C}^2 = I$, $\mathcal{PC} > 0$, and $H\mathcal{C} = \mathcal{C}H$
- $\mathcal{PC}$ is a metric operator
- $$\mathcal{C} = \operatorname*{s-lim}_{N\to+\infty} \sum_{n=1}^{N} d_n \langle \phi_n, \cdot \rangle \psi_n,$$

  with $H\psi_n = E_n\psi_n$, $\|\psi_n\| = 1$, d_n restricted by $\mathcal{C}^2 = I$.

Similarity to self-adjoint operator

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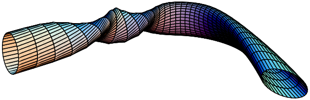
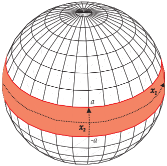
  with $H\psi_n = E_n\psi_n$, $\|\psi_n\| = 1$, d_n restricted by $\mathcal{C}^2 = I$.

Geometric effects in self-adjoint models

Waveguides

- bending - acts as an attractive interaction [ExSe89], [GoJa92], . . .
- twisting - acts as a repulsive interaction [EkKoKr08]





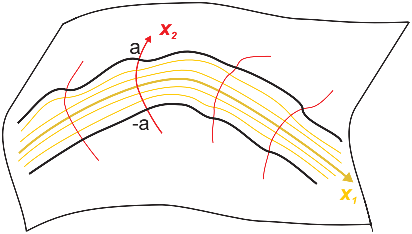
Quantum strips on surfaces [Kr02]

- positive curvature - acts as an attractive interaction
- negative curvature - acts as a repulsive interaction

What is the effect of curvature in $\mathcal{PT}$-symmetric models?

[ExSe89] 1989 Exner, Šeba, *Journal of Mathematical Physics* 30,
[GoJa92] 1992 Goldstone, Jaffe, *Physical Review B* 45,
[EkKoKr08] 2008 Ekholm, Kovařík, Krejčiřík, *Archive for Rational Mechanics and Analysis* 188
[Kr02] 2002 Krejčiřík, *Journal of Geometry and Physics* 45

Fermi coordinates and Hamiltonian



Metric tensor g

$$g = \begin{pmatrix} f(x_1, x_2) & 0 \\ 0 & 1 \end{pmatrix}$$

$$|g| = \det(g)$$

Jacobi equation

$$\partial_2^2 f + K f = 0$$
$$f(\cdot, 0) = 1, \quad \partial_2 f(\cdot, 0) = k$$

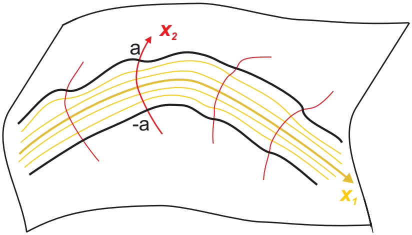
Laplace-Beltrami operator

$$H = -|g|^{-1/2}\partial_i |g|^{1/2} g^{ij} \partial_j \text{ in } L^2\big((-\pi,\pi)\times(-a,a), \mathrm{d}\Omega\big)$$

$$\mathrm{Dom}(H) = W^{2,2} + \text{boundary conditions}$$

$$\mathrm{d}\Omega = |g|^{1/2} \mathrm{d}x_1 \mathrm{d}x_2$$

Fermi coordinates and Hamiltonian



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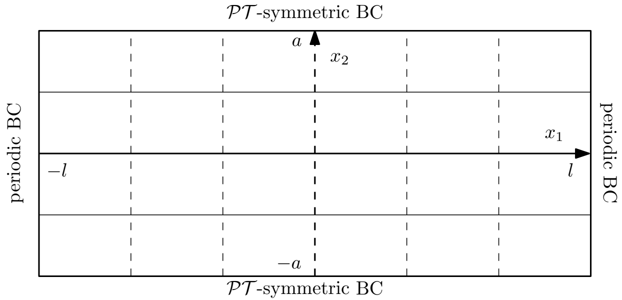
$\mathcal{PT}$-symmetric boundary conditions

