Quasi-Hermitian operators

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Quasi-Hermiticity

Definition

Let $A \in \mathscr{L}(\mathcal{H})$ be densely defined. A is called quasi-Hermitian, if there exists an operator Θ with properties
(i) $\Theta \in \mathscr{B}(\mathcal{H})$,
(ii) $\Theta > 0$,
(iii) $\Theta A = A^* \Theta$.

- mathematics - Diedonné 1961, Proceedings Of The International Symposium on Linear Spaces
- physics - Scholtz, Geyer, Hahne 1992, Annals of Physics

$\dim \mathcal{H} = N$

diagonalizable matrices with real spectrum

- $\Theta > 0 \;\Rightarrow \Theta$ is invertible
- similarity transformation $\varrho = \sqrt{\Theta}$, $\varrho^{-1} A \varrho$ is Hermitian
- modification of scalar product $\langle \cdot, \cdot \rangle_\Theta := \langle \cdot, \Theta \cdot \rangle$, A is Hermitian in $\langle \cdot, \cdot \rangle_\Theta$
- $\left.\begin{array}{rcl} A & = & X^{-1} D X \\ A^* & = & X^* D (X^{-1})^* \end{array}\right\} \Rightarrow X^* X A = A^* X^* X \Rightarrow \Theta = X^* X$
- $\Theta = \sum_{j=1}^{N} \langle \phi_j, \cdot \rangle \phi_j$ $\qquad \{\phi_j\}_{j=1}^{N}$ are eigenvectors of A^*

$\dim \mathcal{H} = \infty$

- $\Theta > 0 \;\Rightarrow \mathrm{Ker}(\Theta) = \{0\}$, i.e. Θ^{-1} exists but can be unbounded equivalently $0 \in \sigma_c(\Theta)$
- if $\Theta^{-1} \in \mathscr{B}(\mathcal{H})$ then A is similar to a self-adjoint operator

Theorem (Dieudonné 1961)

Let A be a bounded quasi-Hermitian operator, $\varrho := \sqrt{\Theta}$. Then there is a uniquely determined bounded Hermitian operator B such that

$$\varrho A = B\varrho \quad \textit{or, equivalently} \quad A^* \varrho = \varrho B.$$

Corollary

Let $A \in \mathscr{B}(\mathcal{H})$ be quasi-Hermitian. Then every eigenvalue of A is real.

Example

- $\mathcal{H} = \oplus_{n=1}^{\infty} \mathcal{H}_n$, where $\dim \mathcal{H}_n = n$
- A_n acting on $\mathcal{H}_n$
- $$A_n = \begin{pmatrix} \alpha_1 & 0 & \dots & & 0 \\ 1 & \alpha_2 & \dots & & 0 \\ \dots & \dots & \dots & & \dots \\ 0 & 0 & \dots & 1 & \alpha_n \end{pmatrix}$$
- α_n are distinct real numbers and $\sum_{j=1}^{n} \alpha_j^2 \leq 1$
- $\|A_n\| \leq 2$
- $\xi \in \mathbb{C}$, $|\xi| = 1$, $x_n := e_1 - \frac{1}{\xi} e_2 + \frac{1}{\xi^2} e_3 - \dots + (-1)^{n-1} \xi^{-n+1} e_n$
- $\|x_n\| = \sqrt{n}$ and $\|(A_n + \xi) \frac{x_n}{\|x_n\|}\| \leq \frac{2}{\sqrt{n}}$

Example

- there exists a bounded operator A such that the restriction on $\mathcal{H}_n$ is A_n
- eigenvalues of A_n are real and distinct $\Rightarrow$ there exists a positive Θ_n such that $\Theta_n A_n = A_n^* \Theta_n$
- we can assume that $\|\Theta_n\| \leq 1 \;\Rightarrow\;$ there exists a bounded positive operator Θ such that restriction on $\mathcal{H}_n$ is Θ_n and $\Theta A = A^* \Theta$
- every $|\xi| = 1$ is in the spectrum of A for $\|(A+\xi)\frac{x_n}{\|x_n\|}\| \leq \frac{2}{\sqrt{n}}$
- $I + A^2$ has not bounded inverse

Pseudo-Hermiticity

further $\Theta^{-1} \in \mathscr{B}(\mathcal{H})$ only

Definition

Let $A \in \mathscr{L}(\mathcal{H})$ be densely defined. A is called pseudo-Hermitian, if there exists an operator η with properties
(i) $\eta, \eta^{-1} \in \mathscr{B}(\mathcal{H})$,
(ii) $\eta = \eta^*$,
(iii) $A = \eta^{-1} A^* \eta$.

