Krein spaces in de Sitter quantum theories

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Motivation

De Sitter solution of Einstein equations

- corresponds to the experimental observation of accelerated expansion of the Universe
- approximates the inflation period in the early Universe
- positive cosmological constant
- maximally symmetric solution
- $SO_0(1,4)$ invariance

Quantum field theory

- quantum elementary systems are associated with unitary irreducible representations of $SO_0(1,4)$
- classification of UIR 1961 Dixmier, 1963 Takahashi, ...
- unsolved problem - quantization of fields for $\Pi_{p,0}$

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Structure of the talk

- de Sitter basics
- origins of "zero-mode problem" and possible ways out
- Gupta-Bleuler like quantization
  - indecomposable representations on Krein space
  - Gupta-Bleuler triplet in the standard form
  - G-R-T construction
  - description of cohomology
- conclusions and references

de Sitter basics

Space-time and coordinates

- hyperboloid embedded in a 4+1-dimensional Minkowski space $\mathbb{M}_5$

$$M_H \equiv \{x \in \mathbb{M}_5;\ x^2 := x \cdot x = \eta_{\alpha\beta}\ x^\alpha x^\beta = -H^{-2}\},$$
$$\alpha, \beta = 0, 1, 2, 3, 4, \quad (\eta_{\alpha\beta}) = \mathrm{diag}(1, -1, -1, -1, -1)\,,$$
$$x := (x^0, \vec{x}, x^4) \ \text{ ambient coordinates}$$

- conformal coordinates

$$x = (H^{-1} \tan\rho, (H\cos\rho)^{-1}\, u)\,, \quad \rho \in (-\frac{\pi}{2}, \frac{\pi}{2}), \quad u \in S^3$$

De Sitter group $SO_0(1,4)$

- $SO_0(1,4)$ - ten parameters
- classification of representations using Casimir operators
- in Dixmier notation, parameters p, q:

$$\mathcal{C}_2 = (-p(p+1) - (q+1)(q-2))\mathbb{I},$$
$$\mathcal{C}_4 = (-p(p+1)q(q-1))\mathbb{I}$$

- p, q represent spin and mass
- our interest - discrete scalar representations $\Pi_{p,0}$ $p \in \mathbb{N}$

Wave equation and modes

Wave equation

- scalar representation $q = 0 \;\Rightarrow \mathcal{C}_4 = 0$
- wave equation $\mathcal{C}_2 = -p(p+1)\mathbb{I}$

  $\mathcal{C}_2$ is proportional to Laplace-Beltrami operator on dS space
- in conformal coordinates:

$$\Box = \frac{1}{\sqrt{g}}\partial_\nu \sqrt{g} g^{\nu\mu}\partial_\mu = H^2 \cos^4\rho \frac{\partial}{\partial\rho}(\cos^{-2}\rho \frac{\partial}{\partial\rho}) - H^2 \cos^2\rho\, \Delta_3$$

  - $\Delta_3 = \frac{\partial^2}{\partial\alpha^2} + 2\cot\alpha \frac{\partial}{\partial\alpha} + \frac{1}{\sin^2\alpha}\frac{\partial^2}{\partial\theta^2} + \cot\theta \frac{1}{\sin^2\alpha}\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\alpha \sin^2\theta}\frac{\partial^2}{\partial\phi^2}$
  - Δ_3 is Laplace operator on S^3
- solutions of wave equation - carrier space of the representation

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Wave equation and modes

Wave equation

- separation of variables

$$\phi(x) = \chi(\rho) D(u), \quad \rho \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \; u \in \mathrm{S}^3$$

- system of equations

$$\begin{aligned} [\Delta_3 + L(L+1)] D(u) &= 0, \\ \left(\cos^4 \rho \frac{d}{d\rho} \cos^{-2} \rho \frac{d}{d\rho} + L(L+1) \cos^2 \rho + (p+2)(1-p)\right) \chi(\rho) &= 0. \end{aligned}$$

Modes

- $D(u) = \mathrm{Y}_{Llm}(u) = C_{Ll} 2^l l! \, (\sin\alpha)^l \, C_{L-l}^{l+1}(\cos\alpha) \, Y_{lm}(\theta, \phi)$ for $(L, l, m) \in \mathbb{N}_0 \times \mathbb{N}_0 \times \mathbb{Z}$ with $0 \leq l \leq L$ and $-l \leq m \leq l$.
- $\chi(\rho) = e^{-i(L+1-p)\rho} (\cos\rho)^{1-p} \; {}_2F_1(-p, L-p+1; L+2; -e^{-2i\rho})$