Strip-like geometries



$\mathcal{PT}$-symmetric boundary conditions

Classification of $\mathcal{PT}$-symmetric b. c.: separated and connected [AlFeKu02]

$$\partial_2\Psi(x_1,a) + (\mathrm{i}\alpha(x_1) + \beta(x_1))\Psi(x_1,a) = 0$$
$$\partial_2\Psi(x_1,-a) + (\mathrm{i}\alpha(x_1) - \beta(x_1))\Psi(x_1,-a) = 0$$

[AlFeKu02] 2002 Albeverio, Fei, and Kurasov, *Letters in Mathematical Physics* 59

General results

Theorem

Let $\alpha, \beta, \in W^{1,\infty}\big((-\pi,\pi)\big)$ *and* $f(\cdot, x_2) \in W^{1,\infty}\big((-\pi,\pi)\big)$ *for every* $x_2 \in (-a,a)$*.*
Then
1. H *is an m-sectorial operator*
2. the adjoint operators H^* *can be found as*

$$H^*(\alpha, \beta) = H(-\alpha, \beta)$$

3. the resolvent of H *is compact.*

Proposition

Let smoothness assumptions be valid and let $f(x_1, x_2) = f(x_1, -x_2)$*. Then* H *is*
1. $\mathcal{PT}$*-symmetric:* $\mathcal{PT}H \subset H\mathcal{PT}$,
2. $\mathcal{P}$*-self-adjoint:* $H^* = \mathcal{P}H\mathcal{P}$,
3. $\mathcal{T}$*-self-adjoint:* $H^* = \mathcal{T}H\mathcal{T}$,
where $(\mathcal{P}\psi)(x_1,x_2) := \psi(x_1,-x_2)$ *and* $(\mathcal{T}\psi)(x_1,x_2) := \overline{\psi(x_1,x_2)}$*.*

Corollary

$\lambda \in \sigma(H) \Leftrightarrow \overline{\lambda} \in \sigma(H)$

Solvable models

Constant curvature

$$K = 0$$

$$g_{(0)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\mathrm{d}\Omega_{(0)} = \mathrm{d}x_1 \mathrm{d}x_2$$

$$K = 1$$

$$g_{(+1)} = \begin{pmatrix} \cos^2 x_2 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\mathrm{d}\Omega_{(+1)} = \cos x_2 \mathrm{d}x_1 \mathrm{d}x_2$$

$$K = -1$$

$$g_{(-1)} = \begin{pmatrix} \cosh^2 x_2 & 0 \\ 0 & 1 \end{pmatrix}$$

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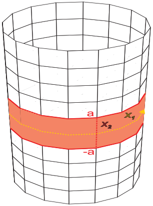

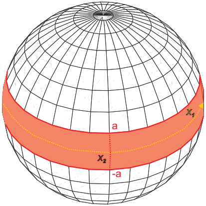

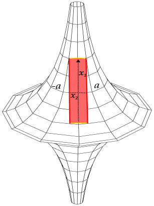

Constant interaction

$$\partial_2 \Psi(x_1, a) + (\mathrm{i}\alpha + \beta)\Psi(x_1, a) = 0$$

$$\partial_2 \Psi(x_1, -a) + (\mathrm{i}\alpha - \beta)\Psi(x_1, -a) = 0$$

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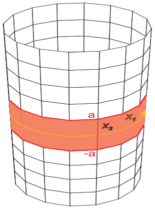

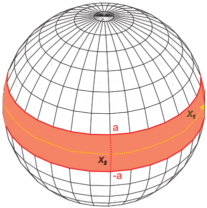

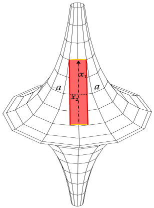

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Solvable models

Operators

$$H_{(K)} = \begin{cases} -\dfrac{1}{\cosh^2 x_2}\partial_1^2 - \dfrac{1}{\cosh x_2}\partial_2 \cosh x_2 \partial_2 & \text{if} \quad K = -1\,, \\ -\partial_1^2 - \partial_2^2 & \text{if} \quad K = 0\,, \\ -\dfrac{1}{\cos^2 x_2}\partial_1^2 - \dfrac{1}{\cos x_2}\partial_2 \cos x_2 \partial_2 & \text{if} \quad K = 1\,. \end{cases}$$