- pseudo-Hermitian operators are closed
- spectra of A and A^* are equal
- pseudo-Hermiticity does not exclude non-empty residual spectrum

Antilinear symmetry

Definition

Let A be a closed densely defined operator on $\mathcal{H}$. We say that A has an antilinear symmetry if there exists an antilinear bijective operator C and the relation $AC\psi = CA\psi$ holds for all $\psi \in \mathrm{Dom}(A)$.

- $\lambda \in \sigma_{p,c,r}(A) \Longleftrightarrow \overline{\lambda} \in \sigma_{p,c,r}(A)$
- antilinear symmetry does not exclude non-empty residual spectrum
- pseudo-Hermiticity and antilinear symmetry are equivalent properties for matrices
- the equivalence is not valid even for bounded operators
- pseudo-Hermiticity together with antilinear symmetry $\Rightarrow$ empty residual spectrum

PT-symmetric example

Example

- $\mathcal{H} = L_2(\mathbb{R})$
- $H = -\frac{d^2}{dx^2} + V(x), \ V(x) = \overline{V(-x)}$
- $\mathcal{P}$-pseudo-Hermitian $H^* = \mathcal{P} H \mathcal{P}$
- antilinear symmetry $\mathcal{PT}$, $[\mathcal{PT}, H] = 0$
- $\mathcal{P}$ parity, $(\mathcal{P}\psi)(x) = \psi(-x)$
- complex conjugation $\mathcal{T}$, $(\mathcal{T}\psi)(x) = \overline{\psi}(x)$
- $H = \mathcal{T} H^* \mathcal{T}$

J-self-adjointness

Definition

Let A be a densely defined operator on $\mathcal{H}$. Let J be an antilinear isometric involution, i.e. $J^2 = I$ and $\langle Jx, Jy\rangle = \langle y, x\rangle$ for all $x, y \in \mathcal{H}$. A is called J-symmetric if $A \subset JA^*J$. A is called J-self-adjoint if $A = JA^*J$.

- J-self-adjointness $\Rightarrow$ empty residual spectrum
- $\mathcal{PT}$-symmetry + $\mathcal{P}$-pseudo-Hermiticity $\Rightarrow$ $\mathcal{T}$-self-adjointness (previous example)

Criterion of similarity to the self-adjoint operator

Theorem (1984, Naboko)

Let $A \in \mathscr{L}(\mathcal{H})$. A is similar to a self-adjoint operator if and only if

$$\sup_{\varepsilon>0} \varepsilon \int_{-\infty}^{\infty} \|(A-\lambda I)^{-1}\psi\|^2 \mathrm{d}\xi \leq M\|\psi\|^2,$$

$$\sup_{\varepsilon>0} \varepsilon \int_{-\infty}^{\infty} \|(A^*-\lambda I)^{-1}\psi\|^2 \mathrm{d}\xi \leq M\|\psi\|^2,$$

where $\lambda = \xi + \mathrm{i}\,\varepsilon, \psi \in \mathcal{H}$ and the integration is carried along an arbitrary straight line, parallel to the real axis, in the upper half plane.

- pseudo-Hermiticity $\Rightarrow$ only one inequality is needed
- pseudo-Hermitian operator with real spectrum is not automatically quasi-Hermitian
- how to construct similarity transformation or 'metric' Θ?