Klein-Gordon inner product

- solutions of wave equation $\phi^p_{Llm}(x) = \chi^p_L(\rho) Y_{Llm}(u)$
- Klein-Gordon inner product

$$\langle \Phi_1, \Phi_2 \rangle = \frac{i}{H^2} \int_{\rho=0} \overline{\Phi_1(\rho, u)} \overleftrightarrow{\partial}_\rho \, \Phi_2(\rho, u) \, du \,,$$

where $du = \sin^2 \alpha \, \sin \theta \, d\alpha \, d\theta \, d\phi$ is the invariant measure on S^3

- Klein-Gordon inner product is dS invariant $\langle \pi(g)\cdot, \pi(g)\cdot \rangle = \langle \cdot, \cdot \rangle$
- we need $\langle \phi^p_{Llm}, \phi^p_{L'l'm'} \rangle = \delta_{LL'} \delta_{ll'} \delta_{mm'}$
- orthogonality is satisfied, normalization?

Normalization

- $\|\phi^p_{Llm}\|^2 = \frac{2^3}{H^2} \frac{\Gamma(L+p+2)}{(\Gamma(p+2))^2 \Gamma(L-p-1)}$
- for $p = 0$ and $L = 0$: $\|\phi^0_{000}\| = 0$
- for $p > 1$ we have $N = p(p+1)(2p+1)/6$ zero norm solutions $(L < p)$
- origin of so-called “zero-mode” problem
- no-go result by Allen, 1985
- we need non-degenerate, $SO_0(1,4)$-invariant set of modes

Set of modes

- for $p = 0$: $\phi_{000} \equiv \psi_g = const.$
- $\{\phi_{Llm}\}_{L>0}$ is not dS invariant
- $\{\phi_{Llm}\}_{L>0} \cup \{\psi_g\}$ is degenerate (for K-G product)
- possible ways out:
  - only $O(4)$-invariance Allen 1985
  - Gupta-Bleuler like quantization Gazeau, Renaud, Takook 2000

Construction of carrier space

- to obtain zero mode with non-zero K-G norm - add the second solution of wave equation ($L = 0$)

$$\psi_g = \frac{H}{2\pi}$$
$$\psi_s = -i\frac{H}{2\pi}\left(\rho + \frac{1}{2}\sin 2\rho\right)$$

- $\phi_0 := \psi_g + \psi_s/2$, $\langle \psi_g, \psi_s \rangle = 1 \ \Rightarrow \|\phi_0\| = 1$
- $\{\phi_{Llm}\}_{L>0} \cup \{\phi_0\}$ is non-degenerate and orthonormal
- $\{\phi_{Llm}\}_{L>0} \cup \{\phi_0\}$ is not dS invariant!
- it is necessary to include $\{\overline{\phi_{Llm}}\}_{L>0} \cup \{\overline{\phi_0}\}$
- $\|\overline{\phi_{Llm}}\| < 0, \|\overline{\phi_0}\| < 0$
- change of notation $\{\phi_{Llm}\}_{L\geq 0} \ \rightarrow \ \{\psi_n\}_{n\in\mathbb{N}_0}$

Construction of carrier space

- dS invariant, non-degenerate carrier space $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$

$$\mathcal{H}_+ := \{ \sum_{n\in\mathbb{N}_0} c_n \psi_n | \sum_{n\in\mathbb{N}_0} |c_n|^2 < \infty \}$$

$$\mathcal{H}_- := \{ \sum_{n\in\mathbb{N}_0} d_n \overline{\psi_n} | \sum_{n\in\mathbb{N}_0} |d_n|^2 < \infty \}$$

- subspaces

$$\begin{aligned}
\mathcal{N} &:= \mathbb{C}\psi_g, \quad \psi_g = 1/2(\psi_0 + \overline{\psi_0}), & \|\psi_g\| = 0, \\
\mathcal{S} &:= \mathbb{C}\psi_s, \quad \psi_s = 1/(2i)\,(\psi_0 - \overline{\psi_0}), & \|\psi_s\| = 0, \\
\mathcal{K}^+ &:= \{ \sum_{n\in\mathbb{N}} c_n \psi_n | \sum_{n\in\mathbb{N}} |c_n|^2 < \infty \}, & \|\psi_n\| > 0, \ n \in \mathbb{N}, \\
\mathcal{K}^- &:= \{ \sum_{n\in\mathbb{N}} d_n \overline{\psi_n} | \sum_{n\in\mathbb{N}} |d_n|^2 < \infty \}, & \|\overline{\psi_n}\| < 0, \ n \in \mathbb{N}
\end{aligned}$$