Partial wave decomposition

$$H_{(K)} = \bigoplus_{m\in\mathbb{Z}} H_{(K)}^m B^m,$$

with $B^m\Psi(x_1,x_2) := \langle \phi_m, \Psi(\cdot,x_2)\rangle_{(-\pi,\pi)}\phi_m, \quad \phi_m(x_1) := (2\pi)^{-1/2}e^{imx_1},$

$$H_{(K)}^m := \begin{cases} -\dfrac{1}{\cosh x_2}\partial_2 \, \cosh x_2 \, \partial_2 + \dfrac{m^2}{\cosh^2 x_2} & \text{if} \quad K = -1\,, \\ -\partial_2^2 + m^2 & \text{if} \quad K = 0\,, \\ -\dfrac{1}{\cos x_2}\partial_2 \, \cos x_2 \, \partial_2 + \dfrac{m^2}{\cos^2 x_2} & \text{if} \quad K = 1\,, \end{cases}$$

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Partial wave decomposition

Proposition

$D := \bigcap_{m\in\mathbb{Z}} \varrho\left(H_{(K)}^m\right)$ is non-empty and $D \subset \varrho\left(H_{(K)}\right)$. For every $z \in D$,

$$(H_{(K)} - z)^{-1} = \bigoplus_{m\in\mathbb{Z}} \left(H_{(K)}^m - z\right)^{-1} B^m.$$

Corollary

$$\sigma\left(H_{(K)}\right) = \bigcup_{m\in\mathbb{Z}} \sigma\left(H_{(K)}^m\right)$$

Proposition

For every $m \in \mathbb{Z}$ and $K \in \{-1, 0, 1\}$:

1. *The families of operators $H_{(K)}^m(\alpha, \beta)$ are holomorphic with respect to parameters α, β entering the boundary conditions.*
2. *The spectrum of $H_{(K)}^m$ is discrete consisting of simple eigenvalues (i.e. the algebraic multiplicity being one), except of finitely many eigenvalues of algebraic multiplicity two and geometric multiplicity one that can appear for particular values of α, β.*

Partial wave decomposition

Proposition

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Zero curvature

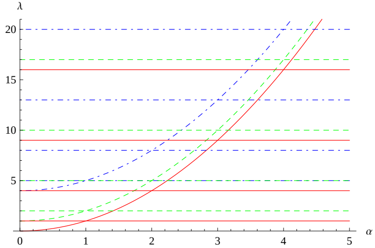
$H^m_{(0)}$ eigenvalue problem

$$\begin{cases} -\psi'' + m^2\psi = \lambda\psi, \\ \psi'(\pm a) + (\mathrm{i}\alpha \pm \beta)\psi(\pm a) = 0 \end{cases}$$

Spectrum of $H^m_{(0)}$, $\beta = 0$

[KrBiZn06]

$$\lambda_{j,m} = \begin{cases} \alpha^2 + m^2, \\ \left(\frac{j\pi}{2a}\right)^2 + m^2 \end{cases}$$



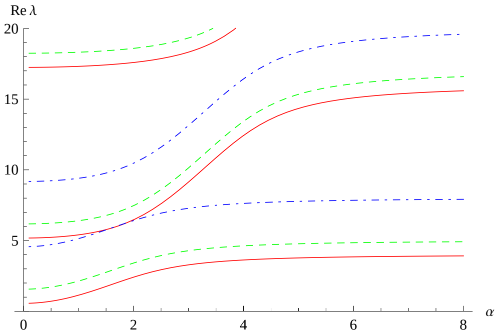
[KrBiZn06] 2006 Krejčiřík, Bíla, Znojil, *Journal of Physics A: Mathematical and General* 39