Metric

Proposition

Let A be densely defined quasi-Hermitian operator with discrete spectrum and 'metric' Θ. Let $\{\phi_n\}_{n=1}^{\infty}$ be eigenvectors of A^* and $\|\phi_n\| = 1$. Then

$$\Theta = \operatorname*{s-lim}_{N\to\infty} \sum_{j=1}^{N} c_j \langle \phi_j, \cdot \rangle \phi_j,$$

where c_j are positive numbers satisfying $0 < m \leq c_j \leq M < \infty$.

- antilinear symmetry + pseudo-Hermiticity + real discrete spectrum $\Rightarrow$ quasi-Hermiticity ?
- NO

Why?

- Θ-sum does not converge for all $\psi \in \mathcal{H}$
- $0 \in \sigma_c(\Theta)$
- non-completness of $\{\phi_n\}$

Little help

- If ηC is a antilinear isometric involution (A is ηC-self-adjoint) then existence and invertibility of Θ implies $\Theta^{-1} \in \mathscr{B}(\mathcal{H})$
- example $\mathcal{PT}$-symmetry + $\mathcal{P}$-pseudo-Hermiticity

Example

Example (2006, Krejčiřík, Bíla, Znojil)

- $\mathcal{H} = L_2(0,d), \quad (\mathcal{P}\psi)(x) := \psi(d-x)$
- $\mathcal{PT}$-symmetric and $\mathcal{P}$-pseudo-Hermitian point interaction
- $H = -\frac{d^2}{dx^2}$ and $\psi'(0) + i\alpha\psi(0) = 0, \; \psi'(d) + i\alpha\psi(d) = 0$
- $\sigma(H) = \{\alpha^2\} \cup \{\frac{j\pi}{d}\}_{j=1}^{\infty}$
- $\psi_0(x) = A_0 \exp(-i\alpha x), \quad \psi_j(x) = A_j \left(\cos(\frac{j\pi}{d}x) - i\frac{\alpha d}{j\pi}\sin(\frac{j\pi}{d}x)\right)$

Example

- $\Theta = I + \phi_0\langle\phi_0, \cdot\rangle + \Theta_0 + i\alpha\Theta_1 + \alpha^2\Theta_2$,
  where
  - $\phi_0 = \sqrt{\frac{1}{d}\exp(i\alpha x)}$,
  - $(\Theta_0\psi)(x) = -\frac{1}{d}(J\psi)(d)$,
  - $(\Theta_1\psi)(x) = 2(J\psi)(x) - \frac{x}{d}(J\psi)(d) - \frac{1}{d}(J^2\psi)(d)$,
  - $(\Theta_2\psi)(x) = -(J^2\psi)(x) + \frac{x}{d}(J^2\psi)(d)$,
  - with $(J\psi)(x) = \int_0^x \psi$.
- if $\alpha = \frac{j\pi}{d}$ then $\mathrm{Ker}(\Theta) \neq \{0\}$

Example with continuous spectrum

Example (2004, Albeverio, Kuzhel)

- $\mathcal{H} = L_2(\mathbb{R})$ $\mathcal{PT}$-symmetric point interaction at origin
- $H = -\frac{d^2}{dx^2} + V(x)$
- $V = a\langle\delta, \cdot\rangle\delta + b\langle\delta', \cdot\rangle\delta + c\langle\delta, \cdot\rangle\delta' + d\langle\delta', \cdot\rangle\delta'$
- spectrum of H is $[0, \infty)$ plus at most two eigenvalues (real or complex conjugate pair)
- H is not similar to the self-adjoint operator for all choices of parameters (a, b, c, d) (real spectrum)

Example with continuous spectrum

Example

- metric

$$\Theta_{\tau,\omega} = \frac{\tau^2+1}{2\tau} I + \frac{\tau^2-1}{4\tau} (\operatorname{sign} x) \left(e^{i\omega}(I+\mathcal{P}) + e^{-i\omega}(I-\mathcal{P}) \right),$$

  where

$$\tau = \sqrt{\frac{(|b|+2)^2+ad}{(|b|-2)^2+ad}}, \quad e^{i\omega} = \frac{|b|}{b},$$

  where $b \neq 0$ and $((|b|^2+2)^2+ad)((|b|-2)^2+ad) > 0$.

- $\sqrt{\Theta}$ maps H to a self-adjoint point interaction