Construction of carrier space

Krein space

- $\mathcal{H}$ is a Krein space $(\mathcal{H}, \langle\cdot,\cdot\rangle, J)$, J - fundamental symmetry
  - $\langle\cdot,\cdot\rangle$ is indefinite product (K-G)
  - $\langle\cdot, J\cdot\rangle$ is positive product and
    $J^2 = I, \quad \langle\cdot, J\cdot\rangle = \langle J\cdot, \cdot\rangle, \quad |\langle\cdot,\cdot\rangle| \leq \langle\cdot, J\cdot\rangle$
  - $(\mathcal{H}, \langle\cdot, J\cdot\rangle)$ is a Hilbert space

Krein space as a carrier space

- fundamental symmetry J

$$J\psi_n := \psi_n, \quad n \in \mathbb{N}, \qquad J\psi_g := \psi_s$$

$$J\overline{\psi_n} := -\overline{\psi_n}, \quad n \in \mathbb{N}, \qquad J\psi_s := \psi_g$$

- $\mathcal{H} = \mathcal{N} \oplus_J \mathcal{S} \oplus_J \mathcal{K}^+ \oplus_J \mathcal{K}^- = (\mathcal{N} \dotplus \mathcal{S}) \oplus \mathcal{K}^+ \oplus \mathcal{K}^-$
- n.b. $\langle\psi_g, \psi_s\rangle = 1$

Construction of carrier space

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Cohomology - definitions

Definition

A representation π on $\mathcal{H}$ is called irreducible if there is no non-trivial closed invariant subspace.

$$\mathcal{W} \subset \mathcal{H}, \quad \pi(G)\mathcal{W} \subset \mathcal{W} \ \Rightarrow \ \mathcal{W} = \mathcal{H} \text{ or } \mathcal{W} = 0.$$

π is called topologically indecomposable if there are no non-zero closed invariant subspaces $\mathcal{U}$ and $\mathcal{V}$ of $\mathcal{H}$ such that $((\mathcal{U}+\mathcal{V})^{\perp})^{\perp} = \mathcal{H}$ and $\mathcal{U} \cap \mathcal{V} = 0$

Example

$$\pi(g) = \begin{pmatrix} \pi_1(g) & c_{12}(g) & c_{13}(g) \\ 0 & \pi_2(g) & c_{23}(g) \\ 0 & 0 & \pi_3(g) \end{pmatrix}$$

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Cohomology - definitions

Definition

Let π_1, π_2 be representations on $\mathcal{H}_1, \mathcal{H}_2$. A function $c(g_1, ..., g_n)$ of $g_k \in G$ with values in $\mathscr{L}(\mathcal{H}_1, \mathcal{H}_2)$, *i.e.* in the set of everywhere-defined linear mappings from $\mathcal{H}_1$ into $\mathcal{H}_2$, is called n-cochain. The set of all n-cochains is denoted by $C^n(\pi_1, \pi_2)$.

Definition

The coboundary operation δ (satisfying $\delta^2 = 0$)

$$(\delta c_n)(g_1, ..., g_{n+1}) := \pi_2(g_1) c_n(g_2, ..., g_n) + \sum_{i=1}^{n} (-1)^n c_n(g_1, ..., g_i g_{i+1}, ..., g_{n+1}) + (-1)^{n+1} c_n(g_1, ..., g_n) \pi_1(g_{n+1}),$$

where $c_n \in C^n(\pi_1, \pi_2), g_1, ..., g_n \in G$.