Zero curvature

Spectrum of $H^m_{(0)}$, $\beta > 0$

$$(k^2 - \alpha^2 - \beta^2)\sin 2ka - 2\beta k \cos 2ka = 0$$

only real eigenvalues



Zero curvature

Spectrum of $H^m_{(0)}$, $\beta < 0$

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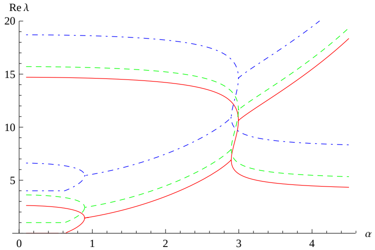
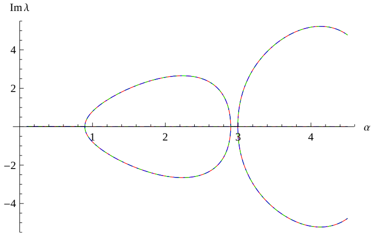
pairs of complex conjugated eigenvalues appear





Positive curvature

$H^m_{(+1)}$ eigenvalue problem

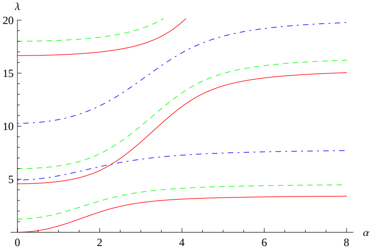
$$\begin{cases} -\psi''(x) + \tan x \psi'(x) + \frac{m^2}{\cos^2 x}\psi(x) = \lambda\psi(x), \\ \psi'(\pm a) + \mathrm{i}\alpha\psi(\pm a) = 0 \end{cases}$$

Spectrum of $H^m_{(+1)}$

$$\begin{vmatrix} \dot{P}_n^{(m)}(b) + \mathrm{i}\alpha P_n^{(m)}(b) & \dot{Q}_n^{(m)}(b) + \mathrm{i}\alpha Q_n^{(m)}(b) \\ \dot{P}_n^{(m)}(-b) + \mathrm{i}\alpha P_n^{(m)}(-b) & \dot{Q}_n^{(m)}(-b) + \mathrm{i}\alpha Q_n^{(m)}(-b) \end{vmatrix} = 0,$$

$b = \sin a, \;\; n = 1/2(-1+\sqrt{1+4\lambda})$

all but finitely many eigenvalues are real, bounded perturbation of $H^m_{(0)}$ with $\beta > 0$



Negative curvature

$H^m_{(-1)}$ eigenvalue problem

$$\begin{cases} -\psi''(x) - \tanh x\psi'(x) + \frac{m^2}{\cosh^2 x}\psi(x) = \lambda\psi(x), \\ \psi'(\pm a) + \mathrm{i}\alpha\psi(\pm a) = 0 \end{cases}$$

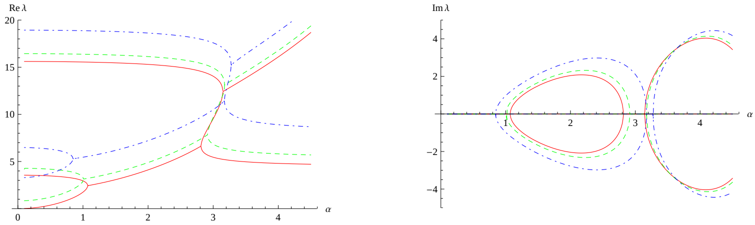
Spectrum of $H^m_{(-1)}$

$$\begin{vmatrix} \left(\dot{\frac{P_k^{(l)}}{\sqrt{\cosh}}}\right)(c) + \mathrm{i}\alpha\frac{P_k^{(l)}}{\sqrt{\cosh}}(c) & \left(\dot{\frac{Q_k^{(l)}}{\sqrt{\cosh}}}\right)(c) + \mathrm{i}\alpha\frac{Q_k^{(l)}}{\sqrt{\cosh}}(c) \\ \left(\dot{\frac{P_k^{(l)}}{\sqrt{\cosh}}}\right)(-c) + \mathrm{i}\alpha\frac{Q_k^{(l)}}{\sqrt{\cosh}}(-c) & \left(\dot{\frac{Q_k^{(l)}}{\sqrt{\cosh}}}\right)(-c) + \mathrm{i}\alpha Q_k^{(l)}(-c) \end{vmatrix} = 0,$$

$c = \tanh a, \quad k = \sqrt{1-4\lambda}, \quad l = m\mathrm{i} - 1/2$

pairs of complex conjugated eigenvalues appear,
bounded perturbation of $H^m_{(0)}$ with $\beta < 0$

Negative curvature



Similarity to self-adjoint / normal operators

Proposition

For every $m \in \mathbb{Z}$ and $K \in \{-1, 0, 1\}$: If all the eigenvalues are simple, then

1. *the eigenvectors of $H^m_{(K)}$ form a Riesz basis in $L^2((-a,a), \mathrm{d}\nu_{(K)})$,*
2. *$H^m_{(K)}$ is similar to a normal operator, i.e., for every m there exists a bounded operator ϱ with bounded inverse such that $\varrho H^m_{(K)} \varrho^{-1}$ is normal,*
3. *if moreover all eigenvalues are real, then $H^m_{(K)}$ is similar to a self-adjoint operator, i.e., $\varrho H^m_{(K)} \varrho^{-1}$ is self-adjoint.*
4. *Let us denote by $\left\{\psi_{i,m}\right\}_{i \in \mathbb{N}}$ the eigenfunctions of $H^m_{(K)}$. The set of eigenfunctions $\mathscr{B} := \left\{\phi_m \psi_{i,m}\right\}_{m \in \mathbb{Z}, i \in \mathbb{N}}$, where ϕ_m were introduced before, forms a Riesz basis of $L^2(\Omega_0, G)$.*

Quasi-Hermiticity, $K = 0$ and $\beta = 0$

Metric operator Θ

- $\Theta, \Theta^{-1} \in \mathscr{B}(\mathcal{H}), \Theta > 0, \quad \Theta H_{(0)}^m = \left(H_{(0)}^m\right)^* \Theta$
- first closed formulae in [KrBiZn06], [Kr08]
- $\Theta = I + L$, L is an integral operator with kernel
- $$\mathcal{L}(x,y) = \frac{e^{i\alpha(x-y)} - 1}{2a} + i\frac{\alpha}{2a}(y-x) - \frac{\alpha^2}{2a}xy - \frac{\alpha^2}{2}(x+y) + \frac{\alpha^2}{2}a + \left(-i\alpha + \frac{\alpha^2}{2}(x-y)\right)\operatorname{sgn}(y-x)$$

$\mathcal{C}$ operator

- $\mathcal{C} \in \mathscr{B}(\mathcal{H}), \mathcal{C}^2 = I, \mathcal{PC} > 0, \quad \mathcal{C} H_{(0)}^m = H_{(0)}^m \mathcal{C}.$
- $\mathcal{C} = \mathcal{P} + M$, M is an integral operator with kernel
- $\mathcal{M}(x,y) = \alpha e^{-i\alpha(x+y)}\left(\tan(\alpha a) - i\cos(\alpha a)\operatorname{sgn}(x+y)\right)$

[KrBiZn06] 2006 Krejčiřík, Bíla, Znojil, *Journal of Physics A: Mathematical and General* 39,
[Kr08] 2008 Krejčiřík, *Journal of Physics A: Mathematical and Theoretical* 41.

Conclusions

Results

- spectrum of Laplace-Beltrami operators subject to $\mathcal{PT}$-symmetric b.c.
- effects of curvature: positive curvature – real spectrum, negative curvature – pairs of complex eigenvalues
- similarity to self-adjoint (normal) operators – existence of metric and $\mathcal{C}$ operators
- closed formula for the metric and $\mathcal{C}$ operator for zero curvature

Open questions and further research

? reality of all eigenvalues for positive curvature

? non-constant curvature

? non-constant interaction functions

? unbounded strips, zero curvature [BoKr08]

? metric and $\mathcal{C}$ operators for non-zero curvature

? similar self-adjoint operators (partial results for zero curvature)

[BoKr08] 2008 Borisov, Krejčiřík, *Integral Equations Operator Theory*, 62