2-cochain

$$(\delta c_1)(g_1, g_2) = \pi_2(g_1) c_1(g_2) + c_1(g_1) \pi_1(g_2)$$

Cohomology - definitions

Definition

Cocycle:
$Z^n(\pi_1, \pi_2) := \left\{ c_n \in C^n \,\middle|\, \delta c_n = 0 \right\}$
Coboundary:
$B^n(\pi_1, \pi_2) := \delta C^{n-1}(\pi_1, \pi_2), \quad B^0(\pi_1, \pi_2) := 0$
Cohomology:
$H^n(\pi_1, \pi_2) := Z^n(\pi_1, \pi_2)/B^n(\pi_1, \pi_2)$

Definition

Let $c_1 \in C^m(\pi_2, \pi_3)$ and $c_2 \in C^n(\pi_1, \pi_2)$ we define $\times$ operation as

$$(c_1 \times c_2)(g_1, ..., g_{m+n}) := c_1(g_1, ..., g_m)c_2(g_{m+1}, ..., g_{m+n})$$

Indecomposable representation and cohomology

Example

$$\pi(g) = \begin{pmatrix} \pi_1(g) & c_{12}(g) & c_{13}(g) \\ 0 & \pi_2(g) & c_{23}(g) \\ 0 & 0 & \pi_3(g) \end{pmatrix}$$

π is a representation

- $\delta c_{12} = \delta c_{23} = 0$
- $\delta c_{13} = -c_{12} \times c_{23}$

Invariant complements

- $\mathcal{H}_1$ does not have any invariant complement iff $c_{12} \notin B^1(\pi_2, \pi_1)$.
- the existence of indecomposable representation implies $H^1(\pi_2, \pi_1) \neq 0$, $H^1(\pi_3, \pi_2) \neq 0$

Gupta-Bleuler triplet

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- Gupta-Bleuler triplet

$$\left.\begin{array}{c} \pi_3 \to \pi_2 \to \pi_1 \\ \mathcal{H}_3 \supset \mathcal{H}_2 \supset \mathcal{H}_1 \end{array}\right\}$$

- Indecomposable representation

$$\pi(g) = \begin{pmatrix} \pi_1(g) & c_{12}(g) & c_{13}(g) \\ 0 & \pi_2(g) & c_{23}(g) \\ 0 & 0 & \pi_3(g) \end{pmatrix}$$

- π_j is a representation on $\mathcal{R}_j := \mathcal{H}_{j+1}/\mathcal{H}_j$
  $\mathcal{H}_3 := \mathcal{H}$ and $\mathcal{R}_1 := \mathcal{H}_1$

G-B triplet in the standard form

Definition

- definition [1985 Araki, Comm. Math. Phys.]

$$\left.\begin{array}{l}\pi_1^{\#} \to \pi_2 \to \pi_1 \\ \mathcal{H}_3 \supset \mathcal{H}_2 \supset \mathcal{H}_1\end{array}\right\}$$

- π_3 is a conjugate of π_1

$$\forall g \in G,\ \phi \in \mathcal{H}_1^{\#},\ \psi \in \mathcal{H}_1,\ \langle \pi_1^{\#}(g^{-1})\phi, \psi\rangle = \langle \phi, \pi_1(g)\psi\rangle$$

Massless minimally coupled field

- for mmcf

$$\mathcal{H}_1 = \mathcal{N}, \quad \mathcal{H}_2 := \mathcal{H}_1^{\perp} = \mathcal{N} \oplus \mathcal{K}^- \oplus \mathcal{K}^+, \ \mathcal{H} \equiv \mathcal{H}_3 = (\mathcal{N} \dotplus \mathcal{S}) \oplus \mathcal{K}^- \oplus \mathcal{K}^+$$

$$\mathcal{R}_1 = \mathcal{N}, \qquad \mathcal{R}_2 = \mathcal{H}_2/\mathcal{H}_1 \simeq \mathcal{K}^- \oplus \mathcal{K}^+, \ \mathcal{R}_3 = \mathcal{H}_3/\mathcal{H}_2 \simeq \mathcal{S}$$

- 'one particle sector' ($\mathcal{R}_2$) contains also modes with negative norm

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Proposition

The representation of the de Sitter group corresponding mmc in the Krein space $\mathcal{H}$ defines a Gupta-Bleuler triplet in the standard form

$$\pi(g) = \pi_1^{\#}(g) \to \pi_2(g) \to \pi_1(g)$$

on the space

$$\mathcal{H} = \mathcal{N} \oplus_J \left(\mathcal{K}^+ \oplus_J \mathcal{K}^-\right) \oplus_J \mathcal{S},$$

with subspaces $\mathcal{H}_j$ and $\mathcal{R}_j$

$$\mathcal{H}_1 = \mathcal{N},\ \mathcal{H}_2 = \mathcal{N} \oplus_J \left(\mathcal{K}^+ \oplus_J \mathcal{K}^-\right),\ \mathcal{H}_3 = \mathcal{H} = \mathcal{N} \oplus_J \mathcal{K}^+ \oplus_J \mathcal{K}^- \oplus_J \mathcal{S}$$
$$\mathcal{R}_1 = \mathcal{H}_1 = \mathcal{N},\ \mathcal{R}_2 = \mathcal{H}_2/\mathcal{H}_1 \simeq \mathcal{K}^+ \oplus_J \mathcal{K}^-,\ \mathcal{R}_3 = \mathcal{H}_3/\mathcal{H}_2 \simeq \mathcal{S}$$

Operator valued matrix elements act as

$$\begin{aligned}
\pi_1(g)\psi_g &= \psi_g, \\
\pi_2(g)\psi_2 &= P_2(\pi(g)\psi_2),\ \psi_2 \in \mathcal{R}_2 \\
\pi_3(g)\psi_s &= \langle \psi_g, \pi(g)\psi_s\rangle \psi_s = \psi_s, \\
c_{12}(g)\psi_2 &= \langle \psi_s, \pi(g)\psi_2\rangle \psi_g,\ \psi_2 \in \mathcal{R}_2 \\
c_{23}(g)\psi_s &= P_2(\pi(g)\psi_s - \psi_s), \\
c_{13}(g)\psi_s &= \langle \psi_s, \pi(g)\psi_s\rangle \psi_g,
\end{aligned}$$

where $\pi(g)\psi(x) := \psi(g^{-1}.x)$, P_2 denotes the OG (in $\langle\cdot, J\cdot\rangle$) projector on $\mathcal{R}_2$.

GRT construction

Other invariant subspaces

- $\mathcal{N} \oplus \mathcal{K}^+$ is invariant
- $\left(\mathcal{N} \oplus \mathcal{K}^+\right)^\perp = \mathcal{N} \oplus \mathcal{K}^-$ is also invariant

Matrix representation

$$\pi(g) = \begin{pmatrix} \pi_1(g) & c_{12}^+(g) & c_{12}^-(g) & c_{13}(g) \\ 0 & \pi_2^+(g) & 0 & c_{23}^+(g) \\ 0 & 0 & \pi_2^-(g) & c_{23}^-(g) \\ 0 & 0 & 0 & \pi_1^*(g) \end{pmatrix}$$

- Krein space decomposition

$$\mathcal{H}_1 = \mathcal{N},\ \mathcal{H}_2^+ = \mathcal{N} \oplus_J \mathcal{K}^+,\ \mathcal{H}_2^- = \mathcal{N} \oplus_J \mathcal{K}^+ \oplus_J \mathcal{K}^-,\ \mathcal{H}_3 = \mathcal{H}$$
$$\mathcal{R}_1 = \mathcal{N},\ \mathcal{R}_2^+ = \mathcal{H}_2^+/\mathcal{H}_1 \simeq \mathcal{K}^+,\ \mathcal{R}_2^- = \mathcal{H}_2^+/\mathcal{H}_2^- \simeq \mathcal{K}^-,\ \mathcal{R}_3 = \mathcal{H}_3/\mathcal{H}_2^- \simeq \mathcal{S}$$

- Matrix elements

$$\pi_2^\pm(g)\psi_2^\pm = P_2^\pm(\pi(g)\psi_2^\pm),\ \psi_2^\pm \in \mathcal{R}_2^\pm = \mathcal{K}^\pm$$
$$c_{12}^\pm(g)\psi_2^\pm = \langle \psi_s, \pi(g)\psi_2^\pm \rangle \psi_g,\ \psi_2^\pm \in \mathcal{R}_2^\pm = \mathcal{K}^\pm$$
$$c_{23}^\pm(g)\psi_s = P_2^\pm(\pi(g)\psi_s - \psi_s)$$

Summary

Results

- Zero mode problem can be solved by Gupta-Bleuler like quantization
- G-R-T construction is Gupta-Bleuler triplet in the standard form with richer inner structure
- the existence of Gupta-Bleuler triplet can be formulated in cohomological conditions
- the method for $p = 1$ can be extended to higher representations, $\dim(\mathcal{N}) > 1$

Next direction

- detailed study of finite dimensional representations on zero-norm subspaces
- quantum field theory based on indecomposable representations

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